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**AUSTRALIAN ATOMIC ENERGY COMMISSION**  
**RESEARCH ESTABLISHMENT**  
**LUCAS HEIGHTS**

**COMPOUND NUCLEUS FORMULATION OF REACTION MATRIX THEORY**

by

**J.L. COOK**  
**W.K. BERTRAM**

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ABSTRACT

It is shown that multilevel resonance parameters for each element of the reaction matrix cannot be determined from available data. However, additional constraints may be introduced without affecting agreement with experiment. The Bohr compound nucleus hypothesis is applied to the T-matrix and it is found, as in Newton's model, that the channel matrix can be inverted analytically to provide simple formulae for cross sections, for both the real Wigner-Eisenbud reaction matrix and Moldauer's complex reaction matrix. Wigner-Eisenbud theory leads directly to Newton's strong correlation model and its unacceptable consequences. Moldauer's theory does not and can explain cross section behaviour adequately while being consistent with Bohr's hypothesis.

Abstract (cont.)

Abstract (cont.)

Cross sections can be written as a sum of single level contributions, as in the Adler-Adler formulation. Finally, Moldauer's statistical theory is shown to be applicable, and expressions are derived for the averaged cross sections as functions of the complex Moldauer resonance parameters.

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## 1. INTRODUCTION

The concept of a compound nucleus was introduced by Bohr (1936), and for some time it was regarded as the principal nuclear reaction mechanism. At about the same time Breit and Wigner (1936) proposed their celebrated single level formula, which gave excellent agreement with isolated levels in cross sections, and satisfied the Bohr compound nucleus postulate. Much later, Wigner and Eisenbud (1947) presented a more rigorous multilevel theory which was capable of explaining the strong interference between levels observed in fission cross sections. This theory did not employ the Bohr postulate concerning the resonant cross sections, except to assume the existence of a single internal compound state wave function and its associated Hamiltonian. There occurred in this theory a troublesome channel matrix which had to be inverted, and Wigner and Eisenbud gave an expansion for this inversion. Later refinements of the multilevel formalism by Vogt (1958) and Reich and Moore (1958) used the concept of replacing the channel matrix to be inverted by a level matrix and then employing approximations to evaluate the inverse. None of these multilevel theories uses the Bohr assumption.

A multilevel theory which does lead directly to the Bohr assumption is the theory employing strong correlation conditions put forward by Teichmann (1950) and Newton (1952). This theory was discarded by later authors because the correlations assumed were not observed in experimentally determined widths, and because all cross sections are predicted to vanish between resonances. Lane and Thomas (1958) concluded from this that the Bohr postulate must be incorrect as an exact relation, and instead holds for averaged cross sections over a large number of resonances. It is pointed out in this paper that this is not the only alternative. If future experiments should reveal that the Bohr postulate is in fact an exact relationship, one can accommodate this conclusion within the framework of reaction matrix theory by using the complex R-matrix defined by Moldauer (1964). This does not imply the strong correlations assumed by Newton and the only difficulty encountered is that the single level approximation no longer satisfies unitarity, which requires compensating multilevel effects.

From a pragmatic point of view, the theories of Vogt and Reich and Moore suffer from the defect that the formulae obtained are so complicated that they are generally not useful for analytical calculations of resonance absorption in reactors, and analytic expressions for the Doppler-broadened cross sections do not exist. However, the formalism

proposed by Adler and Adler (1963) is equivalent to the above theories, does allow us to write analytic formulae for Doppler-broadened cross sections, and lends itself readily to calculations of resonance absorption in reactors, as it represents cross sections as sums over single level contributions. We show that if we employ the Bohr hypothesis, the theory of Adler and Adler remains exact over restricted energy ranges.

Most authors would agree that in neutron resonance reactions, the measurements available are insufficient to determine every element of the reaction matrix. This is discussed in detail and a transformation between equivalent reaction matrices is given. Furthermore with the Bohr hypothesis, and unitarity, we can construct a reduced transition matrix of rank one, in which all channels have the same phase, called the compound phase. It follows that the Wigner-Eisenbud or Moldauer reaction matrix is of rank one. Using this property, the channel matrix is inverted to yield an expression for the scattering matrix in which the compound phase is given as a function of the reaction matrix. Formulae for cross sections are then derived which are manifestly consistent with Bohr's hypothesis.

Conditions for the general reaction matrix are derived which ensure that the collision matrix is symmetric and unitary. The Wigner-Eisenbud boundary conditions lead inevitably to the strong correlation model of Newton with its unsatisfactory features, but Moldauer's complex boundary conditions permit the absence of cross section zeroes and do not imply the strong correlation conditions.

The poles and residues of the S-matrix are evaluated and multilevel effects are found to alter the positions of complex poles from their values in the single level approximation. This approximation violates unitarity in the Moldauer theory. It is demonstrated that the cross sections can be parameterised exactly as a sum of single level contributions with asymmetric correction terms.

The formal statistical theory given by Moldauer (1964) is applicable to our representation of cross sections. His equations are reviewed and presented in our terminology.

## 2. STANDARD THEORY

In terms of the S matrix, the cross section for a reaction from incident channel,  $\alpha$ , to final channel,  $\alpha'$ , is as given in Preston (1962).



$$\sigma_{cc'} = \frac{\pi}{k_c^2} \sum_{\ell j J j' \ell'} g_J \left| \delta_{c\ell j, c'\ell' j'} - S_{c\ell j, c'\ell' j'}^J \right|^2 \quad \dots(1)$$

where

$$k_c^2 = 2\mu_c E_c$$

$E_c$  = the energy in channel  $\alpha$   
 $\mu_c$  = the reduced mass in channel  $\alpha$   
 $\ell$  = the orbital angular momentum  
 $j$  = the channel spin  
 $I$  = the spin of the target nucleus  
 $J$  = the total angular momentum

$$g_J = \frac{2J+1}{2(2I+1)}$$

The total cross section may be expressed as

$$\sigma_{\sigma T} = \frac{2\pi}{k_c^2} \sum_{\ell j J} g_J \left[ 1 - \text{Re} \left( S_{c\ell j, c\ell j}^J \right) \right] \quad \dots(2)$$

The transition matrix  $\underline{T}$  is defined as

$$\underline{T} = \frac{1}{2i} (\underline{S} - \underline{I}) \quad \dots(3)$$

The  $\underline{S}$  matrix is unitary,

$$\text{that is } \underline{S}^+ \underline{S} = \underline{S} \underline{S}^+ = \underline{I} \quad \dots(4)$$

from which it follows that

$$\text{Im } \underline{T} = \underline{T}^+ \underline{T} \quad \dots(5)$$

The  $\underline{S}$  matrix is related to the reaction matrix  $\underline{R}$  by the equation

$$\underline{S} = \underline{\Omega} \left[ \underline{I} + 2i \underline{P}^{\frac{1}{2}} (\underline{I} - \underline{R} \underline{L}^{\circ})^{-1} \underline{R} \underline{P}^{\frac{1}{2}} \right] \underline{\Omega} \quad \dots(6)$$

where

$$R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E} \quad \dots(7)$$

$\gamma_{\lambda c}$  = real constants in the Wigner-Eisenbud theory, and complex constants in the Moldauer theory.

$E_{\lambda}$  = energy at poles in  $\underline{R}$ . These are real constants in the Wigner-Eisenbud theory and complex constants in the Moldauer theory.

$$\underline{L}^{\circ} = \underline{L} - \underline{B} \quad ,$$

$$\underline{L} = \underline{S} + i \underline{P} \quad .$$

$\underline{\Omega}$ ,  $\underline{P}$ ,  $\underline{S}$  are diagonal matrices with elements

$$\Omega_c = e^{-i\phi_c}$$

$P_c$  = penetration factor

$S_c$  = level shift

$B_{cc'}$  = boundary condition matrix

In the above two theories

$$B_{cc'} = B_c \delta_{cc'}$$

where  $B_c$  is real in the Wigner-Eisenbud theory and complex in the Moldauer theory. The evaluation of these quantities has been reported by many authors (e.g. Preston (1962), Lane and Thomas (1958), Lynn (1968)).

The transition matrix is then given by

$$\underline{T} = \frac{1}{2i} (\underline{\Omega}^2 - \underline{I}) + \underline{\Omega} \underline{P}^{\frac{1}{2}} (\underline{I} - \underline{R} \underline{L}^{\circ})^{-1} \underline{R} \underline{P}^{\frac{1}{2}} \underline{\Omega} \quad \dots(8)$$

It is also convenient to define the reduced transition matrix,  $\underline{T}'$ , through the relation

$$\underline{T} = \frac{1}{2i} (\underline{\Omega}^2 - \underline{I}) + \underline{\Omega} \underline{T}' \underline{\Omega} \quad \dots(9)$$

Then

$$\underline{T}' = \underline{P}^{\frac{1}{2}} (\underline{I} - \underline{R} \underline{L}^{\circ})^{-1} \underline{R} \underline{P}^{\frac{1}{2}} \quad \dots(10)$$

The 'reduced  $\underline{S}$  matrix'  $\underline{V}$  obeys

$$\underline{V} = \underline{\Omega}^{-1} \underline{S} \underline{\Omega} \quad \dots(11)$$

$$= \underline{I} + 2i \underline{T}' \quad \dots(12)$$

and is unitary. Therefore  $\underline{T}'$  satisfies an equation similar to equation (5).

$$\text{Im } \underline{T}' = \underline{T}' + \underline{T}' \quad \dots(13)$$

The complexity of multichannel expressions for cross sections arose from the difficulty in inverting the channel matrix  $(\underline{I} - \underline{R} \underline{L}^{\circ})^{-1}$ .

$$\text{Let } \underline{Z}_1 = \underline{P}^{\frac{1}{2}} \underline{R} (\underline{L} - \underline{B}) \underline{P}^{-\frac{1}{2}} \quad \dots(14a)$$

$$\underline{Z}_2 = \underline{P}^{\frac{1}{2}} \underline{R} (\underline{L}^* - \underline{B}) \underline{P}^{-\frac{1}{2}} \quad \dots(14b)$$

Then it follows from (8) and (11) that

$$\underline{V} = (\underline{I} - \underline{Z}_1)^{-1} (\underline{I} - \underline{Z}_2) \quad \dots(15)$$

Applying the unitary condition

$$\underline{V}^+ \underline{V} = \underline{I} \quad \dots(16)$$

one finds after some manipulation that the conditions under which (16) should hold are that

$$(\underline{I} - \underline{Z}_1) (\underline{I} - \underline{Z}_1^+) = (\underline{I} - \underline{Z}_2) (\underline{I} - \underline{Z}_2^+)$$

which yields

$$\underline{R} - \underline{R}^+ = \underline{R} (\underline{B} - \underline{B}^+) \underline{R}^+ \quad \dots(17)$$

Similarly, the conditions under which  $\underline{V}$  should be a symmetric matrix are, letting T denote the operation of transposition

$$\underline{V} = \underline{V}^T$$

$$(\underline{I} - \underline{Z}_1) (\underline{I} - \underline{Z}_1^T) = (\underline{I} - \underline{Z}_2) (\underline{I} - \underline{Z}_2^T)$$

which gives the condition

$$\underline{R} - \underline{R}^T = \underline{R} (\underline{B} - \underline{B}^T) \underline{R}^T \quad \dots(18)$$

It follows immediately from (17) and (18) that if the Wigner-Eisenbud boundary conditions are used,

$$\underline{B} = \underline{B}^+ , \quad \underline{B}^T = \underline{B}$$

$$\underline{R} = \underline{R}^+ , \quad \underline{R}^T = \underline{R} \quad \dots(19)$$

and the Wigner-Eisenbud  $\underline{R}$  matrix is real and symmetric. In Moldauer's theory,  $\underline{B}$  is no longer self-adjoint but is still symmetric, so the Moldauer  $\underline{R}$  matrix is a complex symmetric matrix obeying

$$\underline{R} - \underline{R}^* = \underline{R} (\underline{B} - \underline{B}^*) \underline{R}^* ; \quad \underline{R} = \underline{R}^T \quad \dots(20)$$

where the asterisk denotes complex conjugation.

3. THE UNIQUENESS QUESTION

The main value of multilevel theories is that they show how to parameterise the collision matrix as a function of energy. Though it is not possible in neutron reactions to determine every element of the reaction matrix, Vogt and Reich and Moore have shown that within the framework of the level matrix expansion method, sums of reduced widths over channels arise which can be used as parameters. It is clear that in being forced to use these sums as parameters rather than individual reduced widths, we do not have the necessary information from experiment to determine each element of the reaction matrix.

To discuss this in more detail, let us consider the analysis of experimental cross sections whose purpose is to calculate the reaction matrix in some multilevel scheme. For such an analysis, the differential cross section,  $\sigma_{11}$ , for elastic scattering and the various total reaction cross sections,  $\sum_c \sigma_{1c}$ , are usually available. Initially, there are no restrictions on the elements of the  $\underline{S}$  matrix other than those imposed by symmetry and unitarity. If we consider the transition matrix  $\underline{T}$ , this means that at best, from the given data we can determine only the quantities  $\sum_c |T_{1c}|^2$ , or if in the extreme case, each cross section  $\sigma_{1c}$  is known, we know only  $|T_{1c}|$ . For the partial wave amplitude  $T_{11}$ , each phase shift is an eigenstate of  $(l, J)$  is known except for an arbitrary common phase which is usually set equal to zero. Therefore we shall assume that we can determine the phase of  $T_{11}$  but not of any other element. Thus, if an  $\underline{R}$  matrix has been obtained from the given data and if  $\underline{T}$  is the corresponding transition matrix with elements  $T_{cc'} = t_{cc'} e^{i\theta_{cc'}}$ , then in general it is possible to find another reaction matrix  $\underline{\tilde{R}}$ , which leads to a transition matrix  $\underline{\tilde{T}}$ , with elements  $\tilde{T}_{cc'} = \tilde{t}_{cc'} e^{i\theta_{cc'}}$ , such that this too will reproduce the given cross sections. The matrices  $\underline{R}$  and  $\underline{\tilde{R}}$  may then be related by introducing a symmetric matrix  $\underline{A}$  with elements

$$A_{cc'} = a_{cc'} e^{i\phi_{cc'}}$$

such that

$$\underline{\tilde{T}} = \underline{T} + \underline{A} \quad \dots(21)$$

Since  $\underline{S}$  and  $\underline{\tilde{S}}$  must both be unitary,  $\underline{A}$  must satisfy

$$(\underline{S}^+ \underline{A})^+ (\underline{S}^+ \underline{A}) = \text{Im} (\underline{S}^+ \underline{A}) \quad .$$

The matrix  $\underline{\tilde{T}}$  will reproduce the original data if

$$\begin{aligned}
 a_{11} &= 0 \\
 a_{1c} &= t_{1c} \sqrt{2 - 2 \cos (\tilde{\theta}_{1c} - \theta_{1c})} \quad (c \neq 1) \\
 \tan \phi_{1c} &= \frac{\cos \tilde{\theta}_{1c} - \cos \theta_{1c}}{\sin \tilde{\theta}_{1c} - \sin \theta_{1c}}
 \end{aligned}$$

The matrices  $\underline{R}$  and  $\tilde{\underline{R}}$  can then be related through equations (8) and (21).

It therefore follows that even in the extreme case where every reaction cross section is known, the reaction matrix is not uniquely determined by the available data. However the Wigner-Eisenbud form for  $\underline{R}$ , given by equation (7), does introduce additional constraints upon the  $\underline{S}$  matrix. In this matrix, there are  $n$  channels and  $2n^2$  functions. Unitarity and symmetry imply that only  $\frac{1}{2}n(n+1)$  of these functions are independent. Without further constraints, the Wigner-Eisenbud  $\underline{R}$  matrix would be real and symmetric and therefore would consist of  $\frac{1}{2}n(n+1)$  independent functions. However, the special form for  $\underline{R}$  in equation (7) states that if we specify  $n$  components as a function of energy, say along a row, then all of the Wigner-Eisenbud resonance parameters are determined and thereby all elements of  $\underline{R}$  are determined. In this case, there are only  $n$  independent functions which specify the  $\underline{S}$  matrix. This is also the number of cross sections,  $\sigma_{1c}$ , and it follows that the  $n$  independent functions,  $\sigma_{1c}$ , determine the  $\underline{S}$  matrix completely. The Wigner-Eisenbud derivation of equation (7) imposes  $\frac{1}{2}n(n-1)$  constraints in addition to unitarity and symmetry. In this sense, the Wigner-Eisenbud theory is not a general theory, as a general theory would not have these constraints. The generalised Wigner-Eisenbud reaction matrix would be of the form

$$\begin{aligned}
 R_{cc'} &= \sum_{\lambda} \frac{a_{\lambda cc'}}{E_{\lambda cc'} - E} , & \begin{aligned} a_{\lambda cc'} &= a_{\lambda c'c} , \\ E_{\lambda cc'} &= E_{\lambda c'c} , \end{aligned} \quad \dots(22)
 \end{aligned}$$

where  $a_{\lambda cc'}$  and  $E_{\lambda cc'}$  are real constants.

The additional constraints of the Wigner-Eisenbud theory are that the residues  $a_{\lambda cc'}$  form a matrix of rank one in channel space, and the eigenvalues  $E_{\lambda cc'}$  are independent of  $c$  and  $c'$ .

Let us consider the case where the above assumptions are valid. Even so, in actual measurements, we can determine only some number  $p$  of sums over the cross sections  $\sigma_{1c}$ , where  $p < n$ . Therefore,  $n-p$  functions

remain undetermined in the S matrix. For radiative capture processes, usually  $n \sim 100$  and  $p = 2$  or  $3$ , which leads to a large uncertainty in the phases of the T matrix. For Moldauer's theory, there are  $2n$  functions to be determined, and only  $n$  functions  $\sigma_{lc}$ . We would therefore require both the magnitude and phase of each  $T_{lc}$  to determine  $R_{cc'}$ , and this is clearly impossible. We can therefore conclude that to date, and in the foreseeable future, limitations in the data will never allow the determination of a unique R matrix for low energy neutron collisions with nuclei. Instead, one should use multilevel theories as a guide to ascertain the functional form of cross sections, and this form should be simplified as much as possible.

#### 4. THE COMPOUND NUCLEUS ASSUMPTION

Applying Bohr's compound nucleus hypothesis to the resonant part,  $\sigma_{cc'}^r$ , of all cross sections, we can write (Blatt and Weisskopf, 1952).

$$\sigma_{cc'}^r = \sigma_c(c) P_{c'}(c) \quad \dots(23)$$

where  $\sigma_c(c)$  is the cross section for formation of the compound nucleus in channel  $c$  and  $P_{c'}(c)$  is the probability of decay of the compound nucleus via channel  $c'$ ,

$$\sum_c P_c(c) = 1 ;$$

from reciprocity, it follows that

$$P_c(c) = \frac{k_c^2 \sigma_c(c)}{\sum_{c'} k_{c'}^2 \sigma_{c'}(c)}$$

Writing

$$G^2 = \frac{1}{4\pi} \sum_{c'} k_{c'}^2 \sigma_{c'}(c) ,$$

equation (23) may be expressed in the form

$$\sigma_{cc'}^r = \frac{4\pi}{k_c^2} G^2 P_c P_{c'} . \quad \dots(24)$$

We define the resonant part of the cross section to be the contribution from the reduced transition matrix T':

$$\sigma_{cc'}^r = \frac{4\pi}{k_c^2} |T'_{cc'}|^2 . \quad \dots(25)$$

Thus comparing (25) and (24) we find that

$$T_{cc'}^\dagger = \rho_c \rho_{c'} e^{i\theta_{cc'}} \quad , \quad \dots(26a)$$

with

$$\rho_c = \sqrt{G P_c} \quad \dots(26b)$$

The unitarity relation (13) yields the following equations for the phases  $\theta_{cc'}$  :

$$\sum_{c''} \rho_{c''}^2 \cos(\theta_{cc''} - \theta_{c''c'}) = \sin \theta_{cc'} \quad , \quad \dots(27a)$$

$$\sum_{c''} \rho_{c''}^2 \sin(\theta_{cc''} - \theta_{c''c'}) = 0 \quad ; \quad \dots(27b)$$

when  $c = c'$ , (27a) immediately yields

$$\sin \theta_{cc} = \sum_{c''} \rho_{c''}^2 \quad .$$

Following Newton, we define the compound phase by the equations

$$\begin{aligned} \sin \delta &= G = \sum_c \rho_c^2 \\ \cos \delta &= \sqrt{1 - G^2} \quad . \end{aligned}$$

Then the solutions of (27 a,b) for the phases of the diagonal elements of  $T'$  are

$$\theta_{cc} = \delta \quad \text{or} \quad \Pi - \delta \quad .$$

Newton maintained that only the first solution exists. This is not so as can be illustrated by the counter example for  $n=2$ , where equations (27 a,b) can be solved quite easily to yield two distinct solutions, namely

$$\theta_{11} = \theta_{12} = \theta_{22} = \delta$$

which is Newton's solution, and

$$\theta_{11} = \delta, \quad \theta_{22} = \Pi - \delta$$

$$\tan \theta_{12} = \frac{\rho_1^2 - \rho_2^2}{\cos \delta} \quad .$$

As far as calculating cross sections is concerned, these two solutions are completely equivalent as both will give the same value for the resonant part of the reaction cross section,  $\sigma_{cc'}^r$ . In the general case, when the number of channels is arbitrary, equations (27a, b) will have a large but nevertheless finite number of solutions, all of which, as far as the analysis of cross sections is concerned, are completely equivalent. Newton's solution, with all  $\theta_{cc'} = \delta$  is a particularly interesting one, as it leads to a simple R matrix theory. The choice of this solution in fact constitutes the equiphase principle investigated by Cook (1967). This principle, which has been applied to the photo-production of pions in relation to scattering (Bethe and de Hoffman 1955) has also often been applied, although perhaps unwittingly, to low energy nuclear reactions. This is because the most widely used form of R matrix theory, the single level approximation, embodies this same principle.

Thus the reduced transition matrix is of the form

$$T_{cc'}^i = \rho_c \rho_{c'} e^{i\delta} = \sqrt{P_c P_{c'}} e^{i\delta} \sin \delta \quad \dots(28)$$

and is therefore of rank one. Next we show that R, whether real or complex, must necessarily be of unit rank also.

From equation (10) we find

$$\begin{aligned} \underline{R} &= \underline{P}^{-\frac{1}{2}} \underline{T}^i \underline{P}^{-\frac{1}{2}} [\underline{I} + \underline{L}^o \underline{P}^{-\frac{1}{2}} \underline{T}^i \underline{P}^{-\frac{1}{2}}]^{-1} \\ &= \underline{A} [\underline{I} + \underline{C}]^{-1} \end{aligned}$$

where

$$\underline{A} = \underline{P}^{-\frac{1}{2}} \underline{T}^i \underline{P}^{-\frac{1}{2}}, \quad \underline{C} = \underline{L}^o \underline{A} \quad \dots(29)$$

Since the matrices L<sup>o</sup> and P are diagonal, we find that because A and C are both of unit rank

$$\begin{aligned} \underline{R} &= \underline{A} [\underline{I} - \underline{C} + \underline{C}^2 - \underline{C}^3 + \dots] \\ &= \underline{A} [\underline{I} - \underline{C} (1 + \text{trace } \underline{C})^{-1}] \\ &= (\underline{L}^o)^{-1} \underline{C} (1 + \text{trace } \underline{C})^{-1} \end{aligned}$$

It follows that R must also be of unit rank. This leads to the following results:



$$(\underline{R} \underline{L}^\circ)^n = (\text{trace } \underline{R} \underline{L}^\circ)^{n-1} \underline{R} \underline{L}^\circ \quad \dots(30a)$$

$$(\underline{I} - \underline{R} \underline{L}^\circ)^{-1} = \underline{I} + \underline{R} \underline{L}^\circ (1 - \text{trace } \underline{R} \underline{L}^\circ)^{-1} \quad \dots(30b)$$

$$(\underline{I} - \underline{R} \underline{L}^\circ)^{-1} \underline{R} = \underline{R} (1 - \text{trace } \underline{R} \underline{L}^\circ)^{-1} \quad \dots(30c)$$

We see that from the Bohr postulate (23), the previously difficult inversion of the channel matrix in (30c) become a simple analytical result. The  $\underline{S}$  matrix then becomes

$$\underline{S} = \underline{\Omega} [\underline{I} + 2i \underline{P}^{\frac{1}{2}} \underline{R} \underline{P}^{\frac{1}{2}} (1 - \text{trace } \underline{R} \underline{L}^\circ)^{-1}] \underline{\Omega} \quad \dots(31)$$

or

$$\begin{aligned} S_{cc'} &= \Omega_c \Omega_{c'} \delta_{cc'} + 2i \Omega_c \Omega_{c'} \sqrt{P_c P_{c'}} R_{cc'} (1 - \text{trace } \underline{R} \underline{L}^\circ) \\ &= \Omega_c \Omega_{c'} \delta_{cc'} + 2i \Omega_c \Omega_{c'} \sqrt{P_c |R_{cc}|} \sqrt{P_{c'} |R_{c'c'}|} e^{i\alpha_{cc'}} \\ &\quad (1 - \text{trace } \underline{R} \underline{L}^\circ) \end{aligned} \quad \dots(32)$$

where

$$\begin{aligned} R_{cc'} &= |R_{cc'}| e^{i\alpha_{cc'}} \quad , \quad \text{and} \\ |R_{cc}| |R_{c'c'}| &= |R_{cc'}|^2 \end{aligned} \quad \dots(33)$$

since the  $\underline{R}$  matrix is of rank one. In the Wigner-Eisenbud theory,  $\alpha_{cc'} = 0$ . For the Moldauer theory, we can simplify the model by writing the boundary condition matrix as

$$B_{cc'} = B_c \delta_{cc'} + i\epsilon \delta_{cc'} \quad \dots(34)$$

where  $B_c$  and  $\epsilon$  are real constants. Then we note from the unitarity requirement (20) that the equiphase solution for  $\underline{R}$  is

$$R_{cc'} = \sqrt{|R_{cc}| |R_{c'c'}|} e^{i\xi} \quad \dots(35)$$

where

$$\alpha_{cc'} = \xi \quad ; \quad \sin \xi = \epsilon \sum_c |R_{cc}| .$$

This leads immediately to the equiphase solution for (32) of

$$S_{cc'} = \Omega_c \Omega_{c'} \delta_{cc'} + 2i \Omega_c \Omega_{c'} \sqrt{P_c |R_{cc}|} \sqrt{P_{c'} |R_{c'c'}|} \frac{e^{i\xi}}{(1 - \text{trace } \underline{R} \underline{L}^\circ)} \quad \dots(36)$$

5. CROSS SECTIONS

Using the expression (7) for the matrix elements  $R_{cc'}$ , we obtain

$$1 - \text{trace } \underline{R} \underline{L}^\circ = 1 - \sum_c \sum_\lambda \frac{\gamma_{\lambda c}^2 L_c^\circ}{E_\lambda - E} \quad \dots(37)$$

Using the conventional definitions (Lane and Thomas, 1958) for quantities which are now complex, we obtain:

$$\begin{aligned} \text{Partial level shift } \Delta_{\lambda c} &= -\gamma_{\lambda c}^2 (S_c - B_c) \\ &= -(\alpha_{\lambda c}^2 + i\beta_{\lambda c}^2) (S_c - B_c) \end{aligned}$$

$$\text{Partial Width } \Gamma_{\lambda c} = 2P_c \gamma_{\lambda c}^2 = 2P_c (\alpha_{\lambda c}^2 + i\beta_{\lambda c}^2)$$

$$\text{and } L_c^\circ = S_c - B_c + iP_c. \quad \dots(38)$$

We introduce the quantities

$$\text{Total level shift } \Delta_\lambda = \sum_c \Delta_{\lambda c} \quad \dots(39a)$$

$$\text{Total width } \Gamma_\lambda = \sum_c \Gamma_{\lambda c} \quad \dots(39b)$$

to obtain

$$1 - \text{trace } \underline{R} \underline{L}^\circ = 1 + \sum_\lambda \frac{\Delta_\lambda}{E_\lambda - E} - \frac{i}{2} \sum_\lambda \frac{\Gamma_\lambda}{E_\lambda - E}. \quad \dots(40)$$

If the real part of  $B_{cc'}$  is chosen so that  $S_c \approx B_c$  then  $\Delta_{\lambda c} \approx 0$ , and

$$(1 - \text{trace } \underline{R} \underline{L}^\circ) \approx 1 - \frac{i}{2} \sum_\lambda \frac{\Gamma_\lambda}{E_\lambda - E} = Z e^{-i\eta} \quad \dots(41)$$

where

$$Z = \left[ 1 + \frac{1}{4} \left| \sum_\lambda \frac{\Gamma_\lambda}{E_\lambda - E} \right|^2 \right]^{\frac{1}{2}} \sec \eta.$$

The reduced  $\underline{T}$  matrix is then

$$\underline{T} = \underline{\rho} e^{i(\eta + \xi)} = \underline{\rho} e^{i\delta}$$

where  $\delta = \eta + \xi$  and,

$$\rho_{cc'} = \frac{\sqrt{P_c P_{c'}}}{Z} \frac{|R_{cc'}|}{Z} = \frac{\sqrt{P_c} |R_{cc}|}{Z} \frac{\sqrt{P_{c'}} |R_{c'c'}|}{Z} \quad \dots(42)$$

The elements of the  $S$  matrix can then be written as

$$S_{cc'} = e^{-i(\vartheta_c + \vartheta_{c'})} \left( \delta_{cc'} + 2i \rho_{cc'} e^{i(\eta + \xi)} \right) \quad \dots(43)$$

and the probabilities of decay through channels  $c$  become, from (26b)

$$p_c = \frac{P_c |R_{cc}|}{Z \sin \delta} \quad \dots(44)$$

The total cross section is

$$\begin{aligned} \sigma_{cT} &= \frac{2\pi}{k_c^2} \sum_{\ell j J} g_J \left[ 1 - \operatorname{Re} \left( S_{c\ell j, c\ell j} \right) \right] \\ &= \frac{2\pi}{k_c^2} \sum_{\ell j J} g_J \left[ (1 - \cos \vartheta_{c\ell}) - 2 \sin (2\vartheta_{c\ell} - \delta^J) \right. \\ &\quad \left. \rho_{c\ell j, c\ell j}^J \right] \\ &= \frac{4\pi}{k_c^2} \sum_{\ell} (2\ell+1) \sin^2 \vartheta_{c\ell} - \frac{2\pi}{k_c^2} \sum_{\ell j J} g_J \sin (2\vartheta_{c\ell} - \delta^J) \\ &\quad \frac{2 P_{c\ell} |R_{c\ell j, c\ell j}|}{Z^J} \quad \dots(45) \end{aligned}$$

The elastic scattering cross section is

$$\begin{aligned} \sigma_{cc} &= \frac{\pi}{k_c^2} \sum_{\ell j J} g_J \left| 1 - S_{c\ell j, c\ell j}^J \right|^2 \\ &= \frac{\pi}{k_c^2} \sum_{\ell j J} g_J \left\{ 2(1 - \cos 2\vartheta_{c\ell}) + 4 (\rho_{c\ell j, c\ell j})^2 \right. \\ &\quad \left. - 4 \left[ \sin (\delta^J) + \sin (2\vartheta_{c\ell} - \delta^J) \right] \rho_{c\ell j, c\ell j} \right\} \\ &= \frac{4\pi}{k_c^2} \sum_{\ell} (2\ell+1) \sin^2 \vartheta_{c\ell} + \frac{\pi}{k_c^2} \sum_{\ell j J} g_J \left( \frac{2 P_{c\ell} |R_{c\ell j, c\ell j}|}{Z^J} \right)^2 \\ &\quad - \frac{2\pi}{k_c^2} \sum_{\ell j J} g_J \left[ \sin (\delta^J) + \sin (2\vartheta_{c\ell} - \delta^J) \right] \frac{2 P_{c\ell} |R_{c\ell j, c\ell j}|}{Z^J}. \quad \dots(46) \end{aligned}$$

The reaction cross section is

$$\begin{aligned}
 \sigma_{cc} &= \frac{\Pi}{k_c^2} \sum_{l_j J l_j'} g_J |S_{c l_j, c' l_j'}^J|^2 \\
 &= \frac{4\Pi}{k_c^2} \sum_{l_j J l_j'} g_J (\rho_{c l_j, c' l_j'})^2 \\
 &= \frac{\Pi}{k_c^2} \sum_{l_j J l_j'} g_J \frac{(2 P_{c l}^J |R_{c l_j, c l_j}|) (2 P_{c' l'}^J |R_{c' l_j', c' l_j'}|)}{(Z^J)^2} \dots(47)
 \end{aligned}$$

Making use of the probabilities in equation (24) we find that the above cross sections can be written

$$\sigma_{cc} = \sigma_{cc}^p + \sigma_{cc}^i + \sigma_{cc}^r$$

where

$$\sigma_{cc}^p = \frac{4\Pi}{k_c^2} \sum_l (2l+1) \sin^2 \phi_{cl} = \text{hard sphere cross section,} \dots(48a)$$

$$\begin{aligned}
 \sigma_{cc}^i &= -\frac{4\Pi}{k_c^2} \sum_{l_j J} g_J \left[ \sin \delta^J + \sin (2\phi_{cl} - \delta^J) \right] \underline{P}_{c l_j}^J \sin \delta^J \\
 &= \text{interference scattering cross section,} \dots(48b)
 \end{aligned}$$

$$\sigma_{cc}^r = \frac{4\Pi}{k_c^2} \sum_{l_j J} g_J (\underline{P}_{c l_j}^J)^2 \sin^2 \delta^J \text{ and} \dots(48c)$$

$$\sigma_{cc'} = \frac{4\Pi}{k_c^2} \sum_{l_j J l_j'} g_J \underline{P}_{c l_j}^J \underline{P}_{c' l_j'}^J \sin^2 \delta^J \dots(48d)$$

In the form of equations (48) the compound nucleus hypothesis (23) is manifestly exhibited.

In special cases, the above expressions for the cross sections are identical to those of conventional theory. For example, in the single level approximation we have

$$\begin{aligned}
 \epsilon \approx 0; \quad \xi \approx 0, \quad E_\lambda = E_0 \text{ and is real, } \Gamma_\lambda \text{ is real,} \\
 Z = (E_0 - E)^{-1} \left[ (E_0 - E) + \frac{1}{4} (\Gamma_\lambda)^2 \right]^{\frac{1}{2}} \dots(49)
 \end{aligned}$$

and equations (47) become

$$\sigma_{cT} = \frac{4\pi}{k_c^2} \sum_{\ell} (2\ell+1) \sin^2 \phi_{c\ell} + \left\{ \frac{\pi}{k_c^2} \sum_{\ell j} g_J \Gamma_{c\ell j}^J \cos 2\phi_{c\ell} - 2(E_0 - E) \Gamma_{c\ell j}^J \sin 2\phi_{c\ell} \right\} / \left\{ (E_0 - E)^2 + \frac{1}{4} (\Gamma^J)^2 \right\}, \quad \dots(50a)$$

$$\sigma_{cc} = \frac{4\pi}{k_c^2} \sum_{\ell} (2\ell+1) \sin^2 \phi_{c\ell} + \frac{\pi}{k_c^2} \sum_{\ell j} g_J \frac{(\Gamma_{c\ell j}^J)^2}{(E_0 - E)^2 + \frac{1}{4} (\Gamma^J)^2} - \frac{2\pi}{k_c^2} \sum_{\ell j} g_J \frac{(\Gamma_{c\ell j}^J \Gamma^J \sin^2 \phi_{c\ell} + (E_0 - E) \Gamma_{c\ell j}^J \sin 2\phi_{c\ell})}{(E_0 - E)^2 + \frac{1}{4} (\Gamma^J)^2} \quad \dots(50b)$$

and

$$\sigma_{cc'} = \frac{\pi}{k_c^2} \sum_{\ell j \ell' j'} g_J \frac{\Gamma_{c\ell j}^J \Gamma_{c\ell' j'}^J}{(E_0 - E)^2 + \frac{1}{4} (\Gamma^J)^2} \quad \dots(50c)$$

which are the usual expressions (Schmidt, 1966) for an isolated level.

## 6. CORRELATIONS BETWEEN REDUCED WIDTHS

Newton assumed that the  $E_\lambda$  and  $\gamma_\lambda^2$  are real quantities, as in Wigner-Eisenbud theory, and postulated the strong correlation condition

$$\frac{\gamma_{\lambda c}}{\gamma_{\lambda c'}} = \frac{\gamma_{\mu c}}{\gamma_{\mu c'}} \quad \dots(51)$$

which ensures that the reaction matrix (7), without the background term, is of rank one. For a complex equiphase reaction matrix, we have

$$R_{cc} R_{c'c'} = R_{cc'}^2 \quad \dots(52)$$

and substituting equation (7) into (52), after eliminating common terms on each side, we find

$$\sum_{\lambda} \frac{1}{E_\lambda - E} \sum_{\mu \neq \lambda} \frac{(\gamma_{\lambda c} \gamma_{\mu c'} - \gamma_{\lambda c'} \gamma_{\mu c})^2}{E_\mu - E_\lambda} = 0. \quad \dots(53)$$

It follows that the residue at each pole must vanish in order that (53) be zero for all  $E$ , and hence

$$\sum_{\mu \neq \lambda} \frac{(\gamma_{\lambda c} \gamma_{\mu c'} - \gamma_{\lambda c'} \gamma_{\mu c})^2}{E_{\mu} - E_{\lambda}} = 0 \quad \dots(54)$$

Should the  $\gamma_{\lambda c}$  and  $E_{\lambda}$  be real, and there exist a lower bound to the values of  $E_{\lambda}$ , then at this lower bound, all of the weights  $(E_{\mu} - E_{\lambda})$  are positive. Since the numerators are all perfect squares, the sum of these squares with positive weights cannot vanish unless each term vanishes. Newton's correlation (51) follows immediately. If there is no lower bound to  $E_{\lambda}$ , Newton's condition (51) does not follow from the rank one condition (54), since for any finite  $E_{\lambda}$ , the positive terms can cancel the negative terms. Newton's condition also follows if there is an upper bound to  $E_{\lambda}$ .

The probability of decay in channel c can be evaluated using (51) and (54) with  $\epsilon=0$ , to give

$$\frac{\gamma_{\lambda c}}{\gamma_{\lambda c'}} = \frac{\gamma_{\mu c}}{\gamma_{\mu c'}} = \frac{A_c}{A_{c'}} = R_{cc}/R_{c'c'}$$

where the  $A_c$ 's are constants,

$$P_c = \frac{P_c R_{cc}}{\sum_{c'} P_{c'} R_{c'c'}} = \frac{A_c P_c}{\sum_{c'} A_{c'} P_{c'}} \quad \dots(55)$$

and therefore the  $P_c$ 's are independent of  $\gamma_{\lambda}^2$  and  $E_{\lambda}$ . For radiative capture and fission processes,  $P_c$  is approximately constant with energy. This is not what is observed.

Furthermore, Newton pointed out that in the expressions (45), (46) and (47), the reaction matrix has zeroes between each pole at the same energy in all channels. Therefore, for a given spin state, the reaction cross sections should vanish at some energy between each resonance. This is not observed. Therefore Lane and Thomas (1958) rejected Newton's model, and with it the Bohr hypothesis, as not being in accord with experiment. However, this reasoning is faulty, as if the  $\gamma_{\lambda}^2$  and  $E_{\lambda}$ 's are complex, all that equation (54) implies is that a set of equations exist which correlate the imaginary parts of  $\gamma_{\lambda}^2$  and  $E_{\lambda}$  with the real parts. For example, an acceptable special solution to (54) would be, as in Newton's example,

$$\frac{\gamma_{\lambda c}}{\gamma_{\lambda c'}} = \frac{\gamma_{\mu c}}{\gamma_{\mu c'}} ; \quad \frac{\gamma_{\lambda c}^2}{\gamma_{\lambda c'}^2} = \frac{\gamma_{\mu c}^2}{\gamma_{\mu c'}^2} = \frac{A_c + iD_c}{A_{c'} + iD_{c'}} \quad \dots(56)$$

where  $A_c, D_c$  are constants independent of  $\lambda$ , and putting

$$\frac{\alpha_{\lambda c} + i\beta_{\lambda c}}{\alpha_{\lambda c'} + i\beta_{\lambda c'}} = \frac{A_c + iD_c}{A_{c'} + iD_{c'}}$$

we can evaluate the imaginary parts to obtain

$$\beta_{\lambda c} = \frac{\alpha_{\lambda c'}(A_c^2 + D_c^2) - \alpha_{\lambda c}(A_c A_{c'} + D_c D_{c'})}{A_c D_{c'} + A_{c'} D_c}$$

$$\beta_{\lambda c'} = \frac{\alpha_{\lambda c}(A_{c'}^2 + D_{c'}^2) - \alpha_{\lambda c'}(A_c A_{c'} + D_c D_{c'})}{A_c D_{c'} + A_{c'} D_c} \quad \dots(57)$$

Equation (57) represents a solution where the  $\beta_{\lambda c}$  are correlated to the corresponding  $\alpha_{\lambda c}$  through constants depending only on the channels present. The  $\alpha_{\lambda c}$  remain arbitrary. Therefore Newton's correlation condition is a much weaker constraint for a complex  $R$  matrix.

Furthermore, the cross sections (47) can vanish for a given set of quantum numbers ( $l, j, J$ ) only if the real and imaginary parts of  $R_{cc'}$  vanish at the same energy. Such a restriction implies further strong correlations between the complex resonance parameters which are unnecessary in general. There are sufficient parameters in (47) to fit resonant cross sections adequately without restriction on the minimum value between resonances. Therefore, we can encompass the Bohr postulate within the framework of Moldauer's theory without contradicting observed features of cross sections. Finally, we note from equation (44) and (41) that the probability of decay through channel  $c$  is

$$\mathcal{P}_c = \frac{P_c |R_{cc}|}{\sin \delta \sec \eta} = \frac{P_c |R_{cc}|}{\sin \xi + \cos \xi \tan \eta}$$

$$= \frac{P_c |R_{cc}|}{\sin \xi + (\cos \xi) Z / \sqrt{1-Z^2}} \quad \dots(58)$$

where  $Z$  is given by (41) and  $\xi$  is the argument of  $R_{cc'}$ . Clearly, (58) can be a strong function of energy, dependent upon both the real and imaginary parts of the  $E_\lambda$  and  $\gamma_\lambda^2$ .

If as an alternative, the real reaction matrix (22) is used, it is found that equations (45), (46), (47) remain valid, and the cross section must still vanish between resonances. However, the strong correlation condition (51) is no longer meaningful. It appears that Moldauer's theory is necessary to explain Bohr's hypothesis satisfactorily, and implies that the effective potential to use in the Schroedinger equation for the channel wave function must be complex. This may be deduced as a manifestation of channel-channel coupling not present in Wigner-Eisenbud theory.

#### 7. POLES AND RESIDUES OF THE S MATRIX

It is clear from equations (6) and (8) that the poles in the reduced  $\underline{T}$  matrix are also contained in the full  $\underline{S}$  matrix. From equations (10) and (36) we see that

$$T'_{cc'} = \frac{\sqrt{P_c} |R_{cc}| \sqrt{P_{c'}} |R_{c'c'}| e^{i\xi}}{1 - \text{trace}(\underline{R} \underline{L}^o)} \quad \dots(59)$$

The positions of the complex poles are obtained as the values of the complex energy where

$$\text{trace} \underline{R} \underline{L}^o = 1 \quad \dots(60)$$

We may formally represent the spectrum of poles by a vector such that at each pole

$$E = \xi_\mu - \frac{i}{2} Y_\mu \quad \dots(61)$$

Neglecting the level shift contribution with the factor  $(S_c - B_c)$  and setting

$$E_\lambda = U_\lambda + i V_\lambda/2, \quad \Gamma_\lambda = \Gamma_\lambda^r + i \Gamma_\lambda^i = 2 P_c \gamma_{\lambda c}^2, \quad \dots(62)$$

we obtain the two equations from equation (60) of the form

$$\sum_\lambda \frac{\Gamma_\lambda^r (U_\lambda - \xi_\mu) - \Gamma_\lambda^i (V_\lambda - Y_\mu) / 2}{(U_\lambda - \xi_\mu)^2 + (V_\lambda - Y_\mu)^2 / 4} = 0 \quad \dots(63a)$$



$$\sum_{\lambda} \frac{\Gamma_{\lambda}^i (U_{\lambda} - \xi_{\mu}) + \Gamma_{\lambda}^r (V_{\lambda} - Y_{\mu}) / 2}{(U_{\lambda} - \xi_{\mu})^2 + (V_{\lambda} - Y_{\mu})^2 / 4} = -2 \quad \dots(63b)$$

These equations can be summarised in the form

$$\sum_{\lambda} \Gamma_{\lambda} f_{\lambda\mu} = Y_{\mu} \quad \dots(64)$$

where 
$$f_{\lambda\mu} = \frac{-iY_{\mu} / 2}{(U_{\lambda} - \xi_{\mu}) + i(V_{\lambda} - Y_{\mu}) / 2}$$

is a complex 'mixing matrix' which is such that equation (64) states that a linear superposition of the complex widths gives the equivalent real width in the  $\underline{S}$  matrix. When  $V_{\lambda}, Y_{\mu} \ll U_{\lambda} - \xi_{\mu}$  for  $\lambda \neq \mu$  the off-diagonal elements of  $f_{\lambda\mu}$  are approximately zero, and for  $V_{\lambda} \ll Y_{\mu}$ ,  $f_{\lambda\mu}$  becomes the identity matrix.

To get good approximations to the positions of the poles it is more convenient to let

$$\tau(E) = \sum_{\lambda} \frac{\Gamma_{\lambda}}{E_{\lambda} - E} = 2i \text{ trace } \underline{R} \underline{L}^{\circ} \quad \dots(65)$$

and to define

$$\tau_{\mu}(E) = \sum_{\lambda \neq \mu} \frac{\Gamma_{\lambda}}{E_{\lambda} - E} = \tau(E) - \frac{\Gamma_{\mu}}{E_{\mu} - E} \quad \dots(66)$$

To find  $\xi_{\mu}$  and  $Y_{\mu}$  we must solve the equation

$$\tau\left(\xi_{\mu} - \frac{i}{2} Y_{\mu}\right) = -2i \quad \dots(67)$$

Let

$$\xi_{\mu} = E_{\mu} + \Delta_{\mu}$$

where  $\Delta_{\mu}$  is complex. Then

$$\tau\left(E_{\mu} + \Delta_{\mu} - \frac{i}{2} Y_{\mu}\right) = -\frac{\Gamma_{\mu}}{\Delta_{\mu} - \frac{i}{2} Y_{\mu}} + \tau_{\mu}\left(E_{\mu} + \Delta_{\mu} - \frac{i}{2} Y_{\mu}\right)$$

and since  $\tau_{\mu}(E)$  has no singularities at  $E=E_{\mu}$ , we can use the Taylor expansion

$$\tau_{\mu}(E_{\mu} + \Delta_{\mu} - \frac{i}{2} Y_{\mu}) = \tau_{\mu}(E_{\mu}) + (\Delta_{\mu} - \frac{i}{2} Y_{\mu}) \tau'_{\mu}(E_{\mu}) + \dots$$

The nth derivative of  $\tau_{\mu}$  is just

$$\tau_{\mu}^{(n)}(E) = n! \sum_{\lambda \neq \mu} \frac{\Gamma_{\lambda}}{(E_{\lambda} - E)^{n+1}} \quad \dots(68)$$

Equation (65) becomes

$$-2i = -\frac{\Gamma_{\mu}}{\Delta_{\mu} - \frac{i}{2} Y_{\mu}} + \tau_{\mu}(E_{\mu}) + (\Delta_{\mu} - \frac{i}{2} Y_{\mu}) \tau'_{\mu}(E_{\mu}) + \dots \quad \dots(69)$$

that is

$$Y_{\mu} + 2i \Delta_{\mu} - \Gamma_{\mu} + (\Delta_{\mu} - \frac{i}{2} Y_{\mu}) \tau_{\mu}(E_{\mu}) + (\Delta_{\mu} - \frac{i}{2} Y_{\mu})^2 \tau'_{\mu}(E_{\mu}) + \dots \quad \dots(70)$$

Equation (70) can be solved for the real and imaginary parts to any desired order of accuracy.

For an isolated level,  $\tau_{\mu} \approx 0$ , and we have

$$\Gamma_{\mu} \approx Y_{\mu} + 2i \Delta_{\mu}$$

that is  $Y_{\mu} \approx \Gamma_{\mu}^r + V_{\mu}$ , ...(71a)

and  $\xi_{\mu} \approx U_{\mu} + \Gamma_{\mu}^i / 2$ . ...(71b)

In the next approximation, we assume the derivatives of  $\tau_{\mu}$  may be neglected. This yields the simultaneous equations

$$Y_{\mu} (1 + \frac{1}{2} \text{Im } \tau) + (U_{\mu} - \xi_{\mu}) \text{Re } \tau = \Gamma_{\mu}^r + V_{\mu} [1 + \frac{1}{2} \text{Im } \tau];$$

$$Y_{\mu} (-\frac{1}{2} \text{Re } \tau) + (U_{\mu} - \xi_{\mu}) (2 + \text{Im } \tau) = \Gamma_{\mu}^i \quad \dots(72)$$

which have solutions

$$Y_{\mu} = \frac{(1 + \frac{1}{2} \text{Im } \tau)(\Gamma_{\mu}^r + V_{\mu}(1 + \frac{1}{2} \text{Im } \tau)) - \frac{1}{2} \text{Re } \tau \Gamma_{\mu}^i}{(1 + \frac{1}{2} \text{Im } \tau)^2 + \frac{1}{4} (\text{Re } \tau)^2},$$

$$\xi_{\mu} = U_{\mu} - \frac{\frac{1}{2} \Gamma_{\mu}^i (1 + \frac{1}{2} \text{Im } \tau) + \frac{1}{4} \text{Re } \tau (\Gamma_{\mu}^r + V_{\mu} (1 + \frac{1}{2} \text{Im } \tau))}{(1 + \frac{1}{2} \text{Im } \tau)^2 + \frac{1}{4} (\text{Re } \tau)^2} \quad \dots(73)$$

The residue of the reduced  $\underline{T}$  matrix at each pole is obtained by using the fact that if  $G(E)$  is a function with a simple pole at  $E = z$ , the residue at that pole is given by

$$[\text{Res } G(z)]^{-1} = \lim_{E \rightarrow z} \frac{d}{dE} \left( \frac{1}{G(E)} \right) \quad \dots(74)$$

Let

$$\begin{aligned} G(E)^{-1} &= 1 - \text{trace } \underline{R} \underline{L}^{\circ} \\ &= 1 - \frac{i}{2} \sum_{\lambda} \frac{\Gamma_{\lambda}}{(E_{\lambda} - E)} \end{aligned} \quad \dots(75)$$

Equation (75) tells us that if we put

$$\frac{d}{dE} \left( \frac{1}{G(E)} \right) = -\frac{i}{2} \sum_{\lambda} \frac{\Gamma_{\lambda}}{(E_{\lambda} - E)^2} \quad \dots(76)$$

by neglecting the local energy variation of each  $\Gamma_{\lambda}$  the residue of  $T'_{cc'}$  at the pole  $\xi_{\mu} - \frac{i}{2} Y_{\mu}$  is  $\frac{1}{2} a_{\mu cc'}$  where

$$\begin{aligned} a_{\mu cc'} &= 4 i \frac{\sqrt{P_c P_{c'}} R_{cc'}(z_{\mu})}{\sum_{\lambda} \frac{\Gamma_{\lambda}}{(E_{\lambda} - z_{\mu})^2}} \\ &= 4 i \frac{\sqrt{P_c |R_{cc}(z_{\mu})|} \sqrt{P_{c'} |R_{c'c'}(z_{\mu})|} e^{i\xi(z_{\mu})}}{\sum_{\lambda} \frac{\Gamma_{\lambda}}{(E_{\lambda} - z_{\mu})^2}} \end{aligned} \quad \dots(77)$$

where  $z_{\mu} = \xi_{\mu} - \frac{i}{2} Y_{\mu}$ .

From equations (35) and (36) we see that

$$T'_{cc'} = \frac{\frac{1}{2} \sum_{\lambda} \sqrt{|\Gamma_{\lambda c}| |\Gamma_{\lambda c'}|} e^{i\xi}}{1 - \frac{i}{2} \sum_{\lambda} \frac{\Gamma_{\lambda}}{E_{\lambda} - E}} \quad \dots(78)$$

We can approximate (78) at a given energy to an arbitrary degree of accuracy by considering the sum over  $\lambda$  to extend to a finite number  $N$ , of terms. By taking the sums over  $\lambda$  over a common denominator we see that  $T'_{cc'}$  can be expressed in the form

$$T'_{cc'} = \frac{p_{N-1}(E)}{q_N(E)} \quad \dots(79)$$

where  $p_N$  and  $q_N$  are polynomials in  $E$  of order  $N$  with complex coefficients. The polynomial  $q_N(E)$  in (79) can be factorized into a product of the form

$$q_N(E) = \prod_{\lambda} \left( \xi_{\lambda} - E - \frac{i}{2} Y_{\lambda} \right) \quad \dots(80)$$

Hence, equation (77), by partial fractions, can be reduced to a sum of simple pole terms such that

$$T'_{cc'} = \frac{1}{2} \sum_{\lambda} \frac{a_{\lambda cc'}}{\xi_{\lambda} - E - \frac{i}{2} Y_{\lambda}} \quad \dots(81)$$

where the  $a_{\lambda cc'}$  are the complex residues given by equation (77). Equation (81) states that the reduced  $\underline{T}$  matrix can be expressed as a coherent sum of effective single level terms. Sailor (1955) used this property as an assumption in fitting fission cross sections, and it was first conjectured by Feshbach, Porter and Weisskopf (1954). The  $\underline{S}$  matrix can therefore be written as

$$S_{cc'} = \Omega_c \Omega_{c'} \left\{ \delta_{cc'} + i \sum_{\lambda} \frac{a_{\lambda cc'}}{\xi_{\lambda} - E - \frac{i}{2} Y_{\lambda}} \right\} \quad \dots(82)$$

## 8. CROSS SECTIONS AS SUMS OF SINGLE LEVEL TERMS

It was shown by Cook (1967) that the multilevel expressions for the fission cross section used by Sailor could be written as a sum of single level contributions. Clayton (1970) showed that the same result holds for the scattering cross section. Using these formulae we obtain results of the type derived by Adler and Adler (1963). In our case, where we have complex residues, the  $\underline{S}$  matrix can be written as

$$S_{cc'} = e^{-i(\phi_c + \phi_{c'})} \left\{ \delta_{cc'} + i \sum_{\lambda} \frac{a_{\lambda cc'} \left( \xi_{\lambda} - E + \frac{i}{2} Y_{\lambda} \right)}{\left( \xi_{\lambda} - E \right)^2 + Y_{\lambda}^2 / 4} \right\} \quad \dots(83)$$

giving for  $c \neq c'$

$$\begin{aligned} \sigma_{cc'} &= \frac{\Pi}{k_c^2} S_{cc'}^* S_{cc'} \\ &= \frac{\Pi}{k_c^2} \sum_{\lambda} \frac{|a_{\lambda cc'}|^2}{(\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4} \\ &+ \frac{\Pi}{k_c^2} \sum_{\substack{\lambda, \mu \\ \lambda \neq \mu}} \frac{a_{\lambda cc'} a_{\mu cc'}^* (\xi_{\lambda} - E + i Y_{\lambda}) (\xi_{\mu} - E - \frac{i}{2} Y_{\mu})}{\left[ (\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4 \right] \left[ (\xi_{\mu} - E)^2 + Y_{\mu}^2 / 4 \right]} \end{aligned} \quad \dots(84)$$

The last term can be separated into single level contributions by partial fractions to yield

$$\sigma_{cc'} = \frac{\Pi}{k_c^2} \sum_{\lambda} \frac{\sigma_{\lambda cc'}}{(\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4} - \frac{2\Pi}{k_c^2} \sum_{\lambda} \frac{(\xi_{\lambda} - E) \eta_{\lambda cc'}}{(\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4} \quad \dots(85)$$

where

$$\sigma_{\lambda cc'} = |a_{\lambda cc'}|^2 + \sum_{\mu \neq \lambda} \frac{Y_{\lambda} \beta_{\lambda \mu cc'}}{(\xi_{\lambda} - \xi_{\mu})^2 + (Y_{\lambda} + Y_{\mu})^2 / 4} \quad \dots(85a)$$

$$\eta_{\lambda cc'} = \sum_{\mu \neq \lambda} \frac{\alpha_{\lambda \mu cc'}}{(\xi_{\lambda} - \xi_{\mu})^2 + (Y_{\lambda} + Y_{\mu})^2 / 4} \quad \dots(85b)$$

and

$$a_{\lambda cc'} a_{\mu cc'}^* \left[ (\xi_{\lambda} - \xi_{\mu}) + \frac{i}{2} (Y_{\lambda} + Y_{\mu}) \right] = \alpha_{\lambda \mu cc'} + i \beta_{\lambda \mu cc'} \quad \dots(85c)$$

Equations (85) have the form of a sum of single level terms, where the first sum is over the terms symmetric about the resonance energy  $\xi_{\lambda}$ , and the second sum is over corresponding asymmetric contributions.

Similarly, the elastic scattering cross section can be expressed as,

$$\begin{aligned}
\sigma_{cc} &= \frac{\Pi}{k_c^2} |1 - S_{cc}|^2 \\
&= \frac{4\Pi}{k_c^2} \sin^2 \vartheta_c + \frac{\Pi}{k_c^2} \sum_{\lambda} \frac{\sigma_{\lambda cc}}{(\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4} - \frac{2\Pi}{k_c^2} \sum_{\lambda} \frac{(\xi_{\lambda} - E) \eta_{\lambda cc}}{(\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4} \\
&- \frac{2\Pi}{k_c^2} \sum_{\lambda} \frac{1}{(\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4} \left\{ \sin 2\vartheta_c \cdot \operatorname{Re} \left[ a_{\lambda cc} \left( \xi_{\lambda} - E + \frac{i}{2} Y_{\lambda} \right) \right] - \right. \\
&- \left. \cos 2\vartheta_c \cdot \operatorname{Im} \left[ a_{\lambda cc} \left( \xi_{\lambda} - E + \frac{i}{2} Y_{\lambda} \right) \right] \right\} \quad \dots(86)
\end{aligned}$$

where  $\sigma_{\lambda cc}$  and  $\eta_{\lambda cc}$  are as defined by equations (84a) and (84b). The total cross section is found from (2) by evaluating

$$\begin{aligned}
\sigma_{cT} &= \frac{2\Pi}{k_c^2} \left[ 1 - \operatorname{Re} S_{cc} \right] \\
&= \frac{4\Pi}{k_c^2} \sin^2 \vartheta_c + \frac{2\Pi}{k_c^2} \operatorname{Im} \left\{ e^{-2i\vartheta_c} \sum_{\lambda} \frac{a_{\lambda cc} \left( \xi_{\lambda} - E + \frac{i}{2} Y_{\lambda} \right)}{(\xi_{\lambda} - E) + Y_{\lambda}^2 / 4} \right\} \\
&= \frac{4\Pi}{k_c^2} \sin^2 \vartheta_c - \frac{2\Pi}{k_c^2} \sum_{\lambda} \frac{1}{(\xi_{\lambda} - E)^2 + Y_{\lambda}^2 / 4} \left\{ \sin 2\vartheta_c \cdot \operatorname{Re} \left[ a_{\lambda cc} \left( \xi_{\lambda} - E + \frac{i}{2} Y_{\lambda} \right) \right] - \right. \\
&- \left. \cos 2\vartheta_c \operatorname{Im} \left[ a_{\lambda cc} \left( \xi_{\lambda} - E + \frac{i}{2} Y_{\lambda} \right) \right] \right\} \quad \dots(87)
\end{aligned}$$

## 9. STATISTICAL THEORY

From equation (82) we see that the S matrix can be written as

$$S_{cc'} = S_{cc'}^0 - i \sum_{\lambda} \frac{g_{\lambda c} g_{\lambda c'}}{E - \xi_{\lambda} - \frac{i}{2} Y_{\lambda}} \quad \dots(88)$$

for an equiphase R matrix, where

$$S_{cc'}^0 = \Omega_c \Omega_{c'} \delta_{cc'} = e^{-i(\vartheta_c + \vartheta_{c'})} \delta_{cc'} \quad \dots(89a)$$

and

$$g_{\lambda c} = 2i \left[ \frac{i P_c R_{cc}(z)}{\sum_{\mu} \frac{\Gamma_{\mu}}{(E_{\mu} - z)^2}} \right]^{\frac{1}{2}} \Omega_c \quad \dots(89b)$$

Equation (88) is precisely the form postulated by Moldauer (1964) to calculate average cross sections. If we assume that the apparent widths  $Y_{\lambda}$  and apparent resonance energies  $E_{\lambda}$  have the same statistical properties as the widths  $|\Gamma_{\lambda}|$  and eigenvalues  $\text{Re}(E_{\lambda})$ , which we assume to have the usual statistical properties, then we may employ the relations derived by Moldauer to calculate the following average cross sections;

$$\langle \sigma_{cT} \rangle = \frac{2\pi}{k_c^2} (1 - \text{Re} \langle S_{cc} \rangle) \quad \dots(90a)$$

$$\langle \sigma_{cc'} \rangle = \frac{\pi}{k_c^2} \langle |\delta_{cc'} - S_{cc'}|^2 \rangle \quad \dots(90b)$$

$$\langle \sigma_{cc'}^{\text{direct}} \rangle = \frac{\pi}{k_c^2} |\delta_{cc'} - \langle S_{cc'} \rangle|^2 \quad \dots(90c)$$

$$\langle \sigma_c \rangle_{\text{absorption}} = \frac{\pi}{k_c^2} (1 - |\langle S_{cc} \rangle|^2) = \frac{\pi}{k_c^2} \mathcal{J}_c \quad \dots(90d)$$

where  $\mathcal{J}_c$  is the average transmission coefficient.

$$\sigma_{cc'}(\text{fluctuation}) = \frac{\pi}{k_c^2} \left\{ \langle |S_{cc'}|^2 \rangle - |\langle S_{cc'} \rangle|^2 \right\} \quad \dots(90e)$$

These cross sections can be evaluated by using the following relationships reported by Moldauer:

$$\langle S_{cc'} \rangle = S_{cc'}^0 - \pi \left\langle \frac{g_{\lambda c} g_{\lambda c'}}{D} \right\rangle_{\lambda} \quad \dots(91a)$$

$$\langle |S_{cc'}|^2 \rangle = |\langle S_{cc'} \rangle|^2 + \frac{2\pi}{D} \left\langle \frac{|g_{\lambda c}|^2 |g_{\lambda c'}|^2}{Y_{\lambda}} \right\rangle_{\lambda} - M_{cc'} \quad \dots(91b)$$

where  $D$  = average level spacing, and

$$M_{cc'} = \frac{2\pi^2}{D^2} \left\{ \left| \langle g_{\lambda c} g_{\lambda c'} \rangle_{\lambda} \right|^2 - \left\langle g_{\lambda c} g_{\lambda c'} g_{\mu c} g_{\mu c'} \Phi_0 \left( \frac{Y_{\lambda} + Y_{\mu}}{D} \right) \right\rangle_{\lambda \neq \mu} \right\} \quad \dots(91c)$$

is a function defined by Moldauer.

Assuming that the  $g_{\lambda c}$  for different channels are uncorrelated, we find that

$$\langle g_{\lambda c} g_{\lambda c'} \rangle_{\lambda} = \delta_{cc'} \langle g_{\lambda c}^2 \rangle_{\lambda} \quad \dots(92)$$

and since  $\Phi_0$  is a slowly varying function of its argument, we obtain

$$M_{cc'} \approx \delta_{cc'} \frac{2\Pi^2}{D^2} |\langle g_{\lambda c}^2 \rangle_{\lambda}|^2 (1 - \Phi_0) \quad \dots(93)$$

and

$$\langle S_{cc'} \rangle \approx \left[ e^{-2i\phi_c} - \frac{\Pi}{D} \langle g_{\lambda c}^2 \rangle_{\lambda} \right] \delta_{cc'} \quad \dots(94a)$$

$$\begin{aligned} \langle |S_{cc'}|^2 \rangle &\approx \delta_{cc'} \left\{ 1 - \frac{2\Pi}{D} \operatorname{Re} \left[ e^{2i\phi_c} \langle g_{\lambda c}^2 \rangle_{\lambda} \right] \right. \\ &+ \left. \frac{\Pi^2}{D^2} |\langle g_{\lambda c}^2 \rangle_{\lambda}|^2 \right\} + \frac{2\Pi}{D} \left\langle \frac{|g_{\lambda c}|^2 |g_{\lambda c'}|^2}{Y_{\lambda}} \right\rangle_{\lambda} - M_{cc'} \cdot \end{aligned} \quad \dots(94b)$$

Where  $\langle |\Gamma_{\lambda}| \rangle \ll D$ , the approximate solutions to (81) are found to be

$$\begin{aligned} g_{\lambda c} &\approx \sqrt{\Gamma_{\lambda c}} e^{-i\phi_c} \\ Y_{\lambda} &\approx \Gamma_{\lambda}^r + V_{\lambda} \end{aligned} \quad \dots(95)$$

It follows from equations (94), (90) and (91) that

$$\begin{aligned} \langle \sigma_{cc} \rangle &= \frac{4\Pi}{k_c^2} \sin^2 \phi_c - \frac{2\Pi^2}{k_c^2 D} \left[ (1 - \cos 2\phi_c) \operatorname{Re} \langle \Gamma_{\lambda c} e^{-2i\phi_c} \rangle_{\lambda} \right. \\ &+ \left. \sin 2\phi_c \operatorname{Im} \langle \Gamma_{\lambda} e^{-2i\phi_c} \rangle_{\lambda} \right] + \frac{\Pi^2}{D^2} |\langle \Gamma_{\lambda c} \rangle_{\lambda}|^2 \\ &+ \frac{2\Pi}{D} \langle |\sqrt{\Gamma_{\lambda c}}|^4 / Y_{\lambda} \rangle_{\lambda} \end{aligned} \quad \dots(96a)$$

$$\langle \sigma_{cc'} \rangle = \frac{2\Pi^2}{k_c^2 D} \left\langle \frac{|\Gamma_{\lambda c}| |\Gamma_{\lambda c'}|}{Y_{\lambda}} \right\rangle_{\lambda} \quad \dots(96b)$$



$$\langle \sigma_{cT} \rangle = \frac{2\pi}{k_c^2} \left( 1 - \cos 2\phi_c + \frac{\pi}{D} \operatorname{Re} \left\langle \sqrt{\Gamma_{\lambda c} \Gamma_{\lambda c'}} e^{-2i\phi_c} \right\rangle_{\lambda} \right) \dots(96c)$$

which are generalisations of the equations in conventional statistical theory.

#### 10. CONCLUSION

It has been demonstrated that existing multilevel theories, when applied to present experiments, cannot yield a unique set of reaction matrix parameters. When Bohr's compound nucleus hypothesis is applied, the Wigner-Eisenbud theory cannot give agreement with experiment, but Moldauer's reaction matrix theory can. Expressions for multilevel cross sections are derived which are consistent with the Bohr hypothesis and Moldauer's reaction matrix theory and shown to be equivalent to the assumption that the  $S$  matrix consists of a sum of simple poles times hard sphere phase factors. Although the pole representation is not manifestly in accordance with the Bohr assumption, with the equations derived here, it is necessary that a subtle connection exists. It is pointed out (Cook, 1967) that the single level form for cross sections can be modified readily to allow for Doppler broadening.

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