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LUCAS HEIGHTS

SOLUTIONS OF THE RELATIVISTIC TWO-BODY PROBLEM

II QUANTUM MECHANICS

by

J.L. COOK

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ABSTRACT

This second paper of a series discusses the formulation of the quantum mechanical equivalent of the relative time classical theory put forward in Part I. The relativistic wave function is derived and a covariant addition theorem put forward which allows a covariant scattering theory to be established. The free particle eigenfunctions are not plane waves and a covariant partial wave analysis is given.

A means is given by which wave functions which yield probability densities in 4-space can be converted to ones yielding the equivalent 3-space density. Bound states are considered and covariant analogues are given of the harmonic oscillator potential, Coulomb potential, the square well potential, and two-body fermion interactions.

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ANGULAR MOMENTUM; BOSONS; BOUND STATE; COULOMB FIELD; CROSS SECTIONS; EIGENFUNCTIONS; FERMIONS; FOURIER TRANSFORMATION; HAMILTONIAN FUNCTION; HARMONIC POTENTIAL; PARTIAL WAVES; QUANTUM MECHANICS; RELATIVITY THEORY; SCATTERING; SCHROEDINGER EQUATION; SPINORS; SQUARE–WELL POTENTIAL; SYMMETRY; TENSORS; TWO–BODY PROBLEM; WAVE FUNCTION

CONTENTS

1. INTRODUCTION

In Part I the relativistic two-body problem was discussed and a system of calibrating proper times put forward which permits the simple evaluation of many standard problems in a fully covariant way. This paper deals with the Schrodinger quantization of the proper time theory and examines the properties of various relativistic models whose classical covariant solutions were obtained in Part I.

In the first section the relativistic two-body wave equation and the properties of angular momentum operators are derived. Then a covariant addition theorem is derived which permits the configuration space and momentum space eigenfunctions to be coupled to give a covariant wave function. This theorem is applied to the construction of the two-body free particle wave function which is found not to be a plane wave. This relativistic wave formalism is used to define the covariant cross section and scattering matrix and an expansion into covariant partial waves is derived.

Usually relativistic wave functions cannot be interpreted as defining probability densities in ordinary space. It is shown that this is because we are working in a four-space of hyperbolic symmetry where features of wave propagation are unfamiliar. If we convert the wave function to those eigenfunctions appropriate to spherical symmetry in three-space with an additional time co-ordinate, familiar and meaningful wave functions are obtained. A general symmetry conversion procedure is given and is shown to yield plane waves in three-space for the case of free particles. The conversion is applied to general scattering from potentials and formulae for the scattering matrix are derived.

The problem of bound states is next considered. Four standard models, the harmonic oscillator, motion under the influence of the inverse cube law of force, the covariant Coulomb field and the square well potential, are treated. Only the boson-boson model of these interactions is solved. However, a model for the boson-fermion and fermion-fermion systems is put forward.

2. SCHRODINGER QUANTIZATION

The notation of Part I will be used throughout this work. Let us examine the form of the proper time calibration theory when Schrodinger quantization is applied. The quantized relative four-momentum of the two-body system is (Schiff 1949),

$$Q_\mu = i\hbar \frac{\partial}{\partial R_\mu} \quad . \tag{1}$$

When this is substituted into the component of the Hamiltonian which describes the relative motion,

$$2\mu\,(\mathcal{H} - \mathcal{V}) - Q^2 = 0 \quad , \tag{2}$$

such that the equation becomes an operator equation acting on a covariant wave function $\Psi(R)$, we find the covariant two-body wave equation

$$\left(\frac{2\mu}{\hbar^2}\,(\mathcal{H} - \mathcal{V}) + \Box^2\right) \Psi(\underline{R}) = 0 \tag{3}$$

where

$$\Box^2 = G_{\mu\nu} \frac{\partial}{\partial R_\mu} \frac{\partial}{\partial R_\nu}$$

$G_{\mu\nu}$ = the metric tensor

μ = the reduced mass

$\mathcal{V}$ = the covariant interaction.

Using the co-ordinates (30)* of Part I and assuming hypercentral forces such that $\mho$ is a function only of the hyper-radius S, we can separate (3) into the component eigenfunction equations,

(i) $$\left[S^2 \frac{d^2}{dS^2} + 3S \frac{d}{dS} + Q^2 S^2 - \Lambda^2\right] \Psi_S = \mho \Psi_S$$

(ii) $$(1-y^2)\frac{d^2\Psi_\gamma}{dy^2} + \left[L^2 - \frac{\Lambda^2+1}{1-y^2}\right]\Psi_\gamma = 0 , \quad y = \tanh\gamma$$

(iii) $$(1-x^2)\frac{d^2\Psi_\theta}{dx^2} - 2x\frac{d\Psi_\theta}{dx} + \left[L^2 - \frac{m^2}{1-x^2}\right]\Psi_\theta = 0 , \quad x = \cos\theta$$

(iv) $$\frac{d^2\Psi_\phi}{d\phi^2} + m^2\Psi_\phi = 0$$

where

(v) $$\Psi(\underset{\sim}{R}) = \Psi_S(S)\cdot\Psi_\gamma(\gamma)\cdot\Psi_\theta(\theta)\cdot\Psi_\phi(\theta) \qquad (4)$$

and Heaviside units have been introduced $(\hbar = c = 1)$.

Putting

$$\Lambda^2 = \lambda(\lambda+2) , \quad L^2 = \ell(\ell+1) \qquad (5)$$

we find the free particle eigenfunctions with $\mho = 0$ as

(i) $$\Psi_S = A_S \frac{J_{\lambda+1}(QS)}{S} + B_S \frac{N_{\lambda+1}(QS)}{S}$$

(ii) $$\Psi_\gamma = \left[A_\gamma \mathscr{P}_\ell^{\lambda+1}(\tanh\gamma) + B_\gamma \mathscr{Q}_\ell^{\lambda+1}(\tanh\gamma)\right] \operatorname{sech}\gamma$$

(iii) $$\Psi_\theta = A_\theta \mathscr{P}_\ell^{m}(\cos\theta)$$

(iv) $$\Psi_\phi = A_\phi \exp(im\phi) \qquad (6)$$

where (J_ν, N_ν) are the Bessel functions of first and second kind, $(\mathscr{P}_\mu^\nu , \mathscr{Q}_\mu^\nu)$ are the Legendre functions of first and second kind, and the last two solutions are chosen to be the same as in non-relativistic theory. The A's and B's are constants. $\hbar$ and c are shown explicitly wherever their significance in equations is considered to be important.

The wave equation (3) and solutions such as (6) are valid across the surface of equal proper time and are independent of proper time in systems where the centre-of-mass motion can be factorized from the total wave function. It is most important to realize that the relative time co-ordinate implicit in the definition of S and γ applies only to ordinary times lying on the surface of equal proper time and that the wave function defines wave propagation relative to that particular co-ordinate. Therefore the wave function and its eigenvalues have a quite different significance to those used by Feynman (1949) and Tomonaga (1946), in which the particle times, wave functions and eigenvalues, apply to all possible surfaces, and not just one special surface.

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* Equations (30) are the first set of equations on p. 7 of Part I. The numbering was omitted there in error.

The operators

(i) $$\underset{\sim}{L} = i\hbar\left[\frac{1}{\sin\theta}\ \underline{e}_\theta\ \frac{\partial}{\partial\phi} - \underline{e}_\phi\ \frac{\partial}{\partial\theta}\right]$$

and

(ii) $$\underset{\sim}{L}^2 = \hbar^2\left[\frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\left(\sin\theta\ \frac{\partial}{\partial\theta}\right) + \operatorname{cosec}^2\theta\ \frac{\partial^2}{\partial\phi^2}\right]$$

$$= \underline{Q}_\theta^2 + \underline{Q}_\phi^2\ \operatorname{cosec}^2\theta$$

involve no derivatives with respect to S and γ, and

(iii) $$\underset{\sim}{L}^2\ \Psi\ (\underline{R}) = \ell(\ell+1)\hbar^2\ \Psi\ (\underline{R}) \tag{7}$$

as in non-relativistic theory. The polar operator

(i) $$\underset{\sim}{A} = -i\hbar\left[\underline{e}_R\ \frac{\partial}{\partial\gamma} + \tanh\gamma\left(\underline{e}_\theta\ \frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\ \underline{e}_\phi\ \frac{\partial}{\partial\phi}\right)\right]$$

satisfies

(ii) $$\underset{\sim}{A}^2 = \hbar^2\left\{\left[\frac{\partial^2}{\partial\gamma^2} + 2\tanh\gamma\ \frac{\partial}{\partial\gamma}\right] + \tanh^2\gamma\ \underset{\sim}{L}^2\right\}$$

so that the operator

$$\Lambda^2 = \underset{\sim}{L}^2 - \underset{\sim}{A}^2 = \underset{\sim}{L}^2\ \operatorname{sech}^2\gamma + \hbar^2\left[\frac{\partial^2}{\partial\gamma^2} + 2\tanh\gamma\ \frac{\partial}{\partial\gamma}\right]$$

$$= \underset{\sim}{L}^2\ \operatorname{sech}^2\gamma - \underline{Q}_\gamma^2$$

has eigenvalues $\lambda(\lambda+2)$ when acting on $\Psi(\underline{R})$. Neither $\underset{\sim}{L}$ nor $\underset{\sim}{A}$ contain derivatives with respect to S. Hence

(iii) $$\underset{\sim}{A}^2\ \Psi\ (\underline{R}) = \left[\ell(\ell+1) - \lambda(\lambda+2)\right]\hbar^2\ \Psi\ (\underline{R})$$

$$= a^2\ \Psi\ (\underline{R})\ . \tag{8}$$

If the Bohr correspondence principle (Schiff 1949) is to hold we should choose $\underset{\sim}{A}^2$ to have positive eigenvalues, and so $\ell > \lambda$.

Now consider the mathematical situation of the theory concerning the expansion of plane waves:

$$\Psi = B\exp(i\ \underline{Q}\cdot\underline{R}) = B\exp(iQSZ) \tag{9}$$

into pseudo-spherically symmetric eigenfunctions. The following formulae are contained in Erdelyi et al. (1953). The first is

$$\exp(iQSZ) = 2^\nu\Gamma(\nu)\sum_{n=0}^{\infty} i^n(\nu+n)(QS)^{-\nu}J_{\nu+n}(QS)\ C_n^\nu\ (Z) \tag{10}$$

where ν is arbitrary and C_n^ν is the Gegenbauer function. Comparing (10) with the solutions (6) it is seen that $n=\lambda$, $\nu=1$ are the appropriate choices for a plane wave solution.

Furthermore

$$Z = \cosh\gamma\cosh\delta\,.\,z - \sinh\gamma\ \sinh\delta \tag{11}$$

where $\sinh\delta = \epsilon/Q$, $\cosh\delta = q/Q$, $z = \underset{\sim}{q}\cdot\underset{\sim}{R}/qR$

and therefore we can use the Gegenbauer addition theorem to define solutions in terms of the $\mathcal{Q}_\ell^{\lambda+1}$ (tanh γ). However, these solutions have eigenvalues a^2 from (8) which are negative and must therefore be rejected as not satisfying the correspondence principle. What then is the alternative to the plane wave expansion?

3. THE COVARIANT ADDITION THEOREM AND FREE PARTICLE SOLUTIONS

The volume element in the hyperspace is

$$dV = S^3 \cosh^2\gamma \, \sin\theta \, dS \, d\gamma \, d\theta \, d\phi \quad . \tag{12}$$

Using the $\mathcal{P}_\ell^{\lambda+1}$ solution, we have

$$\int_{-1}^{1} \frac{dt}{1-t^2} \, \mathcal{P}_\ell^{\lambda+1}(t) \cdot \mathcal{P}_\ell^{\lambda'+1}(t) = \frac{(\lambda+\ell+1)! \, \delta_{\lambda\lambda'}}{(\lambda+1)(\ell-\lambda+1)!} \tag{13}$$

from Erdelyi et al., and therefore these eigenfunctions form an incomplete orthonormal set with $\ell \geqslant \lambda + 1$, and hence $a^2 > 0$.

From the expansion properties in the three-dimensional case, we expect an expansion of the form

$$\Psi(\underline{Q}, \underline{R}) = \sum_{\lambda=-1}^{\infty} \sum_{\ell=\lambda+1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\lambda\ell m} \frac{J_{\lambda+1}(QS)}{QS} \times$$

$$\times \, \mathcal{P}_\ell^{\lambda+1}(\tanh\gamma) \, \operatorname{sech}\gamma \cdot \mathcal{P}_\ell^{m}(\cos\theta) \, \exp(im\phi) \tag{14}$$

to represent the free particle eigenfunction that is Lorentz-invariant. The $Q_{\lambda\ell m}$ are functions of the components of the relative 4-momentum Q. To carry out the summations in (14) it is necessary to establish a covariant addition theorem, and this is indicated in a semi-rigorous way.

The spherical harmonic eigenfunctions

$$\mathcal{Y}_{\ell m}(\theta,\phi) = (-1)^m \left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\right] \mathcal{P}_\ell^{m}(\cos\theta) \, \exp(im\phi) \tag{15}$$

(Edmunds 1957) simplify calculations in non-relativistic theory. To this end, we define its covariant equivalent

$$\mathcal{Y}_{n\ell m}(\gamma,\theta,\phi) = \left[\frac{n(\ell-n)!}{(\ell+n)!}\right] \mathcal{Y}_{\ell m}(\theta,\phi) \, \mathcal{P}_\ell^{n}(\tanh\gamma) \, \operatorname{sech}\gamma \tag{16}$$

which satisfies

$$\iiint \mathcal{Y}^*_{n\ell m}(\gamma,\theta,\phi) \cdot \mathcal{Y}_{n'\ell'm'}(\gamma,\theta,\phi) \, . \, d\Omega = \delta_{nn'} \, \delta_{\ell\ell'} \, \delta_{mm'} \tag{17}$$

where we have used $n = \lambda + 1$. The integral is taken over the whole physical relative 4-space and is easily proved using the orthogonality relation (Edmunds 1957)

$$\int_0^{\pi} \sin\theta \cdot d\theta \int_0^{2\pi} d\phi \cdot \mathcal{Y}^*_{\ell m}(\theta,\phi) \, \mathcal{Y}_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'} \, \delta_{mm'} \tag{18}$$

the integral (13) and the volume element (12), which gives

$$d\Omega = \cosh^2\gamma \, \sin\theta \, d\gamma \, d\theta \, d\phi \quad . \tag{19}$$

The components of $\underline{R}$ are chosen to be (S, $\gamma_1, \theta_1, \phi_1$) and those of $\underline{Q}$ to be (Q, $\gamma_2, \theta_2, \phi_2$) for the purpose of the following argument. The object is to determine a Lorentz-invariant eigenfunction $g_n(Z)$ where

$$Z = \underline{Q} \cdot \underline{R} / QS$$

which is a superposition of the angular components of the wave function (6) that is:

$$g_n(Z) = \sum_{\ell=n}^{\infty} \sum_{m=-\ell}^{\ell} b_{n\ell m}(\gamma_2, \theta_2, \phi_2)\, \mathcal{Y}_{n\ell m}(\gamma_1, \theta_1, \phi_1) \; . \tag{20}$$

However, we will postulate that because Z is invariant under the transformations $\gamma_1 \leftrightarrow \gamma_2$, $\theta_1 \leftrightarrow \theta_2$, $\phi_1 \leftrightarrow \phi_2$, the eigenfunctions on the right-hand side of (20) must be similarly invariant, provided we assume $g_n(Z)$ to be a real function. Therefore we put

$$g_n(Z) = \sum_{\ell=n}^{\infty} \sum_{m=-\ell}^{\ell} a_{n\ell m}\, \mathcal{Y}^*_{n\ell m}(\gamma_2, \theta_2, \phi_2) \cdot \mathcal{Y}_{n\ell m}(\gamma_1, \theta_1, \phi_1) \; . \tag{21}$$

The usual addition theorem (Edmunds 1957)

$$\mathcal{P}_\ell(\cos\omega) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \mathcal{Y}^*_{\ell m}(\theta_2, \phi_2)\, \mathcal{Y}_{\ell m}(\theta_1, \phi_1) \tag{22}$$

where $\cos\omega = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)$

can be used to carry out the summation over m in (21), which yields

$$g_n(Z) = \frac{1}{4\pi} \sum_{\ell=n}^{\infty} (2\ell+1)\, a_{n\ell} \left[\frac{n(\ell-n)!}{(\ell+n)!}\right] \mathcal{P}_\ell^n(\tanh\gamma_1)\,\mathrm{sech}\,\gamma_1 \; \times$$

$$\times \; \mathcal{P}_\ell^n(\tanh\gamma_2)\,\mathrm{sech}\,\gamma_2 \cdot \mathcal{P}_\ell(z) \tag{23}$$

where $z = \cos\omega$ and $a_{n\ell} = a_{n\ell m}$, as required by invariance under rotations in 3-space.

It was found in all applications of the theory in Part I that the geometrical physical region is defined by $|Z| \leqslant 1$. Regions where $|Z| > 1$ are actually accessible from the usual physical ranges of $(\gamma_1, \theta_1, \phi_1)$, $(\gamma_2, \theta_2, \phi_2)$ unless the restriction on Z is taken as a separate kinematic condition. The sum on the right-hand side of (23) is therefore explicitly limited to the region where $|Z| \leqslant 1$. Inserting a Heaviside function, θ, we have

$$g_n(Z)\,\theta(1-Z^2) = \frac{1}{4\pi} \sum_{\ell=n}^{\infty} (2\ell+1)\, a_{n\ell} \left[\frac{n(\ell-n)!}{(\ell+n)!}\right] \times$$

$$\times \; \mathcal{P}_\ell^n(t_1)\sqrt{1-t_1^2}\;\; \mathcal{P}_\ell^n(t_2)\sqrt{1-t_2^2} \cdot \mathcal{P}_\ell(z) \cdot \theta(1-Z^2) \; ;$$

$$\theta(x) = 1, \quad x > 0 \; ;$$

$$\theta(x) = 0, \quad x < 0 \; , \tag{24}$$

where $t_1 = \tanh\gamma_1$, $t_2 = \tanh\gamma_2$. Multiplying both sides by $\mathcal{P}_{\ell'}(z)$ and integrating over 3-space, we obtain

$$\int_{a(Z,z_1)}^{a(Z,z_2)} \mathcal{P}_{\ell'}(z')\, g_n(Z')\, \theta(1-Z'^2)\, dz' = \frac{1}{2\pi} \sum_{\ell=n}^{\infty} \left[\frac{n(\ell-n)!}{(\ell+n)!}\right] \times$$

$$\times \; a_{n\ell}\mathcal{P}_\ell^n(t_1)\sqrt{1-t_1^2}\;\; \mathcal{P}_\ell^n(t_2)\sqrt{1-t_2^2}\;\; (\ell + \tfrac{1}{2}) \int_{a(Z,z_1)}^{a(Z,z_2)} \mathcal{P}_\ell(z')\, \mathcal{P}_{\ell'}(z')\, dz' \tag{25}$$

where a(Z,z) are the limits imposed by $z_1 \leqslant z \leqslant z_2$, since $|Z| \leqslant 1$.

Now

$$z = t_1 t_2 + \sqrt{1-t_1^2}\,\sqrt{1-t_2^2} \tag{26}$$

and behaves somewhat like an azimuthal angle's cosine with respect to z-space. However the well-known addition theorem (22) can be written

$$\mathcal{P}_\ell(z) = \mathcal{P}_\ell(t_1)\,\mathcal{P}_\ell(t_2) + 2\sum_{n=1}^{\ell} \frac{\Gamma(\ell-n+1)}{\Gamma(\ell+n+1)}\,\mathcal{P}_\ell^n(t_1)\,\mathcal{P}_\ell^n(t_2)\cos(n\cos^{-1}Z) \tag{27}$$

provided equation (26) is satisfied. Therefore, if equation (27) is substituted into the left hand side of (25) we obtain

$$\sum_{n'=0}^{\ell} C_{n'\ell'}\mathcal{P}_{\ell'}^{n'}(t_1)\,\mathcal{P}_{\ell'}^{n'}(t_2)\int_{a(z,z_1)}^{a(z,z_2)} dz'\; g_n(Z')\cos(n'\cos^{-1}Z')\,\theta(1-Z'^2)$$

$$= \sum_{n'=0}^{\ell'} C_{n'\ell'}\mathcal{P}_{\ell'}^{n'}(t_1)\,\mathcal{P}_{\ell'}^{n'}(t_2)\int_{-1}^{1} dZ'\,\sqrt{1-t_1^2}\,\sqrt{1-t_2^2}\,\cos(n'\cos^{-1}Z')g_n(Z') \tag{28}$$

where

$$C_{n\ell} = h_n\,\Gamma(\ell-n+1)/\Gamma(\ell+n+1)\;,\quad h_0 = 1\;,\quad h_n = 2\;,\quad n \neq 0\;.$$

Now the eigenfunctions $\mathcal{P}_\ell^n$ form an orthonormal set, so the expression (28) cannot equal the right hand side of (25) unless we choose $g_n(Z)$ as orthogonal to $\cos(n'\cos^{-1}Z)$. It follows that $g_n(Z)$ must be a member of this latter set with an appropriate weight function. Hence

$$g_n(Z) = \frac{\cos(n\cos^{-1}Z)}{\sqrt{1-Z^2}}\;. \tag{29}$$

The Chebyshev polynomial

$$T_n(Z) = \cos(n\cos^{-1}Z)$$

satisfies (Gradshteyn and Ryzhik 1965)

$$\int_{-1}^{1} T_n(Z)\,T_m(Z)\,\frac{dZ}{\sqrt{1-Z^2}} = \frac{\pi}{h_n}\,\delta_{mn}\;. \tag{30}$$

It is apparent that for the special case where $t_1 = t_2$, we have

$$a(1,z_2) = 1,\; a(-1,z_1) = -1 \qquad \text{and therefore}$$

$$a_{n\ell} = \frac{2\pi^2}{n}\,. \tag{31}$$

Although the series (28) diverges in the limit $t_1 \to t_2$, $Z \to \pm 1$, the constant $a_{n\ell}$ is correctly projected from the equation, because the factor $(1 - z'^2)$, arising from the process of evaluating the integral, cancels the infinity in the limit. Obviously, the $n = 0$ case must be dealt with separately. We find the eigenfunction expansion as a result of (31)

$$\frac{\cos(n\cos^{-1} Z)}{\sqrt{1-Z^2}} = \frac{2\pi^2}{n} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathcal{Y}^*_{n\ell m} (\gamma_1, \theta_1, \phi_1)\, \mathcal{Y}_{n\ell m} (\gamma_2, \theta_2, \phi_2)$$

$$= \frac{\pi}{2} \sum_{\ell=0}^{\infty} (2\ell+1) \frac{(\ell-n)!}{(\ell+n)!} \mathcal{P}_\ell^n (t_1)\, \mathcal{P}_\ell^n (t_2) \sqrt{1-t_1^2}\, \sqrt{1-t_2^2}\, \mathcal{P}_\ell (z) \,. \quad (32)$$

The second expansion is used for $n = 0$.

To find the equivalent to the plane wave expansion we consider the form of (10) normally used in two dimensions:

$$\exp(iQSZ) = \sum_{n=0}^{\infty} (i)^n h_n J_n (QS) \cdot \cos(n\cos^{-1} Z) \,. \quad (33)$$

The wave function that is a superposition of free particle solutions is

$$\Psi (\underline{Q},\underline{R}) = \sum_{n=0}^{\infty} a_n \frac{J_n (QS)}{QS} \cdot \frac{\cos(n\cos^{-1} Z)}{\sqrt{1-Z^2}} \,. \quad (34)$$

Comparing (33) with (34) we see that a choice of $a_n = (i)^n h_n$ leads to

$$\Psi (\underline{Q},\underline{R}) = \sum_{n=0}^{\infty} (i)^n h_n \frac{J_n (QS)}{QS} \cdot \frac{\cos(n\cos^{-1} Z)}{\sqrt{1-Z^2}}$$

$$= \exp(i\underline{Q}\cdot\underline{R}) / (QS \sqrt{1-Z^2})$$

$$= \exp(i\underline{Q}\cdot\underline{R}) / \sqrt{Q^2 S^2 - (\underline{Q}\cdot\underline{R})^2} \,, \quad |Z| \leqslant 1 \,.$$

(35)

This wave function has a plane wave period, but is distorted by an amplitude that depends upon both $\underline{S}$ and $\underline{Q}$. To test if (35) is a solution to the wave equation (3), we can note that

(i) $\Box^2 f \exp(i\underline{Q}\cdot\underline{R}) = -Q^2 f \exp(i\underline{Q}\cdot\underline{R}) + i\,(\underline{Q}\cdot\Box f) \exp(i\underline{Q}\cdot\underline{R}) + \exp(i\underline{Q}\cdot\underline{R})\ (\Box^2 f)$.

The factor $(Q^2S^2-(\underline{Q}\cdot\underline{R})^2)^{-1/2}$ satisfies

(ii) $\underline{Q}\cdot\Box\ [(S\sqrt{1-Z^2})^{-1}] = 0$

(iii) $\Box^2\ [(S\sqrt{1-Z^2})^{-1}] = 0$. (36)

It is the $n = 0$ eigenfunction of the homogeneous equation

$$\Box^2 f_{n\ell m} = 0 \,.$$

The centre-of-mass motion factors form the complete wave function. Combining all of these results, we find a physically meaningful two-boson wave function without interaction of

$$\Psi (\underline{r}_1,\underline{P}_1,\underline{r}_2,\underline{P}_2) = \frac{\exp(i(\underline{r}_1\cdot\underline{P}_1 + \underline{r}_2\cdot\underline{P}_2))}{QS\sqrt{1-Z^2}} = \frac{\exp(i(\underline{Q}_1\cdot\underline{R}_1 + \underline{Q}_2\cdot\underline{R}_2))}{Q_2 S\sqrt{1-Z^2}} \quad (37)$$

where $\underset{\sim}{r}_1 \cdot \underset{\sim}{P}_1 + \underset{\sim}{r}_2 \cdot \underset{\sim}{P}_2 = \underset{\sim}{Q}_1 \cdot \underset{\sim}{R}_1 + \underset{\sim}{Q}_2 \cdot \underset{\sim}{R}_2$, which is the special case of equation 4a in Part I

and

$$\underline{R}_1 = \frac{m_1}{M}\underline{r}_1 + \frac{m_2}{M}\underline{r}_2 \quad ; \quad \underline{R}_2 = \underline{r}_1 - \underline{r}_2$$

$$\underline{Q}_1 = \underline{P}_1 + \underline{P}_2 \quad ; \quad \underline{Q}_2 = \mu\left(\frac{\underline{P}_1}{m_1} - \frac{\underline{P}_2}{m_2}\right)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \underline{S} = \underline{R}_2, \quad \underline{Q} = \underline{Q}_2, \quad Z = \underline{Q}_2 \cdot \underline{R}_2 / Q \cdot S \ .$$

4. CROSS SECTIONS

Having established an analogy between the relativistic kinematics in terms of relative co-ordinates, and non-relativistic theory in general, we can almost write down the covariant quantities without proof. To show that this analogy holds for the scattering of bosons, covariant cross sections for scattering are derived. These are not the conventional cross sections associated with two-dimensional areas in three-space, but are three-dimensional cross sections of the volume in relative four-space. The expansion (35) is used for this purpose. The Bessel functions behave for large QS as

$$\text{(i)} \quad J_n(QS) \simeq \left[\frac{2}{\pi QS}\right]^{1/2} \cos\left[QS - \frac{1}{2}n\pi - \frac{\pi}{4}\right]$$

$$\text{(ii)} \quad N_n(QS) \simeq \left[\frac{2}{\pi QS}\right]^{1/2} \sin\left[QS - \frac{1}{2}n\pi - \frac{\pi}{2}\right] \tag{38}$$

or in terms of Hankel functions

$$\text{(i)} \quad H_n^{(1)}(QS) = J_n(QS) + i\,N_n(QS) \simeq \left[\frac{2}{\pi QS}\right]^{1/2} \exp(i(QS - \frac{1}{2}n\pi - \frac{\pi}{4}))$$

$$\text{(ii)} \quad H_n^{(2)}(QS) = H_n^{(1)*}(QS) \ . \tag{39}$$

The free particle wave behaves as

$$\frac{N \cdot \exp(i\underline{Q}\cdot\underline{S})}{QS\sqrt{1-Z^2}} = N \sum_{n=0}^{\infty} (i)^n h_n \frac{J_n(QS)}{QS} \frac{\cos(n\omega)}{\sin(\omega)}$$

$$\simeq N \sum_{n=0}^{\infty} (i)^n h_n (QS)^{-\frac{3}{2}} \left[\exp(-i(QS - \frac{n\pi}{2} - \frac{\pi}{4})) + \exp(i(QS - \frac{n\pi}{2} - \frac{\pi}{4}))\right] \times$$

$$\times \ \cos(n\omega)/\sin(\omega) \tag{40}$$

where N = normalization constant, and $Z = \cos\omega$. The first term in brackets describes an incoming wave and the second an outgoing wave, propagating through the four-dimensional space-time. The presence of a scattering and reacting source modifies the outgoing component. The wave function for such a process becomes

$$\Psi \simeq N \sum_{n=0}^{\infty} (i)^n h_n (QS)^{-\frac{3}{2}} \left[\exp(-i(QS - \frac{n\pi}{2} - \frac{\pi}{4})) + \eta_n \exp(i(QS - \frac{n\pi}{2} - \frac{\pi}{4}))\right] \times$$

$$\times \ \cos(n\omega)/\sin(\omega) \tag{41}$$

where η_n is a complex amplitude. Equation (41) holds in the asymptotic region where QS is large and where it is assumed no interaction takes place. The scattered component of the wave is therefore

$$\Psi_{SC} \simeq N \cdot \sum_{n=0}^{\infty} (i)^n \, h_n \, (QS)^{-\frac{3}{2}} (1-\eta_n) \exp(i(QS - \frac{n\pi}{2} - \frac{\pi}{4})) \cdot \frac{\cos(n\omega)}{\sin(\omega)} \quad . \tag{42}$$

Suppose we confine the region of interaction to a hypersphere of radius S_0, whose surface defines a Lorentz-invariant boundary in four-space. With reference to the C.M. proper time τ, as defined in Part I, the number of particles F_S scattered per second into the solid angle $d\Omega$, is the number scattered through $S_0^3 \, d\Omega$. Hence

$$F_S \, d\Omega = - \underline{J} \; \frac{\underline{S}_0}{S_0} \cdot S_0^3 \, d\Omega \tag{43}$$

where $\quad \underline{J} = - \frac{i\hbar}{2\mu} (\Psi^*_{SC} \Box \Psi_{SC} - \Psi_{SC} \Box \Psi^*_{SC})$

is the current out of the 4-sphere. Substituting the scattered component (42) into the equation (43) for the scattering rate, we find

$$F_S(\Omega) \, d\Omega = \frac{Q}{\mu} |\Psi_{SC}|^2 \, S_0^3 \, d\Omega \quad . \tag{44}$$

Put $V = Q/\mu$, as the magnitude of the relative 4-velocity, and define the covariant cross section as

$$\Sigma_{SC} = F_S / V \; . \tag{45}$$

Using (42),(44) and (45), we obtain a cross section

$$\Sigma_{SC} = \frac{N^2}{Q^3} \left| \sum_{n=0}^{\infty} h_n (1-\eta_n) \frac{\cos(n\omega)}{\sin(\omega)} \right|^2 \quad . \tag{46}$$

It is clear that equation (46) for the covariant cross section behaves as if there was a kinematic singularity at $\omega = 0$ on the boundary of the physical region. This singularity is cancelled by the zero in the Jacobian of the volume element. Showing this explicitly

$$\Sigma_{SC} \, d\Omega = \frac{N^3}{Q^3} \left| \sum_{n=0}^{\infty} h_n (1-\eta_n) \frac{\cos(n\cos^{-1} Z)}{\sqrt{1-Z^2}} \right|^2 \sqrt{1-Z^2} \, dZ \, dz \, d\Phi$$

$$= \frac{N^3}{Q^3} \left| \sum_{n=0}^{\infty} h_n (1-\eta_n) \cos(n\omega) \right|^2 d\omega \, dz \, d\Phi \; . \tag{47}$$

Once again we note that the system behaves as if there is an additional azimuthal angle ω. The form of the cross section (47) applies in any frame of reference. A partial wave analysis of this type, when carried out in the laboratory system, has the same form in the C.M. system, or any other frame of reference. The total scattering cross section is

$$\Sigma_{T,SC} = \iiint d\Omega \cdot \Sigma_{SC} = \frac{4\pi^2 N^2}{Q^2} \sum_{n=0}^{\infty} h_n |1-\eta_n|^2 \quad . \tag{48}$$

A completely analogous derivation of the reaction cross section yields

$$\Sigma_r = \frac{4\pi^2 N^2}{Q^3} \sum_{n=0}^{\infty} h_n (1 - |\eta_n|^2) \quad . \tag{49}$$

The first term in $n = 0$ contains all of the s-wave scattering components, since $n \leqslant \ell$. It contains contributions from other partial waves as well. The scattered intensity cannot exceed the initial intensity, and so $|\eta_n| \leqslant 1$. The definition of the covariant scattering matrix is also wholly analogous to non-relativistic theory. Outside the region of interaction, the wave function satisfying (4) (i) is

$$\text{(i)}\quad \Psi = \sum_{n=0}^{\infty} C_n (\mathcal{I}_n - S_n\, \mathcal{O}_n)\quad ,\ \text{with } C_n \text{ constants,}$$

$$\text{(ii)}\quad \mathcal{I}_n = (i)^n \frac{\cos(n\omega)}{\sin(\omega)} \cdot \frac{I_n}{(QS)^{\frac{3}{2}}} = \text{incoming component}$$

$$\text{(iii)}\quad \mathcal{O}_n = (i)^n \frac{\cos(n\omega)}{\sin(\omega)} \frac{O_n}{(QS)^{\frac{3}{2}}} = \text{outgoing component}$$

$$\text{(iv)}\quad I_n = \left(\frac{\pi QS}{2}\right)^{1/2} H_n^{(2)}(QS)\ ;\ = \left(\frac{\pi QS}{2}\right)^{1/2} H_n^{(1)}(QS) \qquad (50)$$

so that

$$\Psi = \sum_n C_n(\mathcal{I}_n + \mathcal{O}_n) - \sum_n (1-S_n)\,\mathcal{O}_n$$

$$\simeq N_0 \left\{\frac{\exp(i\,\underline{Q}\cdot\underline{R})}{S\sin(\omega)}\right\} - \left\{\frac{\exp(i(QS-\frac{\pi}{4}}{S^{3/2}} \cdot F(Q,Z)\right\}$$

$$= \text{(free wave)} - \text{(scattered component)} \qquad (51)$$

where

$$\text{(i)}\quad F(Q,Z) = \frac{1}{\sqrt{2\pi}} \cdot \frac{N_0(1-S_n)}{Q^{3/2}} \cdot \frac{\cos(n\omega)}{\sin(\omega)}$$

N_0 = normalization constant, and

$$\text{(ii)}\quad \Sigma_{SC} = |F(Q,Z)|^2 \quad . \qquad (52)$$

$F(Q,Z)$ is the covariant scattering amplitude.

If the wave functions $\Psi, \mathcal{I}, \mathcal{O}$ are now taken to be column vectors of channel wave functions, Q as channel momenta and n as covariant angular momenta defined in each channel, S_n becomes a matrix in channel space. From the additive properties of $\Lambda_{\mu\nu}$, and the fact that it commutes with $\mathcal{H}$, we can conclude that n, hence λ, is conserved throughout the reaction, just as ℓ would be in the non-relativistic case.

5. SYMMETRY CONVERSION

The covariant wave functions for bound states lead to convergent integrals for probability densities and reasonably simple expressions for covariant cross sections. However, the quantities

$$P = |\Psi|^2 \quad ; \quad J = \frac{\hbar}{2\mu i}\,[\Psi\,\Box\Psi^* - \Psi^*\,\Box\,\Psi] \qquad (53)$$

derived from the wave equation (3) are densities relative to both ordinary space and the relative times which define the surface of equal proper time. Therefore, the question arises as to how we convert these quantities to the conventional 3-space equivalents. The wave propagation in the relative time direction must be removed in such a way as to leave a relative energy eigenvalue in the 3-space Schrodinger equation equivalent to (3). This can be done by defining the Fourier transforms (Sneddon 1951) at a point in 3-space.

$$\text{(i)}\quad \overline{\Psi}(\underset{\sim}{q}, \xi, \underline{R}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(i\xi T)\,\Psi(\underline{Q},\underline{R}) \cdot dT\ \theta(S^2)$$

$$\text{(ii)}\quad \Psi(\underline{Q},\underline{R}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i\xi T)\,\overline{\Psi}(\underset{\sim}{q},\xi,\underline{R})\,d\xi;\quad \Psi(\underset{\sim}{q},\underset{\sim}{R}) = \overline{\Psi}(\underset{\sim}{q}, \sqrt{q^2-Q^2}, \underline{R}) \qquad (54)$$

Taking the transform of (3), we get

$$\text{(i)}\quad [-\nabla^2 - \epsilon^2]\,\psi + \frac{2\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(i\epsilon T)\ \mathcal{V}\ \Psi \cdot dT = 2\,\mu\,\mathcal{E}\,\psi \tag{55}$$

where

$$\text{(ii)}\quad \mathcal{H}\,\Psi = \mathcal{E}\,\Psi\ .$$

Using pseudospherical co-ordinates, we find

$$\psi = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} d(R\tanh\gamma)\cdot\Psi\cdot\exp(i\,\epsilon R\tanh\gamma) = \frac{R}{\sqrt{2\pi}} \int_{-1}^{1} dt\cdot\Psi\cdot\exp(i\,\epsilon R t) \tag{56}$$

and an interaction term

$$\rho = \frac{2\,\mu R}{\sqrt{2\,\pi}} \int_{-1}^{1} dt\cdot\mathcal{V}\,(\underline{R},\underline{Q})\cdot\Psi\,(\underline{R},\underline{Q})\ . \tag{57}$$

With a small variation in the relative time co-ordinate at a fixed point $\underset{\sim}{R}$, the interaction behaves as

$$\rho = \frac{2\,\mu R}{\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{1}{p!} \left[\frac{\partial^p \mathcal{V}}{\partial t^p}\right]_{t=t'} \int_{-1}^{1} (t'-t)\,\Psi\cdot\exp(i\,\epsilon Rt)\cdot dt\ . \tag{58}$$

If the interaction decreases with increasing R, and vanishes as $R \to \infty$, then provided

$$\frac{\partial}{\partial t}\ [\ell n\,\mathcal{V}\,]\ \xrightarrow[R\to\infty]{} 0\ ,\quad |\,t'| << 1 \tag{59}$$

we will have for large R

$$\rho \approx \frac{2\,\mu R}{\sqrt{2\pi}}\cdot\mathcal{V}\,(R,t') \int_{-1}^{1} \Psi\cdot\exp(i\epsilon Rt)\cdot dt$$

$$\approx 2\,\mu\,\mathcal{V}\,(R,0)\,\psi\ . \tag{60}$$

The wave equation (55) then becomes the non-relativistic Schrodinger equation (Schiff 1949),

$$\nabla^2\,\psi + 2\,\mu\,(E - V)\,\psi = 0 \tag{61}$$

for small values of q, where

$$\mathcal{E} = \frac{Q^2}{2\,\mu} \to E\ \ ;\ \ H\,\psi = E\,\psi \approx \left(\mathcal{H} + \frac{E^2}{2\mu}\right)\psi\ . \tag{62}$$

Therefore, for large R, or slowly varying potentials, the wave function ψ becomes that applicable at low velocities, where q is small, and ϵ is considered to be zero. This would be the case for any weak interaction, implying that particle velocities remain small relative to the velocity of light. The invalidity of truncating the series (58) near the null cone where $t' \approx 1$ indicates that measurements of relative time in this region affect the behaviour of the system violently and enhance the higher order terms of the relativistic interaction.

In the absence of any interaction, we would expect the covariant wave function (35), which represents two free particles, to transform directly to a plane wave at any velocity of the C.M., or any physical relative velocity. This is in fact the case:

$$\psi = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} dT \cdot \exp(i\epsilon T) \frac{\exp(i\,\underline{Q}\cdot\underline{S})}{\sqrt{Q^2 S^2 - (\underline{Q}\cdot\underline{S})^2}} \quad \theta(S^2)\,\theta(1-Z^2)$$

$$= \frac{\exp(i\,\underset{\sim}{q}\cdot\underset{\sim}{R})}{\sqrt{2\pi}} \int_{-R}^{R} dT \cdot (Q^2(R^2 - C^2T^2) - (\underset{\sim}{q}\cdot\underset{\sim}{R} - \epsilon T)^2)^{-\frac{1}{2}}$$

$$= \frac{\exp(i\,\underset{\sim}{q}\cdot\underset{\sim}{R})}{\sqrt{2\pi}} \left(- \frac{1}{d} \left[\sin^{-1} \left(\frac{2dT + b}{\Delta} \right) \right]_{-R}^{R} \right) \tag{63}$$

where
$$\Delta = 4\,ad - b^2$$
$$d = Q^2 + \epsilon^2 = q^2$$
$$b = 2\,\underset{\sim}{q}\cdot\underset{\sim}{R}$$
$$a = Q^2R^2 - (\underset{\sim}{q}\cdot\underset{\sim}{R})^2 \quad .$$

The integrand simplifies to give

$$\psi = \frac{\exp(i\,\underset{\sim}{q}\cdot\underset{\sim}{R})}{\sqrt{2\pi q}} \cdot - \left[\sin^{-1} \left(\frac{t_2 z - t_1}{\sqrt{1-z^2}\sqrt{1-t_2^2}} \right) \right]_{t_1=-1}^{t_1=1}$$

where $z = \underset{\sim}{q}\cdot\underset{\sim}{R}/qR$, $t_1 = \tanh\gamma$, $t_2 = \tanh\delta$. Using

$$z = t_1 t_2 + \sqrt{1-t_1^2}\sqrt{1-t_2^2}\; Z$$

we obtain the value π for the integral, giving

$$\psi = \sqrt{\frac{\pi}{2}} \cdot \exp(i\,\underset{\sim}{q}\cdot\underset{\sim}{R}) \quad . \tag{64}$$

All of formal non-relativistic scattering theory is based on the free particle plane wave function (64). Therefore, allowing for relativistic kinematic factors, the non-relativistic expressions for cross sections, the S-matrix, partial wave expansions, and any formalism independent of the explicit form of the interaction, including reaction matrix theory (Wigner and Eisenbud 1947, Preston 1965, Lane and Thomas 1958), potential theory (Regge 1959) and Regge pole theory, are valid to arbitrarily high energies. These theories become covariant within the relative time formalism, provided no measurement is made to test Lorentz invariance. A test of Lorentz invariance is necessarily an experiment with pseudospherical symmetry in 4-space, and the additional n-degeneracy becomes observable.

It is very instructive to show how the wave fronts propagating in 4-space combine to give the plane wave (64), and in doing so remove the n-degeneracy. Using the eigenfunction expansion (35) and the addition theorem (32) we find

$$\psi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dT \cdot \exp(i\epsilon T)\, \Psi(\underline{R})\, \theta(S^2)$$

$$= \frac{R}{\sqrt{2\pi}} \int_{-1}^{1} dt_1 \; \exp(iqRt_1t_2) \;\; x \;\; \frac{\pi}{2} \sum_{n=0}^{\infty} (i)^n\, h_n \; \frac{J_n(qR\sqrt{1-t_1^2}\;\sqrt{1-t_2^2})}{qR\sqrt{1-t_1^2}\;\sqrt{1-t_2^2}} \quad x$$

$$x \sum_{\ell=n}^{\infty} b_{\ell n}\; \mathcal{P}_\ell^{\,n}(t_1)\sqrt{1-t_1^2}\; \mathcal{P}_\ell^{\,n}(t_2)\sqrt{1-t_2^2} \cdot \mathcal{P}_\ell(z) \quad . \tag{65}$$

There exists a standard Fourier transform (Erdelyi et al. 1953)

$$\sqrt{\frac{2\pi}{y}}\ (i)^p\,(\sin\phi)^{\nu-\frac{1}{2}}\cdot C_p^{\nu}(\cos\phi)\ J_{\nu+p}(y) =$$

$$= \int_0^{\pi} \exp(iy\cos\theta\cos\phi)\ J_{\nu-\frac{1}{2}}(y\sin\theta\sin\phi)\ C_p^{\nu}(\cos\theta)\,(\sin\theta)^{\nu+\frac{1}{2}}d\theta \tag{66}$$

where C_p^{ν} is the Gegenbauer function. It is related to the spherical harmonics by (Erdelyi et al. 1953)

$$C_p^{\nu}(z) = 2^{\nu-\frac{1}{2}}\Gamma(p+2\nu)\cdot\Gamma(\nu+\tfrac{1}{2})\cdot(z^2-1)^{\frac{1}{4}-\frac{\nu}{2}}\cdot \mathcal{P}^{\frac{1}{2}-\nu}_{p+\nu-\frac{1}{2}}(z)\ . \tag{67}$$

Substituting $\nu = n+\frac{1}{2}$, $\cos\theta = t_1$, $y = qR$, $p+\nu = \ell+\frac{1}{2}$, $\cos\phi = t_2$

we obtain

$$\sqrt{\frac{2\pi}{qR}}\ (i)^{\ell-n}\,\mathcal{P}_\ell^n(t_2)\ J_{\ell+\frac{1}{2}}(qR) =$$

$$= \int_{-1}^{1} dt_1\ \exp(iqRt_1t_2)\,J_n\left(qR\sqrt{1-t_1^2}\ \sqrt{1-t_2^2}\right)\ \mathcal{P}_\ell^n(t_1)$$

which, when applied to (65) yields

$$\psi = \frac{\pi}{2}\sum_{n=0}^{\infty}(i)^n\ h_n\sum_{\ell=n}^{\infty} b_{\ell n}\ \frac{(i)^{\ell-n}}{q}\cdot\frac{J_{\ell+\frac{1}{2}}(qR)}{\sqrt{qR}}\left[\mathcal{P}_\ell^n(t_2)\right]^2\mathcal{P}_\ell(z)\ ,$$

$$= \frac{1}{q}\sum_{\ell=0}^{\infty}(i)^\ell\ \frac{J_{\ell+\frac{1}{2}}(qR)}{\sqrt{qR}}\cdot\mathcal{P}_\ell(z)\sum_{n=0}^{\ell}\frac{\pi}{2}\ h_n\,b_{\ell n}\left[\mathcal{P}_\ell^n(t_2)\right]^2\ . \tag{68}$$

Now $b_{\ell n} = (2\ell+1)\,\dfrac{(\ell-n)!}{(\ell+n)!}$, and using the addition theorem the second sum equals $(2\ell+1)$, so

$$\psi = \frac{\pi}{2q}\sum_{\ell=0}^{\infty}(i)^\ell\,(2\ell+1)\ \frac{J_{\ell+\frac{1}{2}}(qR)}{\sqrt{qR}}\ \mathcal{P}_\ell(z)$$

$$= \sqrt{\frac{\pi}{2}}\cdot\exp(i\,\underset{\sim}{q}\cdot\underset{\sim}{R})$$

in agreement with the direct result (64).

When the interaction $\mathcal{V}(S)$ is present, the converted wave function is

$$\psi = \frac{R}{\sqrt{2\pi}}\int_{-1}^{1}dt_1\ \frac{\exp(iqRt_1t_2)}{qR}\cdot\sum_{n=0}^{\infty}a_n\,G_n\left(qR\sqrt{1-t_1^2}\ \sqrt{1-t_2^2}\right)\quad \text{x}$$

$$\text{x}\ \sum_{\ell=n}^{\infty}b_{n\ell}\,\mathcal{P}_\ell^n(t_1)\ \mathcal{P}_\ell^n(t_2)\ \mathcal{P}_\ell(z)$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{q}\sum_{\ell=0}^{\infty}(2\ell+1)\ g_\ell(q,R,t_2)\,\mathcal{P}_\ell(z) \tag{69}$$

where $$g_\ell = \sum_{n=0}^{\ell} a_n \frac{(\ell-n)!}{(\ell+n)!} \bar{G}_n(q,R,t_2)\, \mathcal{P}_\ell^n(t_2)$$

is the 3-space radial wave function,

$$\bar{G}_n = \int_{-R}^{R} dT \cdot \exp(i\epsilon T)\, G_n(QS)\, \mathcal{P}_\ell^n\left(\frac{cT}{R}\right) \tag{70}$$

and $G_n(QS)/QS$ is the solution to the 4-space radial wave equation. Bertram (1969 private communication) has proved the important result

$$2^{-\frac{1}{2}}\sqrt{\pi}\; a^{\frac{1}{2}-\nu} \int_{-1}^{1} (1-y^2)^{\frac{\nu}{2}-\frac{1}{4}}\; C_n^\nu(y)\, N_{\nu-\frac{1}{2}}(a\sqrt{1-y^2})\, \exp(ixy)\, dy$$

$$= (i)^n (x^2+a^2)^{-\frac{n}{2}} C_n^\nu\left(\frac{x}{\sqrt{x^2+a^2}}\right)\left\{(x^2+a^2)^{-\frac{\nu}{2}-\frac{n}{2}} N_{\nu+n}(\sqrt{a^2+x^2})\right\} \tag{71}$$

where $N_\nu(x)$ is the associated Bessel function. Using this result, we find

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dT \cdot \exp(i\epsilon T) \sum_{n=0}^{\infty} (i)^n h_n \frac{B_n(QS)}{QS} \frac{\cos(n\cos^{-1} Z)}{\sqrt{1-Z^2}}$$

$$= \frac{2}{\pi} \cdot \frac{1}{q} \cdot 2\pi \sum_{\ell=0}^{\infty} (i)^\ell (\ell+\tfrac{1}{2}) \frac{B_{\ell+\frac{1}{2}}(qR)}{\sqrt{qR}} \cdot \mathcal{P}_\ell(z) \tag{72}$$

where B_ν is any of the Bessel functions J_ν, N_ν, $H_\nu^{(1)}$, $H_\nu^{(2)}$.

The ordinary representation of the delta function (Goertzel and Tralli 1960) is not useful in this scattering theory. Instead we use the distorted form (40) to obtain

$$\int_0^\infty Q^3\, dQ \int_{-1}^{1} dZ_1 \sqrt{1-Z_1^2} \int_{-1}^{1} dz_1 \int_0^{2\pi} d\Phi_1 \cdot \frac{\exp(i\underline{Q}\cdot(\underline{S}-\underline{S}'))}{Q^2 S S' \sin\omega \sin\omega'} \times \frac{1}{(1-Z_1^2)}$$

$$= 4\pi^2 \sum_{n=0}^{\infty} h_n \cdot \frac{\cos(n(\omega-\omega'))}{\sin\omega \cdot \sin\omega'} \times \int_0^\infty \frac{J_n(QS)\, J_n(QS')\, Q\cdot dQ}{SS'}$$

$$= 4\pi \frac{\delta(S-S')\,\delta(Z-Z')}{S^3\sqrt{1-Z^2}} = 4\pi \frac{\delta(S-S')\,\delta(\omega-\omega')}{S^3 \sin^2\omega},$$

$$\underline{Q}\cdot\underline{S}/QS = \cos(\omega-\omega_1), \quad \underline{Q}\cdot\underline{S}'/QS' = \cos(\omega'-\omega_1) \tag{73}$$

where the rotation theorem

$$\int_0^\pi d\omega_1 \cdot \cos(n(\omega_1-\omega)) \cdot \cos(n'(\omega_1-\omega')) = \frac{\pi}{h_n} \delta_{nn'} \cdot \cos(n(\omega-\omega')) \tag{74}$$

has been used. In these equations

$$\underset{\sim}{Q}\cdot\underline{S}_1/QS_1 = Z_1 = \cos\omega_1\ ,\quad Z = \cos\omega\ ,\quad Z' = \cos\omega'$$

and $\underline{S}_1$ is a reference 4-vector.

Following standard derivations (Goertzel and Tralli 1960) the equation for the scattering state is

$$\Psi(\underline{S},\underline{Q}) = \frac{\exp(i\underline{Q}\cdot\underline{S})}{QS\sqrt{1-Z^2}} + \iiiint S'^3 dS' d\omega' dz' d\Phi' \times G(\underline{S},\underline{S}')\,\Psi(\underline{S}',\underline{Q}) \qquad (75)$$

and G is the Green's function

$$G(\underline{S},\underline{S}') = \frac{1}{4\pi}\int_0^\infty Q'^3 dQ' \int_{-1}^{1} \frac{dZ_1}{\sqrt{1-Z_1^2}} \int_{-1}^{1} dz_1 \int_0^{2\pi} d\Phi_1 \quad \times$$

$$\times \quad \frac{\exp(i\underline{Q}\cdot(\underline{S}-\underline{S}'))}{Q'^2 SS' \sin\omega\ \sin\omega'} \cdot \left(\frac{1}{Q^2-Q'^2}\right)$$

$$= \frac{1}{4}\ \Sigma\ g_n(S,S')\ h_n\ \frac{\cos n(\omega-\omega')}{\sin\omega\cdot\sin\omega'}$$

$$= \frac{1}{4\pi}\ \frac{H_0^{(1)}(Q\,|\underline{S}-\underline{S}'|)}{SS'\ \sin\omega\cdot\sin\omega'} \qquad (76)$$

where

$$g\ (S,S') = \int_0^\infty \frac{J_n(Q'S)\,J_n(Q'S')\,Q'dQ'}{SS'\ [\,Q'^2-Q^2\,]}$$

$$= \mathrm{Re}\left[\frac{H_n^{(1)}(QS')\,J_n(QS)}{SS'}\right],\quad S < S'$$

$$= \mathrm{Re}\left[\frac{H_n^{(1)}(QS)\,J_n(QS')}{SS'}\right],\quad S > S'\ .$$

Taking the Fourier transform of (75) and using (64) we find the 3-space wave function

$$\psi(q,R) = \exp(i\underset{\sim}{q}\cdot\underset{\sim}{R}) - \frac{1}{4\pi}\iiiint dS'\ \frac{\exp(iq|R-R'|)}{|R-R'|}\ \mho(S')\,\Psi(\underline{S}',\underline{Q})\ . \qquad (77)$$

From (77) we find the conventional scattering amplitude as in Schiff (1949) of

$$f(\theta,\phi) = \frac{1}{4\pi}\iiiint dS'\ \exp(i\underset{\sim}{q}'\cdot\underset{\sim}{R}')\ \mho(\underline{S}')\,\Psi(\underline{S}',\underline{Q}) \qquad (78)$$

where $\underset{\sim}{q}' = \underset{\sim}{q} - \underset{\sim}{q}_0$, $\underset{\sim}{q}_0$ is the vector representing the initial beam momentum, and $\underset{\sim}{q}$ is the final beam momentum in the direction, (θ,ϕ). Models for scattering and perturbation expansion can be evaluated from equation (78).

6. BOUND STATES

There are some specific points concerning the eigenvalues of the covariant angular momentum tensor which require some elucidation. The most important is the question of its role in the energy eigenvalues for discrete levels in bound states. Five models are given here to make this role apparent.

6.1 Coulomb Two-Boson Atom

The Hamiltonian for this problem, as in equation (3) can be written in the notation of Part I,

$$\mathcal{H}\Psi = \mathcal{E}\Psi = \left\{ \frac{(\underline{Q}_1 - \underline{A}_1)^2}{2M} + \frac{(\underline{Q}_2 - \underline{A}_2)^2}{2\mu} \right\} \Psi \quad . \tag{79}$$

We assume, as in Part I, that the electromagnetic potentials are given by

(i) $\underline{A}_1 = \frac{e^2}{C} \cdot \frac{1}{S} \underline{G}_1$

where $\underline{G}_1$ is an operator which is assumed to obey the eigenvalue equation

(ii) $\underline{G}_1 \Psi = \underline{U}_{CM} G_1 \Psi \quad , \quad [\underline{G}_1, \mathcal{H}] \equiv \underline{G}_1 \mathcal{H} - \mathcal{H} \underline{G}_1 = 0$

where the square brackets denote the commutation relationship, and G_1 is a scalar. $\underline{U}_{CM}$ is the 4-velocity of the C.M. Similarly

$$\text{(iii)} \quad \underline{A}_2 = \frac{e^2}{C} \cdot \frac{1}{S} \underline{G}_2 \quad ; \quad \underline{G}_2 \Psi = \underline{U}_{CM} G_2 \Psi \quad . \tag{80}$$

When these potentials are substituted into (79) we find

$$\mathcal{H} = \frac{Q_1^2}{2M} + \frac{1}{2M} \{\underline{A}_1 \cdot \underline{Q}_1 + \underline{Q}_1 \cdot \underline{A}_1\} - \frac{A_1^2}{2M} + \frac{Q_2^2}{2\mu}$$

$$- \frac{1}{2\mu} \{\underline{A}_2 \cdot \underline{Q}_2 + \underline{Q}_2 \cdot \underline{A}_2\} + \frac{A_2^2}{2\mu}$$

$$= \frac{Q_1^2}{2M} + \frac{Q_2^2}{2\mu} + f_1 \frac{e^2}{C} \cdot \frac{1}{S} + f_2 \frac{e^4}{C^2} \cdot \frac{1}{S^2} \tag{81}$$

to order e^6 at least. Dropping the suffix C.M. from the four-velocity, the following commutation relations must be valid for the above equality to hold

$$[\underline{Q}_1, \underline{U}] = 0 \quad , \quad [\underline{Q}_2, \underline{U}] = 0 \quad , \quad [\underline{Q}_1, \frac{1}{S} G_1] = 0 \quad , \quad [\underline{Q}_2, \frac{1}{S} G_2] = 0$$

$$[\underline{Q}_2, G_2] = \frac{-i}{S} \underline{e}_S G_2 \quad ; \quad [G_2, \frac{1}{S}] = 0 \quad , \quad \underline{e}_S = \underline{S}/S$$

and therefore $[\underline{Q}_1, \underline{A}_1] = 0 \quad , \quad [\underline{Q}_2, \underline{A}_2] = 0$.

These conditions are really only necessary to simplify equation (79). We define two operators (F_1, F_2) with constant eigenvalues (f_1, f_2)

$$F_1 \Psi = f_1 \Psi \quad ; \quad F_2 \Psi = f_2 \Psi$$

such that

$$\frac{\underline{Q}_1}{M} G_1 - \underline{V} \cdot \underline{U} G_2 = F_1 \quad , \quad \underline{V} = \underline{Q}_2/\mu \quad \text{and}$$

$$\frac{\mu}{M} G_1^2 + G_2^2 = F_2 \quad .$$

With these operators, equation (81), on separating the variables yields the radial wave equation

$$\frac{d^2\Psi_S}{dS^2} + \frac{3}{S}\frac{d\Psi_S}{dS} - \frac{(\lambda(\lambda+2) - f_2\,\alpha^2)}{S^2}\,\Psi_S = \left(-\,|E| + \frac{2 f_1\,\alpha\mu}{S}\right)\Psi_S \qquad (82)$$

where $E = 2\mu\left[H - \frac{Q_1^2}{2M}\right] = \frac{\mu}{M}\,(W^2 - M^2)$.

Putting $\beta^2 = 4E$, $\rho = \beta S$

we obtain the equation

$$\frac{d^2\Psi_S}{d\rho^2} + \frac{3}{\rho}\frac{d\Psi_S}{d\rho} + \left[-\frac{\lambda'(\lambda'+2)}{\rho^2} - \frac{1}{4} + \frac{K}{\rho}\right]\Psi_S = 0 \qquad (83)$$

in which (i) $K = 2 f_1\,\alpha\,\beta\mu = f_1\,\frac{e^2}{C}\,\frac{\mu}{\sqrt{-E}}$

(ii) $\lambda'(\lambda'+2) = n'^2 - 1 = \lambda\,(\lambda+2) - f_2\,\alpha^2 = n^2 - 1 - f_2\,\alpha^2$.

Substituting $\Psi_S = W\rho^{-\frac{3}{2}}$ into (83) we obtain

$$\ddot{W} + W\left(-\frac{1}{4} + \frac{K}{\rho} + \frac{\frac{1}{4} - n'^2}{\rho^2}\right) = 0 \qquad (84)$$

This is Whittaker's equation (Erdelyi et al. 1953). The solution which tends to zero as ρ tends to infinity is

$$W_{K,n'} = e^{-\rho/2}\,\rho^{K}\,{}_2F_0\left(\frac{1}{2} - K + n', \frac{1}{2} - K - n'\,;\, -\frac{1}{\rho}\right)$$

$$= e^{-\rho/2}\,\rho^{n'+1/2}\,\Psi\left(\frac{1}{2} - K + n', 2n' + 1;\ \rho\right) \qquad (85)$$

where $\Psi\,(a, c\,;\,x)$ is the confluent hypergeometric function. It behaves asymptotically for large ρ such that

$$\Psi_S \sim e^{-\rho/2}\,\rho^{-n'-1}\left[\sum_{m=0}^{\infty}(-1)^m\,\frac{\Gamma(n' + 1/2 + m - K)}{\Gamma(n' + 1/2 - K)} \times \frac{1}{m!} \times \frac{\Gamma(1/2 - K + m - n')}{\Gamma(1/2 - K - n')}\,\rho^{K-n'-1/2-m}\right]$$

$$\sim (\text{constant})\ e^{-\rho/2}\,\rho^{K-\frac{3}{2}}\ . \qquad (86)$$

Near the origin, no non-singular solution exists unless $(\frac{1}{2} - K + n')$ is a negative integer. We have there

$$\Psi_S \sim \frac{\Gamma(2n')}{\Gamma(1/2 - K + n')}\,\rho^{-n'-1/2}\ .$$

Two discernible cases arise:

(a) $N = -\left(\frac{1}{2} - K + n'\right) =$ an integer.

This solution becomes the same as in the non-relativistic case, namely the Laguerre polynomials times factors. This is because

$$L_N^{2n'}(\rho) = \frac{(-1)^N}{N!} \Psi(-N, 2n'+1; \rho)$$

and these functions are non-singular at the origin. However, we would then have

$$K = \frac{1}{2} + N + n' .$$

Should the integer solutions for n be chosen, K is approximately half-integer, in which case the energy levels are

$$E = -\frac{f_1 \alpha^2}{K^2}$$

and do not tend to the correct non-relativistic limit of the Bohr levels. There are two possible answers to this dilemma. Firstly, we could choose the solutions where n is half-integer. This would imply that the scattering theory developed in Sections 3 and 4 was not applicable to such a pair of particles. An equivalent scattering theory in which n is half-integer can be derived.

(b) From the relation

$$K_{n'}\left(\frac{x}{2}\right) = \sqrt{\pi}\, \exp(-x/2)\, x^{n'} \Psi\left(n' + \frac{1}{2}, 2n' + 1; x\right) \tag{87}$$

it can be seen that as $\alpha \to 0$, the solution with integer N cannot give the free particle eigenfunctions $K_n\left(\frac{\rho}{2}\right)$ at negative energies. We can therefore choose N to be half-integer to preserve this relationship.

In this case the singularity at the origin does not allow us to normalize the solution over the physical volume element. Such two-boson atoms cannot therefore admit point source potentials which yield the Bohr levels and are non-singular at the origin. If we assume the two bosons to be extended sources, it is possible to introduce a surface cutoff at very small values of S. Using Green's theorem, we find

$$[W_1 W_2' - W_2 W_1'] = (K_1 - K_2 \int_\rho^\infty W_1 W_2 \frac{d\rho}{\rho} . \tag{88}$$

Making use of the radial wave equation (82) we see that a cutoff at ρ allows the integral to be written

$$\lim_{\rho \to \rho_0} [W_1 W_2' - W_2 W_1'] = (K_1 - K_2) \int_{S_0}^\infty W_1 W_2 \frac{dS}{S} . \tag{89}$$

This integral will vanish if

$$\rho_0 \frac{W_i'(\rho_0)}{W_i(\rho_0)} = B_n \quad , \quad \frac{\partial B_n}{\partial K} = 0 .$$

The higher terms in the expansion near the origin are found from the relation (85) and the equations (Erdelyi et al. 1953)

(i) $$\Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a,c;x) + x^{1-c} \frac{\Gamma(c-1)}{\Gamma(a)} \Phi(a-c+1, 2-c; x)$$

where

(ii) $$\Phi(a,c;x) = 1 + \frac{a}{c}x + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \ldots \tag{90}$$

is the confluent hypergeometric function.

For small ρ_0 we obtain

$$K\rho_0 = \frac{(1-\frac{c}{2}-B_n)(2-c)}{B_n - 2 + \frac{c}{2}} = \frac{(2n' + 1)(B_n + n' - ½)}{B_n + n' - \frac{3}{2}} . \tag{91}$$

Right at the origin, we would have $\rho_0 = 0$, $B_n = -n' + \frac{1}{2}$. However, S_0 and B_n both have to be independent of K. This can only occur if

(i) $\rho_0 = 2\sqrt{-E}\ S_0 \propto 1/K$

which is satisfied if the energy levels appropriate to the problem are given by

(ii) $$\sqrt{-E} = \frac{\alpha}{K}\mu \quad , \quad f_1 = \mu . \tag{92}$$

The surface cutoff

$$S_0 = \frac{1}{2\alpha}\frac{(2n' - 1)(B_n + n' - ½)}{(B_n + n' - \frac{3}{2})} \tag{93}$$

can be chosen as small as desired, provided the actual limit is never taken to the origin. The normalizations are found from the relation

$$\int_{S_0}^{\infty} W_K W_{K'} \frac{dS}{S} = (2n'^2 - 3n' + 2)(\rho_0)^{1-2n'} \delta_{KK'} \tag{94}$$

and hence

$$\Psi_{S,K,n} = S^{-\frac{3}{2}} (2n^2 - 3n + 2)^{-½} (\rho_0)^{n-½} W_{K,n'}(\rho) . \tag{95}$$

The energy levels are found to be

$$E = -\frac{\alpha^2\mu^2}{K^2} = -\mu^2\left[\frac{\alpha^2}{(N+n)^2} + \frac{\alpha^4 f_2^2}{(N+n)^3 n} + O(\alpha^6)\right] . \tag{96}$$

When one mass becomes very large, we find for $m_2 >> m_1$

$$E_1 = m_1\left(1 - \frac{\alpha^2}{2(N+n)^2} - \frac{\alpha^4}{2(n+N)^3}\cdot\frac{f_2^2}{n} + O(\alpha^6)\right) . \tag{97}$$

Two points of importance arise

(i) The term proportional to $\frac{3}{8}\alpha^4$ is missing.

(ii) The quantum number n has replaced the $(\ell + \frac{1}{2})$ in the normal case of one light boson in a central field.

6.2 Linear Harmonic Oscillator

The force

$$F = -\sum_\nu K_\nu X_\nu \tag{98}$$

can be represented by the potential

$$\mho = \sum_\nu \frac{1}{2} K_\nu X_\nu^2 \tag{99}$$

with a wave equation

$$\left[-\Box^2/2\mu + \frac{1}{2}\sum_\nu K_\nu X_\nu^2\right]\Psi = \mathcal{E}\,\Psi \tag{100}$$

which separates to give four equations

$$\left[\frac{\partial^2}{\partial X_\nu^2} + \frac{1}{2} K_\nu X_\nu^2\right]\Psi_\nu = \mathcal{E}_\nu\ \Psi_\nu\ ,\quad \nu = 1, 2, 3, 4,$$

where $\mathcal{E} = \sum_\nu \mathcal{E}_\nu \ ,\quad \Psi = \prod_\nu \Psi_\nu$. (101)

The K_ν must transform as tensors. Putting

$$\xi_\nu = \alpha_\nu X_\nu,\ \alpha_\nu^4 = \frac{\mu K_\nu}{2}\ ,\ \sigma_\nu = 2\frac{\mathcal{E}_\nu}{\hbar}\sqrt{\frac{\mu}{K_\nu}} = 2\frac{\mathcal{E}_\nu}{\hbar\omega_\nu}$$

we obtain

$$\frac{d^2\Psi_\nu}{d\xi_\nu^2} + (\sigma_\nu - \xi_\nu^2)\,\Psi_\nu = 0\ . \tag{102}$$

The standard solutions are the Hermite polynomials such that

$$\Psi_\nu\,(X_\nu) = N_{n\nu} H_n(\alpha_\nu X_\nu)\,\exp(-\tfrac{1}{2}\alpha_\nu^2 X_\nu^2)\ , \tag{103}$$

$$\sigma_\nu = 2n + 1\ ,\quad \mathcal{E}_{n\nu} = (n + \tfrac{1}{2})\hbar\omega_\nu\ ,\quad n = 0, 1, 2, \ldots$$

$$N_{n\nu} = \sqrt{\frac{\alpha_\nu}{\pi\, 2^n\, n!}}\ .$$

There are therefore zero point energies corresponding to all modes of vibration along the four axes. The time-like vibrations have not been observed and it is not known how they would manifest themselves. These functions form an orthonormal set over the physical volume element.

6.3 Inverse Cube Law of Force

When Goldstein (1953) formulated the Bethe-Salpeter (1951) wave functions for two spinors interacting via the ladder exchange of neutral bosons, he obtained the radial wave equation from the quantum field theory of the form

$$\left(\frac{d^2}{dR^2} + \frac{3}{R}\frac{d}{dR} + 1 + \frac{4\eta}{R^2}\right)\Psi_S = 0 \tag{104}$$

in the case of equal masses. This is a special example of the radial wave equation (4(i)) for a hypercentral inverse cube law of 4-force. Putting $\mho = -\frac{K}{S^2}$ in equation (4) we find

$$\left(\frac{d^2}{dS^2} + \frac{3}{S}\frac{d}{dS} + Q^2 - \frac{(\Lambda^2 + K)}{S^2}\right)\Psi_S = 0 \tag{105}$$

which is identical to Goldstein's equation provided $R = QS$, $4\eta = \Lambda^2 + K = n^2 + K - 1$. In the non-relativistic limit, this force gives the solutions appropriate to the $1/r^2$ potential. It is therefore not surprising that he found no bound states, as no non-relativistic bound states exist (Morse and Feshbach 1953). Goldstein's solutions

$$\Psi_S = \frac{1}{R} H_{n'}^{(2)}(R) \quad ; \quad n' = (1 - 4\eta)^{1/2} \tag{106}$$

are obviously those for the inverse cube law of 4-force in the relative time theory. No bound states exist here as well for $0 \leqslant S \leqslant \infty$.

6.4 Square Well Potential

With an interaction in equation (4) of

$$\begin{aligned} \mathcal{V} &= -\mathcal{V}_0 \ , \quad S < a \ , \quad (\mathcal{V}_0 \ , \ a \ , \text{constants}) \\ \mathcal{V} &= 0 \ , \quad S > a \end{aligned} \tag{107}$$

we have the covariant analogue of the square well potential. The solutions are

$$\begin{aligned} \Psi_S &= A \frac{J_n(\alpha S)}{\alpha S} \ , \quad S < a \\ &= B \frac{H_n^{(1)}(\beta S)}{\beta S} \ , \quad S > a \end{aligned} \tag{108}$$

where $\alpha = [2\mu(\mathcal{V}_0 - \mathcal{E})]^{1/2}$

$\beta = (2\mu\mathcal{E})^{1/2}$.

In the bound state region $\mathcal{E} < 0$, hence

$$\Psi_S = iB \frac{H_n^{(1)}(i\,|\beta|S)}{|\beta|S} = \frac{iB}{|\beta|S} K_n(|\beta|S) \tag{109}$$

is the solution. These functions tend to zero as S tends to infinity. By choosing boundary conditions on each solution such that

$$a \left[\frac{\partial \Psi_S}{\partial S} \Big/ \Psi_S \right]_{S=a} = -(n+1) \tag{110}$$

we also ensure that the internal eigenfunctions form an orthonormal set in the interval $0 \leqslant S \leqslant a$. The energies of the bound states are obtained by noting that the boundary condition (110) is equivalent to the conditions

$$\rho J_n'(\rho) + n J_n(\rho) = \rho J_{n-1}(\rho) = 0 \tag{111}$$

when $\rho = \rho_\nu$ at the zeroes of $J_{n-1}(\rho)$. Hence the energy levels are

$$\mathcal{E}_\nu = \rho_\nu / a \quad . \tag{112}$$

6.5 Two-Fermion Atom

There are many possible models of the two-fermion atom, and the one derived here is chosen mainly for its relative simplicity. The main problem is in finding how to linearise equation (2). We choose the form

$$[\ \mathcal{E} - \mathcal{V} + \rho_1\ \mathfrak{M} + \rho_3\ \gamma \cdot Q\ ]\ \Psi_8 = 0 \tag{113}$$

where Ψ_8 is an eight-component spinor, four components appropriate to one ordinary spinor and four to the other.

$$\mathcal{E} = \frac{\beta}{\sqrt{1-\beta^2}}\ \left(\frac{\mu}{M}\ (W^2 - M^2)\right)^{1/2} ,$$

$$\mathfrak{M} = \frac{1}{\sqrt{1-\beta^2}}\ \left(\frac{\mu}{M}\ (W^2 - M^2)\right)^{1/2} ,$$

$$Q = Q_s ,$$

$$\beta = 1/\sqrt{1 + |V|^2/c^2} ,$$

$$= 1 + \frac{W^2 - M^2}{2M\mu} ,$$

$$V = \text{relative velocity} . \tag{114}$$

The Hamiltonian for the relative motion is

$$\mathcal{H} = \rho_3\ \gamma_\mu\ Q^\mu - \rho_1\ \mathfrak{M} + \mathcal{V}$$

$$= \rho_3\ \gamma_S\ Q_S - \rho_3\ i\ \frac{\gamma_S}{S}\ \rho_3\ K + \mathcal{V} - \rho_1\ \mathfrak{M} \tag{115}$$

in which

$$K = \rho_3 \left(\frac{1}{2}\ \sigma_{\mu\nu}\ \Lambda^{\mu\nu} + \frac{3}{2}\right)$$

$$\gamma_S = \frac{\gamma_\nu\ X^\nu}{S} \tag{116}$$

where we have (Corinaldesi and Strocchi 1963)

$$\gamma_\mu\ \gamma_\nu + \gamma_\nu\ \gamma_\mu = 2\ \delta_{\mu\nu}$$

$$\sigma_{\mu\nu} = \frac{1}{2i} \cdot (\gamma_\mu\ \gamma_\nu - \gamma_\nu\ \gamma_\mu) \tag{117}$$

and the γ_μ's are the usual Dirac matrices. From (117) we can show that

$$\gamma_\mu Q^\mu = \frac{\gamma_\nu X^\nu}{S^2} \, \underline{S} \cdot \underline{Q} + \frac{1}{S^2} \frac{i}{2} \cdot \gamma_\nu \, X^\nu \sigma_{\mu\nu} \Lambda^{\mu\nu} \tag{118}$$

and using

$$Q_S = \frac{1}{S}\left(\underline{S} \cdot \underline{Q} - \frac{3}{2} \, i\right) = i \left(\frac{\partial}{\partial S} + \frac{3}{2S}\right) \tag{119}$$

we can readily prove that the following relations hold:

$$\gamma_S^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = 1 \ , \quad \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} = 12$$

$$\rho_1 \rho_2 \gamma_S + \rho_2 \gamma_S \rho_1 = 0 \ ,$$

$$K \gamma_S - \gamma_S K = 0 \ , \quad K \rho_2 - \rho_2 K = 0 \ ,$$

$$[\mathcal{H}, K] = 0 \ , \quad [\mathcal{H}, K^2] = 0$$

where we choose $\rho_3 = 1 \ , \ \rho_1 = \rho_2 = \gamma_5 \ .$ (120)

The analogue to the total angular momentum is

$$J_{\mu\nu} = \Lambda_{\mu\nu} + \frac{1}{2} \sigma_{\mu\nu} \tag{121}$$

where $\frac{1}{2} \sigma_{\mu\nu}$ is the spin tensor and

$$J^2 = \frac{1}{2} J_{\mu\nu} J^{\mu\nu} = \frac{1}{2} \Lambda_{\mu\nu} \Lambda^{\mu\nu} + \frac{1}{2} \sigma_{\mu\nu} \Lambda^{\mu\nu} + \frac{1}{8} \sigma_{\mu\nu} \sigma^{\mu\nu} = K^2 - \frac{3}{4} \ . \tag{122}$$

We choose $\gamma_S = \gamma_1$. Putting

$$\Psi_S = \begin{pmatrix} \Psi_4 \\ \Psi_4 \end{pmatrix} , \quad \Psi_4 = \frac{1}{S} \begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix} \tag{123}$$

we obtain the following set of differential equations

$$- i S^{\frac{3}{2}} \, Q_S \left(\frac{G_2}{S^{\frac{3}{2}}}\right) - \frac{K}{S} F_2 + (\mathcal{E} + \mathfrak{M} - \mathcal{V}) F_1 = 0$$

$$- i S^{\frac{3}{2}} \, Q_S \left(\frac{G_1}{S^{\frac{3}{2}}}\right) - \frac{K}{S} F_1 + (\mathcal{E} - \mathfrak{M} - \mathcal{V}) F_2 = 0$$

$$i S^{\frac{3}{2}} \, Q_S \left(\frac{F_2}{S^{\frac{3}{2}}}\right) + \frac{K}{S} G_2 + (\mathcal{E} + \mathfrak{M} - \mathcal{V}) G_1 = 0$$

$$i S^{\frac{3}{2}} \, Q_S \left(\frac{F_1}{S^{\frac{3}{2}}}\right) + \frac{K}{S} G_1 + (\mathcal{E} - \mathfrak{M} - \mathcal{V}) G_2 = 0 \ . \tag{124}$$

With the aid of (119) and assuming that, as in non-relativistic theory, K is integer, the equations

$$G_2 = F_2 \quad , \quad G_1 = F_1 \quad ,$$

$$(\mathcal{E} + \mathfrak{M} - \mathcal{V})\, F_1 - \frac{dF_2}{dS} - \frac{K}{S}\, F_2 = 0$$

$$(\mathcal{E} - \mathfrak{M} - \mathcal{V})\, F_2 + \frac{dF_1}{dS} - \frac{K}{S}\, F_1 = 0 \tag{125}$$

are obtained. These are identical to Dirac's (1958) radial equations, and he has shown that they have solutions for bound states provided

$$\mathcal{V} = -e^2/S \quad ,$$

$$\mathcal{E} = \frac{\mathfrak{M}\, c^2}{\left[1 + \dfrac{\alpha^2}{(P + n')^2}\right]^{1/2}} \quad , \quad P = (K^2 - \alpha^2)^{1/2} \quad , \quad n' = \text{integer} \quad . \tag{126}$$

When one mass becomes very large, it can be seen from equation (114) that

$$\mathcal{E} \to E_1 \text{ and } \mathfrak{M} \to m_1 \text{ as } m_2 \to \infty \tag{127}$$

and we are left with Dirac's formula for the fine structure of the hydrogen atom. The relative time theory therefore works very well in this case.

There is one important feature of this model. Using equation (122) and (116) we find

$$K^2 - \frac{3}{4} = \lambda\,(\lambda + 2) + \rho_2\, K \quad . \tag{128}$$

In the non-relativistic approximation, we replace ρ_2 by 1 and obtain

$$\lambda = K - \frac{1}{2} \quad , \quad -K - \frac{3}{2} \tag{129}$$

and λ must be $\frac{1}{2}$ - integer if we are to obtain the correct fine structure. This has the effect of making the radial wave equations the same as in the non-relativistic theory, but the angular eigenfunctions become for example,

$$\mathcal{Y}_{n\ell m} = c\,\sqrt{1 - t^2}\;\, \mathcal{P}_\ell^{K - 1/2}(t)\; \mathcal{Y}_{\ell m}\,(\theta, \phi) \quad . \tag{130}$$

The boson theory given in previous sections does not apply to spinors and the mathematical theorems such as the covariant addition theorem, and the equations for symmetry conversion for these eigenfunctions need to be investigated.

Finally we shall indicate how the energy levels in the relative time theory are related to the spectrum of photons obtained from decaying discrete states. Let $(W_a, \underset{\sim}{0})$ be the initial 4-momentum in the C.M. frame of a two-body system where the W_a are discrete mass levels. Let this system decay with emission of a photon of 4-momentum $(\nu, \underset{\sim}{\nu})$ leaving the system in a state of 4-momentum $(W'_b, -\underset{\sim}{\nu})$. By conservation of energy

$$W_a = W'_b + \nu \quad ,$$

$$W_b^2 = W_b'^2 - \nu^2$$

where W_b is the C.M. energy in the rest frame of the recoiling system.

Therefore

$$(W_a - \nu)^2 - \nu^2 = W_b^2 ,$$

$$\nu = \frac{W_a^2 - W_b^2}{2 W_a} = \frac{M}{\mu} \frac{(Q_a^2 - Q_b^2)}{2\sqrt{Q_a^2 + M^2}} \tag{131}$$

where Q_a and Q_b are the relative 4-momenta in initial and final states respectively. For example, in the Coulomb field case the levels given by (126) become

$$\nu = \frac{M}{\mu} \frac{(Q_a^2 - Q_b^2)}{2\sqrt{Q_a^2 + M^2}} = \frac{M}{\sqrt{M^2 + 2\mu(\gamma_a - 1)}} \cdot [\gamma_a - \gamma_b]$$

where $$\gamma_a = \frac{1}{\left[1 + \frac{\alpha^2}{(p_a + n_a')^2}\right]^{1/2}} , \quad \gamma_b = \frac{1}{\left[1 + \frac{\alpha^2}{(p_b + n_b')^2}\right]^{1/2}} . \tag{132}$$

(p_a, n_a'), (p_b, n_b') are the covariant quantum numbers appropriate to states (a) and (b). Compared with other effects the correction to the levels obtained from the Dirac equation are too small to be tested.

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APPENDIX

It is of some interest to show how the partial waves of covariant angular momentum are related to the partial waves of ordinary angular momentum. In order that the wave function Ψ in equation (50) should be converted by the Fourier transform to the equivalent wave function ψ in a particular Lorentz frame, we must have

$$\psi = \sum_{\ell} O_{\ell} \left(\mathcal{I}_{\ell} + S_{\ell} \mathcal{O}_{\ell} \right) \quad ,$$

where

$$\mathcal{I}_{\ell} = \frac{h^{(2)}_{\ell+\frac{1}{2}}(qR)}{\sqrt{qR}} \mathcal{P}_{\ell}(z), \quad \mathcal{O}_{\ell} = \frac{h^{(1)}_{\ell+\frac{1}{2}}(qR)}{\sqrt{qR}} \cdot \mathcal{P}_{\ell}(z)$$

$$S_{\ell} = \sum_{n=0}^{\ell} h_n \frac{(\ell-n)!}{(\ell+n)!} \left[\mathcal{P}_{\ell}^{n}(t_2) \right]^2 S_n \quad .$$

Therefore from equations (32) and (46),

$$\Sigma_{SC} = \frac{1}{Q^3} \left| \sum_{n=-\infty}^{\infty} (1 - S_n) \cdot \frac{\cos(n\omega)}{\sin(\omega)} \right|^2$$

$$= \frac{\pi^2}{4Q^3} \left| \sum_{\ell=0}^{\infty} (2\ell+1) \sum_{n=0}^{\ell} (1 - S_n)\, h_n \frac{(\ell-n)!}{(\ell+n)!} \left[\mathcal{P}_{\ell}^{n}(t_1) \right]^2 (1 - t_1^2)\, \mathcal{P}_{\ell}(z) \right|^2$$

$$= \frac{\pi^2}{4Q} \left(1 - \frac{\epsilon^2}{q^2} \right) \sigma_{SC}(q,z) \quad ,$$

where $\sigma_{SC} = \left| \sum_{\ell=0}^{\infty} (2\ell+1)(1 - S_{\ell}) \mathcal{P}_{\ell}(z) \right|^2$ is the conventional differential cross section.