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ABSTRACT

The statistical distribution of a product of variables which have a Gaussian distribution is investigated. These distributions are found to be given, in general, by special functions. Expansions for these functions for small values of the variable and their asymptotic behaviour are derived. The functions are tabulated for products of up to seven variates. Some simple integrals related to the functions are given.

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BOUNDARY CONDITIONS; DIFFERENTIAL EQUATIONS; DISTRIBUTION FUNCTIONS; GAMMA FUNCTION; GAUSSIAN PROCESSES; INTEGRALS; MELLIN TRANSFORM

CONTENTS

1. INTRODUCTION

In many physical theories of random motion and in the theory of error distributions in experimental measurements, one encounters a class of functions which represents those statistical distribution functions obtained for the product of a fixed number of variates, each of which has a normal distribution with zero mean. These functions do not appear to have been the subject of general investigation in the literature, though they form a special case of the Meijer (1936) G-function.

Expansions valid in the vicinity of the origin and asymptotic expansions for large values of the variable are derived and shown to be adequate for computing the distribution of products of up to seven variates. An integral relation between the functions is given and permits an alternative means of computing the functions. The usefulness of these functions is shown by expressing a number of hitherto unevaluated integrals in terms of these functions. The way in which these functions arise in the theory or error distributions is demonstrated. Finally, the functions and the coefficients of the series expansions are tabulated.

2. DERIVATION OF FUNCTION

Consider a set of n characteristic probability distribution functions for a set of n independent variables, x_i, of the form

$$P_i\ (x_i) = \sqrt{\frac{\alpha_i}{\pi}}\ e^{-\alpha_i\ x_i^2} \quad -\infty < x_i < \infty \qquad \ldots(1)$$

which represents a normal distribution of zero mean and variance $1/2\alpha_i$. It is readily shown, for the purpose of simplifying the following arguments, that results pertaining to Equation (1) can be derived from the distribution

$$P_i\ (x_i) = 2\sqrt{\frac{\alpha_i}{\pi}}\ e^{-\alpha_i\ x_i^2} \quad 0 \leqslant x_i \leqslant \infty\ . \qquad \ldots(2)$$

Let u be defined by

$$u = \prod_{i=1}^{n} x_i \qquad \ldots(3)$$

We require the characteristic probability distribution function for u.

Let

$$\text{(i)}\quad w = \log u\ ; \qquad \text{(ii)}\quad y_i = \log x_i \qquad \ldots(4)$$

so

$$w = \sum_{i=1}^{n} y_i \quad . \qquad \ldots(5)$$

A well known theorem of statistics (Cramer 1945) states that the characteristic function of a sum of independent variables is equal to the product of the characteristic functions of the terms. Removing all degrees of freedom except u from the joint distribution obtained by applying this theorem to Equations (5) and (2), we obtain, after integrating over all degrees of freedom except w, the joint distribution function

$$P_n\,(w) = \left(\frac{2}{\sqrt{\pi}}\right)^n \left(\prod_{i=1}^{n} \alpha_i\right)^{½} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dy_1 \ldots dy_{n-1} \quad X$$

$$X \exp\left\{ -\sum_{i=1}^{n-1} \alpha_i\, e^{2y_i} - \alpha_n\, e^{2(w - \sum_{i=1}^{n-1} y_i)} \right\} , \qquad \ldots(6)$$

or in terms of the original variables

$$P_n(u) = \left(\frac{2}{\sqrt{\pi}}\right)^n \left(\prod_{i=1}^{n} \alpha_i\right)^{½} \int_{0}^{\infty} \cdots \int_{0}^{\infty} dx_1 \ldots dx_{n-1} \quad X$$

$$X \left(\prod_{i=1}^{n-1} \alpha_i x_i\right)^{-1} \exp\left\{ -\sum_{i=1}^{n-1} \alpha_i x_i^2 - \frac{\alpha_n\, u^2}{\prod_{i=1}^{n-1} x_i^2} \right\} . \qquad \ldots(7)$$

One can see, from a change of variables in Equation (7), that no generality is lost by putting all α_i equal to unity. We therefore define the function $C_n(x)$ by the multiple integral

$$C_n(u) = \left(\frac{2}{\sqrt{\pi}}\right)^n \int_{0}^{\infty} \cdots \int_{0}^{\infty} dx_1 \ldots dx_{n-1} X \left(\prod_{i=1}^{n-1} x_i\right)^{-1} \quad X$$

$$X \quad \exp\left\{ -\sum_{i=1}^{n-1} x_i^2 - u^2 / \prod_{i=1}^{n-1} x_i^2 \right\} . \qquad \ldots(8)$$

If Equation (8) is multiplied on both sides by the function

$$\frac{2}{\sqrt{\pi}\, u} \exp\left(-v^2/u^2\right)$$

and the equation integrated over the range $0 \leqslant u \leqslant \infty$, then with $u = x_n$ one finds by comparing with Equation (8) that the result is $C_{n+1}(v)$, *i.e.*

$$C_{n+1}(v) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{du}{u}\, e^{-v^2/u^2}\, C_n(u) \quad . \qquad \ldots(9)$$

Equation (9) is an integral recurrence relation for the functions.

3. DIFFERENTIAL EQUATION

From Equation (9) we obtain

$$\left(x \frac{d}{dx}\right)^m C_{n+1}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dy}{y}\, e^{-x^2/y^2} \left(y \frac{d}{dy}\right)^m C_n(y)$$

$$m = 0,\ 1,\ 2 \ldots\ldots \qquad \ldots(10)$$

so that any linear superposition of the functions $(x \frac{d}{dx})^m\, C_n(x)$ obeys Equation (9). Putting $m = n + 1$, we find

$$\left(x \frac{d}{dx}\right)^{n+1} C_{n+1}(x) = -2x^2 \int_0^\infty dy\, e^{-x^2/y^2}\, y^{-3} \left(y \frac{d}{dy}\right)^n C_n(y), \qquad \ldots(11)$$

after integrating by parts. It can now be proved by induction that $C_n(x)$ satisfies the n^{th} order differential equation

$$\left(x \frac{d}{dx}\right)^n C_n(x) - (-2)^n\, x^2\, C_n(x) = 0 \quad . \qquad \ldots(12)$$

Suppose we put

$$\left(y \frac{d}{dy}\right)^n C_n(y) = a_n\, y^2\, C_n(y) \qquad \ldots(13)$$

in the integrand of Equation (11). This immediately yields from Equation (9)

$$\left(x \frac{d}{dx}\right)^{n+1} C_{n+1}(x) = -2a_n\, x^2\, C_{n+1}(x) \quad . \qquad \ldots(14)$$

But for $n = 1$, $C_1 = \frac{2}{\sqrt{\pi}} e^{-x^2}$,

$$x \frac{d}{dx}\, C_1(x) = -2x^2\, C_1(x)$$

and therefore $a_1 = -2$. Comparing Equation (13) with Equation (14) gives $a_{n+1} = -2a_n$, which yields $a_n = (-2)^n$, thus proving that Equation (12) is satisfied for all n.

The Meijer G-function (Erdelyi *et al.* 1953a) satisfies the differential equation

$$\left[(-1)^{p-m-n}\, z \prod_{j=1}^{p}\left(z\frac{d}{dz} - a_j + 1\right) - \prod_{j=1}^{q}\left(z\frac{d}{dz} - b_j\right)\right] y = 0 \qquad \ldots(15)$$

where $y = G^{mn}_{pq}\left(z\middle|\begin{matrix}a_r\\ b_r\end{matrix}\right)$.

Substituting $x^2 = z$ in Equation (12) we find that the G-function

$$y = G^{m(N-m)}_{ON}\,(x^2\,|\,0,\ 0,),\quad 0 \leqslant m \leqslant N, \qquad \ldots(16)$$

is a solution where, to avoid confusion with standard notation, we use N to indicate the order of Equation (12). It will be shown from the integral representation of the G-function that Equation (16) is also the solution given by $C_N(x)$. Since (Erdelyi *et al.* 1953a) $p<q$ in the above case, the only singularities are $x = 0, \infty$.

The solution to Equation (12), which is analytic at the origin, is found, from the standard Frobenius method, to be

$$B_n(x) = z_n \sum_{m=0}^{\infty} \frac{(-1)^{mn}\, x^{2m}}{[\Gamma(m+1)]^n} \qquad \ldots(17)$$

where z_n is a normalisation constant to be determined.

If instead of Equation (12) we consider the equation

$$\left(x\frac{d}{dx}\right)^n y - \left((2\nu)^n + (-2)^n x^2\right) y = 0 \quad , \qquad \ldots(18)$$

then the behaviour of the solution near the origin is given by

$$B_{n,\nu}(x) = z_n \sum_{m=0}^{\infty} (-1)^{mn} \frac{x^{2m+2\nu}}{[\Gamma(m+\nu+1)]^n} \quad . \qquad \ldots(19)$$

From the solutions Equation (19), it will be shown that expansions for $C_n(x)$ applicable near the origin can be generated.

4. MELLIN TRANSFORM

It is customary in statistics (Cramer 1945) to define a moment generating function as the one-sided Laplace transform of the probability distribution. In the case of the distributions (Cramer 1945), this is not convenient as some theorems pertaining to the moment generating functions for the distributions of sums of random variables do not hold. Instead we use the Mellin transform (Erdelyi *et al.* 1954).

$$g_n(s) = \int_0^\infty C_n(x)\, x^{s-1}\, dx \qquad \ldots(20)$$

as the definition of a moment generating function. This is essentially the two-sided Laplace transform of the distribution of w in Equation (6). It follows from Equations (3), (4) and (5) that in evaluating Equation (20)

$$g_n(s) = \left(\frac{2}{\sqrt{\pi}}\right)^n \int_{-\infty}^\infty dy\; e^{sy} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty dy_1 \ldots dy_{n-1} \quad \times$$

$$\times \quad \exp\left\{ -\sum_{i=1}^{n-1} e^{2y_i} - e^{2\left(y - \sum_{i=1}^{n-1} y_i\right)} \right\} . \qquad \ldots(21)$$

By changing the variable from y to $y_n = y - \sum_{i=1}^{n-1} y_i$, we find

$$g_n(s) = \left(\frac{2}{\sqrt{\pi}}\right)^n \left[\int_{-\infty}^\infty dy_1\; e^{sy_1} \exp\left\{ - e^{2y_1} \right\} \right]^n = [g_1(s)]^n . \qquad \ldots(22)$$

Therefore, the moment generating function for the joint distribution of n variables is the n^{th} power of the moment generating function for each distribution. The C-function is obtained by inverting the Mellin transform (Erdelyi *et al.* 1954) using Equation (22).

$$C_n(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g_n(s)\, x^{-s}\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} [g_1(s)]^n\, x^{-s}\, ds . \qquad \ldots(23)$$

The definition Equation (2) gives the transform

$$g_1(s) = (\sqrt{\pi}\,)^{-1}\, \Gamma(s/2) \qquad \ldots(24)$$

and accordingly

$$C_n(x) = \pi^{\frac{-n}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\Gamma(\tfrac{s}{2})\right]^n x^{-s} \, ds \quad . \qquad \ldots(25)$$

The gamma function in the integrand contains poles along the negative real axis at zero and negative even integers. The contour is chosen so that the line from c-i∞ to c+i∞ is to the right of all singularities and is, therefore, in the right half-plane where x^{-s} is analytic everywhere. The simplest contour is a semicircle with the straight side given by the above line, which curves almost to the negative real axis, then loops back around the origin to exclude all of the poles. In the limit that the semicircle radius tends to infinity, and the loop sides tend to the negative axis, the only non-zero contribution to Equation (25) comes from the residues at the poles. The integral has the form

$$I = \frac{1}{2\pi i} \int \frac{f(z)}{F^n(z)} \, dz = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dF^{n-1}} \left(\frac{f(z)}{\frac{dF}{dz}} \right)\right]_{F=0} \quad , \qquad \ldots(26)$$

where the zeros in F correspond to the poles in the gamma function. Using the relation (Erdelyi *et al.* 1953b)

$$\Gamma(z) \quad \Gamma(1-z) \quad = \quad \pi \text{ cosec } (\pi z) \qquad \ldots(27)$$

and letting the poles be at $\frac{s}{2} = -m$, we evaluate the residues by substituting

$$\frac{s}{2} = -m-\nu \; ; \quad h = - \Gamma(1 - {}^{s}/2)\cdot\sin\left(\frac{\pi s}{2}\right) = \Gamma(m+\nu+1)\cdot\sin((m+\nu)\,\pi) \quad . \qquad \ldots(28)$$

This gives, from Equations (26) and (25), $\Gamma({}^{s}/2) = -\pi/h$,

$$C_n(x) = \frac{\pi^{\frac{-n}{2}}}{(n-1)!} \sum_{m=0}^{\infty} \left\{ \frac{d^{n-1}}{dh^{n-1}} \left(\frac{x^{2m+2\nu}}{\frac{dh}{ds}} \right) \right\}_{h=0} (-\pi)^n$$

$$= \frac{2(-1)^{n-1} \pi^{\frac{n}{2}}}{(n-1)!} \left[\sum_{m=0}^{\infty} \sum_{p=0}^{n-1} \binom{n-1}{p} D^{(p)}\left(\frac{1}{\frac{dh}{d\nu}} \right) D^{(n-p-1)} (x^{2m+2\nu}) \right] ,$$

$$D^{(n)} = \left(\frac{1}{\frac{dh}{d\nu}} \cdot \frac{d}{d\nu} \right)^n \quad . \qquad \ldots(29)$$

Using the expansion (29) we find

$$C_1(x) = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{m!} = \frac{2}{\sqrt{\pi}} e^{-x^2} \quad , \qquad \ldots(30)$$

$$C_2(x) = \frac{-4}{\pi} \ln x \sum_{m=0}^{\infty} \frac{x^{2m}}{(m!)^2} + \frac{4}{\pi} \sum_{m=0}^{\infty} \psi(m+1) \frac{x^{2m}}{(m!)^2} \quad ,$$

$$= \frac{4}{\pi} K_o (2x) \qquad \ldots(31)$$

in which

$$\psi(m+1) = \frac{d}{dm} (\ln(\Gamma(m+1))) = \sum_{p=1}^{m} \left(\frac{1}{p}\right) - \gamma \quad ,$$

and γ is Euler's constant.

This is the well-known logarithmic derivative of the gamma function, and K_o is the associated Bessel function. The higher order functions cannot be expressed in terms of known functions. For example

$$C_3(x) = \pi\sqrt{\pi} \left\{ \frac{4}{\pi^3} (\ln x)^2 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(m!)^3} - \right.$$

$$- \frac{12}{\pi^3} \ln x \sum_{m=0}^{\infty} (-1)^m \frac{\psi(m+1)}{(m!)^3} x^{2m} +$$

$$\left. + \sum_{m=0}^{\infty} (-1)^m \left(\frac{9\psi^2 - 3\psi'}{\pi^3} + \frac{1}{\pi} \right) \frac{x^{2m}}{(m!)^3} \right\} \quad , \qquad \ldots(32)$$

$$C_4(x) = - \frac{1}{3\pi^2} \left\{ \sum_{m=0}^{\infty} \left[8(\ln x)^3 - 48 (\ln x)^2 \psi + \right. \right.$$

$$+ \ln x (96\psi^2 - 24\psi' + 8 \pi^2) + (-64\psi^3 + 48\psi\psi' -$$

$$\left. \left. - 4\psi'' - 16 \pi^2\psi) \right] \frac{x^{2m}}{(m!)^4} \right\} \quad , \qquad \ldots(33)$$

where

$$\psi^{(n)} = \left[\frac{d}{dz} \quad \psi(z+1)\right]_{z=n} .$$

The formulae became unwieldy for n ⟩ 4 and, therefore, we sought a convenient means of extending the representation of the function in terms of the regular solution (17). We let

$$z_n = -2(-1)^n \pi^{-n/2} \quad . \qquad \text{...(34)}$$

After some tedious manipulation and comparison we find that the functions can be evaluated from the operations,

$$C_n(x) = \lim_{\nu\to 0} \sum_{p=0}^{n-1} a_p \frac{d^p}{d\nu^p} B_{n,\nu}(x) , \qquad \text{...(35)}$$

where the a_p are to be determined. One can show by evaluating the expansion (29) and comparing with the form (35) that in operational notation

$$\begin{aligned}
C_1(x) &= \delta^0 B_1(x) \\
C_2(x) &= \delta^1 B_2(x) \\
C_3(x) &= (\delta^2 + \pi^2\delta^0) \; B_3(x) \\
C_4(x) &= (\delta^3 + 4\pi^2\delta^1) \; B_4(x) \\
C_5(x) &= (\delta^4 + 10\pi^2\delta^2 + 9\pi^4\delta^0) \; B_5(x) \\
C_6(x) &= (\delta^5 + 20\pi^2\delta^3 + 64\pi^4\delta^1) \; B_6(x) \\
C_7(x) &= (\delta^6 + 35\pi^2\delta^4 + 259\pi^4\delta^2 + 225\pi^6\delta^0) \; B_7(x) \\
C_8(x) &= (\delta^7 + 56\pi^2\delta^5 + 784\pi^4\delta^3 + 2304\pi^6\delta^1) \; B_8(x) \qquad \text{...(36)}
\end{aligned}$$

where $\delta^p \equiv \lim_{\nu\to 0} \frac{d^p}{d\nu^p}$, $B_z = B_{n+\nu}$.

It was not possible to find the general coefficient a_p for one can see, from Equation (16), that these are related to the coefficients of the derivatives obtained in inverting an ordinary Taylor expansion. A general expression for the latter has not been derived. The convergence of these series from the point of view of evaluation improves as n increases, owing to the increasing powers of factorials in the denominators.

To obtain the general form for the series expansion we note from Equation (29) that the formal operations lead to

$$C_n(x) = (-1)^{n+1} \frac{2\pi^{n/2}}{(n-1)!} \sum_{m=0}^{\infty} x^{2m} \sum_{j=1}^{n} f_m^{(j)} (2\ell nx)^{n-j}$$

where

$$f_m^{(j)} = \sum_{q=1}^{j} w_{j+1-q}\, p^{q-1} \binom{n+q-j-1}{q-1} . \qquad \ldots(37)$$

w_j is the coefficient in the expansion of

$$\left[1/\frac{dh}{d\nu}\right]^{n-1} = \sum_{j=1}^{n-1} w_j\, D^{n-j}$$

where

$$D^r f = \sum_{q=0}^{r} \binom{r}{q} p^{(q)}\, f^{(r-q)}$$

$$p = \frac{d}{d\nu}\left(1/\frac{dh}{d\nu}\right),\; p^{(q)} = \frac{d}{d\nu}\left(p^{q-1}\right) .$$

Tables of $f_m^{(j)}$ are given in Appendix A.

5. ASYMPTOTIC BEHAVIOUR

The asymptotic series of $C_n(x)$ for large x proved difficult to obtain from the integral relation (9). We indicate the asymptotic form for large x by noting from the differential Equation (12) that a permissible solution for large x is

$$C_n(x) = a_n\, x^{-\nu}\, e^{-\mu x^{\lambda}} \qquad \ldots(38)$$

which will become evident from the proof following. a_n, ν, μ and λ are constants to be determined. Taking the integral relation (9) and substituting

$$w = x^{p-1}\, u,$$

$$p = \lambda/(\lambda + 2) \qquad \ldots(39)$$

we obtain a form suitable for using the saddle-point method (Copson 1946)

$$C_{n+1}(x) = \frac{2}{\sqrt{\pi}} \; a_n \; x^{-2\nu/(\lambda+2)} \quad \times$$

$$\times \int_0^{\infty} \exp\left\{- x^{2p} \; (1/w^2 + \mu \; w^{\lambda})\right\} w^{-\nu-1} \; dw \quad . \qquad \ldots(40)$$

The turning point in the exponent of the integrand occurs when

$$w = w_0 = (2/\lambda\mu)^{1/(\lambda+2)} \quad . \qquad \ldots(41)$$

Since there are two branches $0 \leqslant w \leqslant w_0$, and $w_0 \leqslant w \leqslant \infty$, we must multiply the integral by 2. Let

$$t = 1/w^2 + \mu \; w^{\lambda} - (1/w_0^2 + \mu \; w_0) \qquad \ldots(42)$$

$$w \approx w_0 \; (1 + \Delta) \quad , \qquad \ldots(43)$$

the second equation denoting that Δ is small near the saddle-point. Expanding t about the saddle point we find that in its vicinity

$$t \approx \Delta^2 \; (\lambda + 2)/w_0^2$$

$$\Delta \approx w_0 \; t^{1/2}/\sqrt{\lambda + 2} \qquad \ldots(44)$$

and, therefore, by the saddle-point method

$$C_{n+1}(x) \simeq \frac{2a_n}{\sqrt{\lambda + 2}}\left(\frac{2}{\lambda\mu}\right)^{\frac{1-\nu}{\lambda+2}} x^{-\frac{(\lambda+2\nu)}{\lambda+2}} \quad \times$$

$$\times \; \exp\left\{- x^{2p}\left[\left(\frac{2}{\lambda\mu}\right)^{-\frac{2}{\lambda+2}} + \mu\left(\frac{2}{\lambda\mu}\right)^{\frac{\lambda}{\lambda+2}}\right]\right\} . \qquad \ldots(45)$$

Making use of the known functions $C_1(x)$ and $C_2(x)$ and their asymptotic behaviour, we find by inspection that

$$C_n(x) \simeq \frac{2}{\sqrt{\pi n}}^{(n+1)/2} x^{(1-n)/n} \; e^{-nx^{2/n}} \quad ; \quad x \simeq \infty \qquad \ldots(46)$$

is the appropriate asymptotic limit, where

$$\lambda\mu = 2, \quad \text{for all } n,$$

$$\lambda = 2/n, \ \mu = n\ ,$$

$$\nu = (1-n)/n, \quad \text{and}$$

$$a_n = 2^{(n+1)/2} / \sqrt{\pi n} \ .$$

All of the functions $C_n(x)$ behave in this way.

The asymptotic behaviour in Equation (46) does not allow one to compute $C_n(x)$ to sufficiently small values of x that the series (37) becomes useful. To permit this we postulate an asymptotic form

$$C_n(x) \triangleq \frac{2^{\frac{n+1}{2}}}{\sqrt{n\pi}} x^{(1-n)/n} e^{-y} \sum_{r=0}^{\infty} \frac{b_r^{(n)}}{y^r} \ , \qquad \ldots(47)$$

where $y = n\, x^{2/n}$.

Changing the variable to y in the differential Equation (12) we get

$$\left\{\left(y \frac{d}{dy}\right)^n - (-1)^n y^n\right\} C_n(y) = 0 \ . \qquad \ldots(48)$$

Let

$$C_n(y) = y^{-k} e^{-y} S_n(y) \qquad \ldots(49)$$

$$S_n(y) = \sum_{r=0}^{\infty} \frac{b_r^{(n)}}{y^r} \ , \quad (b_0^{(n)} = 1, \quad k = \frac{n-1}{2}) \ . \qquad \ldots(50)$$

The sum in Equation (50) then has to satisfy the equation

$$\sum_{p=0}^{n} \binom{n}{p} \left[\left(y \frac{d}{dy}\right)^p \left(y^{-k} e^{-y}\right)\right]\left[\left(y \frac{d}{dy}\right)^{n-p} S_n(y)\right] -$$

$$- (-1)^n y^{n-k} e^{-y} S_n(y) = 0 \ , \qquad \ldots(51)$$

where $k = (n-1)/2$ as above.

This yields the form

$$\sum_{p=0}^{n} d_p \left[\left(y \frac{d}{dy}\right)^{n-p} S_n(y)\right] = 0 , \qquad \ldots(52)$$

where $d_0 = 1$

$$d_p = (-1)^p \binom{n}{p} \sum_{\ell=1}^{p+1} a_{p\ell} \; y^{\ell-1} \quad (n > p \geqslant 1)$$

and $d_n = (-1)^n \sum_{\ell=1}^{n} a_{n\ell} \, y^{\ell-1}$.

The values of $b_r^{(n)}$ may be obtained by the usual Frobenius method in which we equate coefficients of powers of y after substituting for $S_n(y)$ in Equation (52). We did not succeed in finding a general form for the a_n and these must first be evaluated by deriving the differential Equation (52) for each value of n.

This procedure yields the following series

$$S_2(y) = \sum_{k=0}^{\infty} \frac{1}{y^k} \cdot \frac{1}{k!} \cdot \frac{\Gamma(\tfrac{1}{2}+k)}{\Gamma(\tfrac{1}{2}-k)} , \quad y = 2x ; \qquad \ldots(53a)$$

$$S_3(y) = 1 - \frac{1}{3y} + \frac{2}{9y^2} - \frac{14}{81y^3} + \ldots. ,$$

$$b_r^{(3)} = -\frac{1}{3r}\left[\left\{(r-2)(r^2-r+1) + 1\right\} b_{r-2}^{(3)} + \left\{3r(r-1) + 1\right\} b_{r-1}^{(3)}\right] ,$$

$$y = 3x^{2/3} ; \qquad \ldots(53b)$$

$$S_4(y) = 1 - \frac{5}{8y} + \frac{65}{128} \cdot \frac{1}{y^2} - \frac{861}{3072} \cdot \frac{1}{y^3} \ldots$$

$$b_r^{(4)} = -\frac{1}{4r}\left[\left\{(r-3)\left[(r-3)(r^2 + \tfrac{9}{2}) + \tfrac{27}{2}\right] + \frac{81}{16}\right\} b_{r-3}^{(4)} + \right.$$

$$\left. + \left\{(r-2)(4r^2 - 4r+5) + 5\right\} b_{r-2}^{(4)} + \left\{6r(r-1) + \frac{5}{2}\right\} b_{r-1}^{(4)}\right] ,$$

$$y = 4x^{1/2} ; \qquad \ldots(53c)$$

$$S_5(y) = 1 - \frac{1}{y} + \frac{1}{y^2} - \frac{2}{5}\frac{1}{y^3} - \frac{28}{10} \cdot \frac{1}{y^4} \quad \cdots ,$$

$$b_r^{(5)} = -\frac{1}{5r}\Big[b_{r-4}^{(5)}\Big\{(r-4)\,[(r-4)\,\Big\{(r-4)\,(r^2+2r+16) + 80\Big\} +$$

$$+ 80] + 32\Big\} + b_{r-3}^{(5)}\Big\{5(r-3)\ [(r-3)(r^2+5) + 15] + 31\Big\} +$$

$$+ 5b_{r-2}^{(5)}\Big\{(r-2)\,[2r(r-1) + 3] + 3\Big\} + 5b_{r-1}^{(5)}\Big\{2r(r-1) + 1\Big\}\Big] ,$$

$$y = 5x^{2/5} \; ; \qquad \ldots(53d)$$

$$S_6(y) = 1 - \frac{35}{24} \cdot \frac{1}{y} + \frac{2065}{1125} \cdot \frac{1}{y^2} + \frac{57337}{82944} \cdot \frac{1}{y^3} -$$

$$- \frac{7.39095}{y^4} + \frac{26.2300}{y^5} \quad \cdots ,$$

$$b_r^{(6)} = -\frac{1}{6r}\Big[b_{r-1}^{(6)}\Big\{15(r-1)^2 + 15(r-1) + \frac{35}{4}\Big\} + b_{r-2}^{(6)}\Big\{20(r-2)^3 +$$

$$+ 60(r-2)^2 + 75(r-2) + 35\Big\} + b_{r-3}^{(6)}\Big\{15(r-3)^4 + 90(r-3)^3 + \frac{335}{2}(r-3)^2 +$$

$$+ \frac{495}{2}(r-3) + \frac{1771}{16}\Big\} + b_{r-4}^{(6)}\Big\{6(r-4)^5 + 60(r-4)^4 + 245(r-4)^3 +$$

$$+ 510(r-4)^2 + \frac{4323}{8}(r-4) + \frac{931}{4}\Big\} + b_{r-5}^{(6)}\Big\{(r-5)^6 + 15(r-5)^5 +$$

$$+ \frac{375}{4}(r-5)^4 + \frac{625}{2}(r-5)^3 + \frac{9375}{16}(r-5)^2 + \frac{9375}{16}(r-5) + \frac{15625}{64}\Big\}\Big] ,$$

$$y = 6x^{1/3} \; ; \qquad \ldots(53e)$$

$$S_7(y) = 1 - \frac{2}{y} + \frac{3}{y^2} - \frac{1}{y^3} - \frac{16}{y^4} + \frac{48.5714}{y^5} + \frac{281.7143}{y^6} + \; \cdots\cdot ,$$

$$y = 7x^{2/7} \; ;$$

$$b_r^{(7)} = -\frac{1}{7r}\Big[\Big\{21(r-1)^2 + 21(r-1) + 14\Big\} b_{r-1}^{(7)}$$

$$+ \Big\{35(r-2)^3 + 105(r-2)^2 + 140(r-2) + 70\Big\} b_{r-2}^{(7)}$$

$$+ \Big\{35(r-3)^4 + 210(r-3)^3 + 525(r-3)^2 + 630(r-3) + 301\Big\} b_{r-3}^{(7)}$$

$$+ \Big\{21(r-4)^5 + 210(r-4) + 875(r-4)^3 + 1890(r-4)^2 +$$

$$+ 2107(r-4) + 966\Big\} b_{r-4}^{(7)}$$

$$+ \Big\{7(r-5)^6 + 105(r-5)^5 + 665(r-5)^4 + 2275(r-5)^3 +$$

$$+ 4431(r-5)^2 + 4655(r-5) + 2059\Big\} b_{r-5}^{(7)}$$

$$+ \Big\{(r-6)^7 + 21(r-6)^6 + 189(r-6)^5 + 945(r-6)^4 +$$

$$+ 2835(r-6)^3 + 5103(r-6)^2 + 5103(r-6)$$

$$+ 2187\Big\} b_{r-6}^{(7)}\Big] \quad . \qquad \ldots(53f)$$

The values for $b_j^{(n)}$ for $n \leqslant 7$ are given in Appendix B.

6. INTEGRALS YIELDING C-FUNCTIONS

Quite a number of fairly common integrals can be expressed in terms of C-functions. For example, by the double application of Equation (9) we find

$$C_{n+2}(x) = \left(\frac{2}{\sqrt{\pi}}\right)^2 \int_0^\infty \frac{dw}{w} \int_0^\infty \frac{du}{u} \times$$

$$\times \exp\left\{-\frac{x^2}{u^2} - \frac{u^2}{w^2}\right\} C_n(w) \quad . \qquad \ldots(54)$$

Making use of the standard form (Gradshteyn & Ryzkik 1965)

$$\int_0^\infty x^{\nu-1} \exp\left\{-\beta x^p - \gamma x^{-p}\right\} dx$$

$$= \frac{2}{p}\left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}} K_{\left(\frac{\nu}{p}\right)}\left(2\sqrt{\beta\gamma}\right) \qquad \ldots(55)$$

one obtains a second integral relation

$$C_{n+2}(x) = \frac{4}{\pi} \int_0^{\infty} \frac{dw}{w} \cdot K_o\left(\frac{2x}{w}\right) C_n(w) \quad . \qquad \ldots(56)$$

From this relation, two integrals of the form

$$C_3(x) = \frac{8}{\pi\sqrt{\pi}} \int_0^{\infty} \frac{dw}{w} K_o\left(\frac{2x}{w}\right) \cdot e^{-w^2} \qquad \ldots(57)$$

and

$$C_4(x) = \frac{16}{\pi^2} \int_0^{\infty} \frac{dw}{w} \cdot K_o\left(\frac{2x}{w}\right) K_o(2w) \qquad \ldots(58)$$

are immediately obtained. Various other integrals are readily derived. For example, using the standard form

$$K_o(x) = \int_0^{\infty} \frac{\cos t}{\sqrt{t^2+x^2}} dt \qquad \ldots(59)$$

one can show

$$C_4(x) = \frac{\pi}{2} \int_0^{\infty} \frac{du}{u} \cos\left(\frac{1}{u}\right) \left[J_o^2 (2ux) + N_o^2 (2ux)\right] ,$$

where J_o and N_o are the Bessel functions. By applying the Faltung theorem for Fourier, Mellin and Laplace transforms upon Equations (9) and (56), various other forms can be derived.

Similarly one may prove readily from Equation (9) that the following convolution theorem holds

$$C_{n+m}(v) = \int_0^{\infty} \frac{du}{u} C_n\left(\frac{v}{u}\right) C_m(u) \quad . \qquad \ldots(60)$$

This form is useful for evaluating higher orders in n.

Tables of $C_n(x)$ for $3 \leqslant n \leqslant 7$ are given in Appendix C. These tables were evaluated from the series expansions and checked using (60) to six decimal places.

7. APPLICATIONS

Let a physical formula be given by an expression

$$x = \prod_{i=1}^{N} x_i \qquad \ldots(61)$$

and after measurements of the x_i are made, suppose that these are each distributed according to a normal distribution with mean $\bar{x}_i$, and variance σ_i. Any such product (61) can be written

$$x = \prod_{i=1}^{N} (x_i - \bar{x}_i) + \sum_{j=1}^{N} \bar{x}_j \prod_{i \neq j}^{N} (x_i - \bar{x}_i) + \ldots .$$

$$\ldots . + \sum_{i=1}^{N} (x_i - \bar{x}_i) \prod_{j \neq i}^{N} \bar{x}_j + \prod_{i=1}^{N} \bar{x}_i \quad . \qquad \ldots (62)$$

The first term in Equation (62) has distribution $C_N\left[\left(\frac{x}{\prod_{i=1}^{N} \sigma_i}\right)^{1/2}\right]$

Each of the second set of terms has a distribution $C_{N-1}\left[\left(\frac{y_j}{\prod_{j=1}^{N} \sigma_j}\right)^{1/2}\right]$

with weight $\bar{x}_j$ and $y_j = \prod_{i \neq j} (x_i - \bar{x}_i)$. It is therefore given as a whole by ordinary convolutions of C-functions. The second last group in Equation (62) is a set of terms with a normal distribution of mean zero and variance σ_i. The mean value of all terms except the last is zero, and therefore $\prod_{i=1}^{N} \bar{x}_i$ is the mean value of x.

In normal physical measurements, the standard deviations σ_i are much less than means $\bar{x}_i$, so one can neglect all terms except the last two. It follows immediately from the central limit theorem that in this case x is distributed approximately as a normal distribution with mean $\prod_{i=1}^{N} \bar{x}_i$ and variance $\sum_{i=1}^{N} \sigma_i \left(\prod_{j \neq i}^{N} \bar{x}_j\right)^2$. However, if the errors become appreciable, this approximation fails owing to the other terms. In the extreme cases where the means $\bar{x}_i$ are much less than the standard deviations σ_i, the first term dominates, and the distribution is approximately distributed as the function C_N. The intermediate region can be examined by expanding the distribution P(x) of (61) into ascending powers of $\mu_i/\sqrt{\sigma_i}$, whence it is found that the leading term is the distribution of the first term in (62).

We have also encountered these functions in other connections, such as in the inversion of random matrices, products of random matrices, and in the theory of nuclear cross sections, where such product distributions arise for certain parameters. We feel sure that they will occur sufficiently frequently in future theories to warrant the investigations made in this paper.

8. REFERENCES

Copson, E.T. (1946) - The asymptotic expansion of a function defined integral or contour integral. SRE/ACS106, Department of Scientific Research and ν Experiment. Admiralty, London, UK. pp.50-53.

Cramer, H. (1945) - Mathematical Methods of Statistics. Princeton University Press,pp. 71, 86, 174, 294.

Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricorni, F.G. (eds.) (1953a) - Higher Transcendental Functions, Vol.I. McGraw-Hill, New York. p.210.

Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricorni, F.G. (eds.) (1953b) - Higher Transcendental Functions, Vol.I. McGraw-Hill, New York. p.3.

Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricorni, F.G. (eds.) (1954) - Tables of Integral Transforms, Vol.I. McGraw-Hill, New York. p.305.

Gradshteyn, I.S. & Ryzkik, I.M. (1965) - Tables of Integrals, Series and Products. Academic Press, New York. p.342.

Meijer, C.S. (1936) - Niew Arch. Wiskunde 18 (2) 10-39.

APPENDIX A

F(M,J) VALUES FOR SERIES EXPANSION

N = 3

M/J	1	2	3
0	3.22515E-02	1.11697E-01	2.55864E-01
1	-3.22515E-02	8.18126E-02	-3.07793E-01
2	4.03144E-03	-2.23209E-02	6.59083E-02
3	-1.49313E-04	1.12533E-03	-3.46683E-03
4	2.33301E-06	-2.10827E-05	6.91064E-05
5	-1.86641E-08	1.91059E-07	-6.63007E-07
6	8.64078E-11	-9.70939E-10	3.54055E-09
7	-2.51918E-13	3.04665E-12	-1.15972E-11
8	4.92027E-16	-6.31952E-15	2.49744E-14
9	-6.74934E-19	9.11871E-18	-3.72480E-17
10	6.74934E-22	-9.52366E-21	4.00646E-20
11	-5.07088E-25	7.43186E-24	-3.21029E-23

N = 4

M/J	1	2	3	4
0	1.02660E-02	7.11082E-02	3.66821E-01	6.92951E-01
1	1.02660E-02	-5.20836E-02	4.13915E-01	-5.84088E-01
2	6.41624E-04	-7.10496E-03	4.85149E-02	-1.14145E-01
3	7.92128E-06	-1.19401E-04	8.85666E-04	-2.43793E-03
4	3.09425E-08	-5.59237E-07	4.50849E-06	-1.36238E-05
5	4.95080E-11	-1.01360E-09	8.76404E-09	-2.83323E-08
6	3.82006E-14	-8.58498E-13	7.86884E-12	-2.68253E-11
7	1.59103E-17	-3.84833E-16	3.70544E-15	-1.31968E-14
8	3.88435E-21	-9.97800E-20	1.00224E-18	-3.70448E-18
9	5.92036E-25	-1.59974E-23	1.66715E-22	-6.36366E-22
10	5.92036E-29	-1.67079E-27	1.79868E-26	-7.06322E-26

N = 5

M/J	1	2	3	4	5
0	3.26776E-03	3.77241E-02	3.24570E-01	1.40215E+00	2.79262E+00
1	-3.26776E-03	2.76312E-02	-3.46906E-01	1.19330E+00	-3.08676E+00
2	1.02118E-04	-1.88465E-03	2.19122E-02	-1.21645E-01	3.04383E-01
3	-4.20237E-07	1.05573E-05	-1.37357E-04	8.91808E-04	-2.45655E-03
4	4.10388E-10	-1.23618E-08	1.77417E-07	-1.26963E-06	3.77564E-06
5	-1.31324E-13	4.48109E-12	-6.95863E-11	5.34952E-10	-1.68961E-09
6	1.68884E-17	-6.32566E-16	1.04740E-14	-8.52163E-14	2.82654E-13
7	-1.00434E-21	4.05080E-20	-7.07533E-19	6.03215E-18	-2.08446E-17
8	3.06654E-26	-1.31287E-24	2.39963E-23	-2.12866E-22	7.61821E-22
9	-5.19320E-31	2.33876E-29	-4.44590E-28	4.08173E-27	-1.50613E-26

N = 6

M/J	1	2	3	4	5	6
0	1.04016E-03	1.80119E-02	2.27421E-01	1.64875E+00	7.20397E+00	1.41839E+01
1	1.04016E-03	-1.31929E-02	2.32003E-01	-1.40077E+00	7.59326E+00	-1.31924E+01
2	1.62525E-05	-4.49927E-04	7.80523E-03	-7.43251E-02	4.14145E-01	-9.98449E-01
3	2.22943E-08	-8.40127E-07	1.66846E-05	-1.86251E-04	1.16070E-03	-3.14066E-03
4	5.44294E-12	-2.45931E-10	5.44692E-09	-6.73180E-08	4.56495E-07	-1.33515E-06
5	3.48348E-16	-1.78297E-14	4.30006E-13	-5.73134E-12	4.14903E-11	-1.28763E-10
6	7.46630E-21	-4.19483E-19	1.08322E-17	-1.53283E-16	1.16969E-15	-3.80770E-15
7	6.34625E-26	-3.83753E-24	1.04841E-22	-1.55860E-21	1.24268E-20	-4.20984E-20
8	2.42090E-31	-1.55469E-29	4.45443E-28	-6.90468E-27	5.71497E-26	-2.00321E-25

N = 7

M/J	1	2	3	4	5	6	7
0	3.31094E-04	8.02672E-03	1.38266E-01	1.47248E+00	1.04773E+01	4.46694E+01	8.83232E+01
1	-3.31094E-04	5.87921E-03	-1.35449E-01	1.24142E+00	-1.05884E+01	4.12406E+01	-9.23608E+01
2	2.58667E-06	-1.00251E-04	2.40520E-03	-3.42030E-02	3.14605E-01	-1.67449E+00	4.05816E+00
3	-1.18275E-09	6.23981E-08	-1.74496E-06	2.91975E-05	-3.04045E-04	1.82689E-03	-4.90909E-03
4	7.21892E-14	-4.56647E-12	1.43618E-10	-2.67227E-09	3.04712E-08	-1.98921E-07	5.76060E-07
5	-9.24021E-19	6.62126E-17	-2.27851E-15	4.58834E-14	-5.60868E-13	3.90024E-12	-1.19657E-11
6	3.30083E-24	-2.59633E-22	9.59616E-21	-2.05724E-19	2.65891E-18	-1.94532E-17	6.25317E-17
7	-4.00809E-30	3.39312E-28	-1.32973E-26	3.00133E-25	-4.06269E-24	3.10069E-23	-1.03634E-22

APPENDIX B

B-VALUES FOR ASYMPTOTIC APPROXIMATION

N = 3					
	1.00000E+00	-3.33333E-01	2.22222E-01	-1.72839E-01	3.29218E-02
	6.03567E-01	-3.27999E+00	1.36280E+01	-4.90874E+01	1.36090E+02
N = 4					
	1.00000E+00	-6.25000E-01	5.07813E-01	-2.80273E-01	-6.92047E-01
	4.07216E+00	-1.16148E+01	1.65938E+00	2.55816E+02	-2.16149E+03
N = 5					
	1.00000E+00	-1.00000E+00	1.00000E+00	-4.00000E-01	-2.80000E+00
	1.20000E+01	-9.12000E+00	-2.32320E+02	1.90656E+03	-5.32454E+03
N = 6					
	1.00000E+00	-1.45833E+00	1.79253E+00	-5.89832E-01	-7.39095E+00
	2.62300E+01	5.27114E+01	-1.14264E+03	5.26778E+03	2.63600E+04
N = 7					
	1.00000E+00	-2.00000E+00	3.00000E+00	-1.00000E+00	-1.60000E+01
	4.85714E+01	2.81714E+02	-3.55714E+03	6.09257E+03	2.32707E+05

APPENDIX C

TABLES OF $C_N(X)$ FOR $3 \leqslant n \leqslant 7$

X	N = 3	N = 4	N = 5	N = 6	N = 7
0.0050	1.49990E+01	2.44284E+01	3.01645E+01	3.07539E+01	2.73099E+01
0.0100	1.09286E+01	1.53838E+01	1.66827E+01	1.51774E+01	1.22014E+01
0.0150	8.86640E+00	1.13762E+01	1.13827E+01	9.66380E+00	7.32103E+00
0.0200	7.54551E+00	9.03347E+00	8.52022E+00	6.88077E+00	4.99606E+00
0.0250	6.60193E+00	7.47492E+00	6.72739E+00	5.22393E+00	3.66992E+00
0.0300	5.88325E+00	6.35580E+00	5.50175E+00	4.13652E+00	2.82872E+00
0.0350	5.31212E+00	5.51035E+00	4.61354E+00	3.37495E+00	2.25604E+00
0.0400	4.84428E+00	4.84797E+00	3.94235E+00	2.81615E+00	1.84590E+00
0.0450	4.45223E+00	4.31458E+00	3.41888E+00	2.39145E+00	1.54072E+00
0.0500	4.11781E+00	3.87576E+00	3.00041E+00	2.05967E+00	1.30674E+00
0.0550	3.82845E+00	3.50844E+00	2.65913E+00	1.79466E+00	1.12297E+00
0.0600	3.57515E+00	3.19656E+00	2.37620E+00	1.57908E+00	9.75747E-01
0.0650	3.35122E+00	2.92859E+00	2.13839E+00	1.40100E+00	8.55819E-01
0.0700	3.15161E+00	2.69598E+00	1.93615E+00	1.25196E+00	7.56732E-01
0.0750	2.97241E+00	2.49231E+00	1.76240E+00	1.12581E+00	6.73856E-01
0.0800	2.81052E+00	2.31259E+00	1.61181E+00	1.01799E+00	6.03796E-01
0.0850	2.66348E+00	2.15295E+00	1.48027E+00	9.25021E-01	5.44013E-01
0.0900	2.52927E+00	2.01030E+00	1.36457E+00	8.44248E-01	4.92573E-01
0.0950	2.40624E+00	1.88215E+00	1.26219E+00	7.73588E-01	4.47981E-01
0.1000	2.29304E+00	1.76647E+00	1.17107E+00	7.11391E-01	4.09067E-01
0.1100	2.09167E+00	1.56614E+00	1.01637E+00	6.07358E-01	3.44740E-01
0.1200	1.91790E+00	1.39904E+00	8.90492E-01	5.24288E-01	2.94126E-01
0.1300	1.76642E+00	1.25787E+00	7.86541E-01	4.56857E-01	2.53589E-01
0.1400	1.63322E+00	1.13731E+00	6.99610E-01	4.01350E-01	2.20626E-01
0.1500	1.51520E+00	1.03338E+00	6.26116E-01	3.55101E-01	1.93469E-01
0.1600	1.40997E+00	9.43041E-01	5.63389E-01	3.16158E-01	1.70839E-01
0.1700	1.31558E+00	8.63957E-01	5.09400E-01	2.83059E-01	1.51791E-01
0.1800	1.23050E+00	7.94277E-01	4.62586E-01	2.54695E-01	1.35615E-01
0.1900	1.15345E+00	7.32531E-01	4.21721E-01	2.30208E-01	1.21767E-01
0.2000	1.08340E+00	6.77531E-01	3.85835E-01	2.08927E-01	1.09827E-01
0.2100	1.01947E+00	6.28310E-01	3.54149E-01	1.90320E-01	9.94656E-02
0.2200	9.60935E-01	5.84074E-01	3.26033E-01	1.73963E-01	9.04205E-02
0.2300	9.07169E-01	5.44163E-01	3.00973E-01	1.59510E-01	8.24817E-02
0.2400	8.57649E-01	5.08025E-01	2.78542E-01	1.46682E-01	7.54796E-02
0.2500	8.11920E-01	4.75195E-01	2.58388E-01	1.35247E-01	6.92753E-02
0.2600	7.69591E-01	4.45279E-01	2.40215E-01	1.25014E-01	6.37548E-02
0.2700	7.30323E-01	4.17942E-01	2.23776E-01	1.15823E-01	5.88235E-02
0.2800	6.93819E-01	3.92894E-01	2.08859E-01	1.07541E-01	5.44024E-02
0.2900	6.59820E-01	3.69889E-01	1.95284E-01	1.00053E-01	5.04255E-02
0.3000	6.28097E-01	3.48711E-01	1.82897E-01	9.32649E-02	4.68366E-02
0.3100	5.98447E-01	3.29173E-01	1.71568E-01	8.70928E-02	4.35884E-02
0.3200	5.70692E-01	3.11110E-01	1.61179E-01	8.14665E-02	4.06403E-02
0.3300	5.44672E-01	2.94381E-01	1.51634E-01	7.63254E-02	3.79575E-02
0.3400	5.20243E-01	2.78858E-01	1.42844E-01	7.16166E-02	3.55102E-02
0.3500	4.97280E-01	2.64429E-01	1.34734E-01	6.72946E-02	3.32725E-02
0.3600	4.75666E-01	2.50997E-01	1.27237E-01	6.33193E-02	3.12219E-02
0.3700	4.55297E-01	2.38473E-01	1.20295E-01	5.96558E-02	2.93388E-02
0.3800	4.36082E-01	2.26779E-01	1.13856E-01	5.62733E-02	2.76062E-02
0.3900	4.17934E-01	2.15844E-01	1.07874E-01	5.31449E-02	2.60089E-02
0.4000	4.00778E-01	2.05607E-01	1.02307E-01	5.02464E-02	2.45338E-02

X	N = 3	N = 4	N = 5	N = 6	N = 7
0.4100	3.84342E-01	1.96010E-01	9.71200E-02	4.75569E-02	2.31692E-02
0.4200	3.69165E-01	1.87002E-01	9.22800E-02	4.50573E-02	2.19047E-02
0.4300	3.54586E-01	1.78538E-01	8.77575E-02	4.27309E-02	2.07312E-02
0.4400	3.40753E-01	1.70576E-01	8.35266E-02	4.05627E-02	1.96405E-02
0.4500	3.27618E-01	1.63078E-01	7.95636E-02	3.85391E-02	1.86254E-02
0.4600	3.15135E-01	1.56011E-01	7.58471E-02	3.66482E-02	1.76792E-02
0.4700	3.03263E-01	1.49343E-01	7.23580E-02	3.48791E-02	1.67962E-02
0.4800	2.91964E-01	1.43045E-01	6.90787E-02	3.32219E-02	1.59711E-02
0.4900	2.81203E-01	1.37093E-01	6.59935E-02	3.16679E-02	1.51992E-02
0.5000	2.70947E-01	1.31461E-01	6.30882E-02	3.02089E-02	1.44761E-02
0.5100	2.61167E-01	1.26129E-01	6.03494E-02	2.88379E-02	1.37981E-02
0.5200	2.51835E-01	1.21076E-01	5.77653E-02	2.75480E-02	1.31617E-02
0.5300	2.42924E-01	1.16284E-01	5.53250E-02	2.63335E-02	1.25637E-02
0.5400	2.34411E-01	1.11736E-01	5.30185E-02	2.51888E-02	1.20011E-02
0.5500	2.26274E-01	1.07416E-01	5.08367E-02	2.41088E-02	1.14715E-02
0.5600	2.18493E-01	1.03311E-01	4.87711E-02	2.30891E-02	1.09724E-02
0.5700	2.11046E-01	9.94074E-02	4.68141E-02	2.21255E-02	1.05015E-02
0.5800	2.03917E-01	9.56922E-02	4.49585E-02	2.12141E-02	1.00570E-02
0.5900	1.97089E-01	9.21544E-02	4.31978E-02	2.03514E-02	9.63704E-03
0.6000	1.90546E-01	8.87834E-02	4.15261E-02	1.95342E-02	9.23986E-03
0.6100	1.84273E-01	8.55695E-02	3.99376E-02	1.87595E-02	8.86397E-03
0.6200	1.78256E-01	8.25036E-02	3.84273E-02	1.80246E-02	8.50794E-03
0.6300	1.72483E-01	7.95773E-02	3.69904E-02	1.73269E-02	8.17050E-03
0.6400	1.66940E-01	7.67826E-02	3.56225E-02	1.66641E-02	7.85042E-03
0.6500	1.61618E-01	7.41124E-02	3.43195E-02	1.60341E-02	7.54660E-03
0.6600	1.56505E-01	7.15596E-02	3.30775E-02	1.54347E-02	7.25802E-03
0.6700	1.51591E-01	6.91180E-02	3.18930E-02	1.48643E-02	6.98373E-03
0.6800	1.46866E-01	6.67815E-02	3.07628E-02	1.43210E-02	6.72287E-03
0.6900	1.42322E-01	6.45446E-02	2.96838E-02	1.38033E-02	6.47461E-03
0.7000	1.37949E-01	6.24021E-02	2.86531E-02	1.33097E-02	6.23820E-03
0.7100	1.33741E-01	6.03490E-02	2.76680E-02	1.28388E-02	6.01295E-03
0.7200	1.29690E-01	5.83808E-02	2.67260E-02	1.23892E-02	5.79821E-03
0.7300	1.25788E-01	5.64931E-02	2.58249E-02	1.19599E-02	5.59336E-03
0.7400	1.22029E-01	5.46819E-02	2.49625E-02	1.15497E-02	5.39785E-03
0.7500	1.18406E-01	5.29435E-02	2.41366E-02	1.11575E-02	5.21114E-03
0.7600	1.14914E-01	5.12741E-02	2.33454E-02	1.07823E-02	5.03276E-03
0.7700	1.11547E-01	4.96706E-02	2.25872E-02	1.04233E-02	4.86225E-03
0.7800	1.08298E-01	4.81296E-02	2.18601E-02	1.00796E-02	4.69916E-03
0.7900	1.05164E-01	4.66482E-02	2.11627E-02	9.75041E-03	4.54312E-03
0.8000	1.02140E-01	4.52236E-02	2.04934E-02	9.43493E-03	4.39374E-03
0.8100	9.92201E-02	4.38532E-02	1.98510E-02	9.13249E-03	4.25067E-03
0.8200	9.64007E-02	4.25343E-02	1.92339E-02	8.84242E-03	4.11358E-03
0.8300	9.36775E-02	4.12648E-02	1.86411E-02	8.56410E-03	3.98217E-03
0.8400	9.10467E-02	4.00422E-02	1.80713E-02	8.29695E-03	3.85615E-03
0.8500	8.85043E-02	3.88645E-02	1.75235E-02	8.04041E-03	3.73524E-03
0.8600	8.60469E-02	3.77297E-02	1.69967E-02	7.79397E-03	3.61919E-03
0.8700	8.36712E-02	3.66360E-02	1.64897E-02	7.55714E-03	3.50776E-03
0.8800	8.13737E-02	3.55814E-02	1.60018E-02	7.32945E-03	3.40072E-03
0.8900	7.91514E-02	3.45642E-02	1.55320E-02	7.11049E-03	3.29787E-03
0.9000	7.70015E-02	3.35830E-02	1.50796E-02	6.89983E-03	3.19899E-03

X	N = 3	N = 4	N = 5	N = 6	N = 7
0.9100	7.49210E-02	3.26361E-02	1.46437E-02	6.69710E-03	3.10391E-03
0.9200	7.29073E-02	3.17221E-02	1.42236E-02	6.50192E-03	3.01243E-03
0.9300	7.09579E-02	3.08396E-02	1.38186E-02	6.31395E-03	2.92441E-03
0.9400	6.90703E-02	2.99872E-02	1.34281E-02	6.13287E-03	2.83966E-03
0.9500	6.72422E-02	2.91639E-02	1.30513E-02	5.95837E-03	2.75805E-03
0.9600	6.54714E-02	2.83682E-02	1.26878E-02	5.79015E-03	2.67943E-03
0.9700	6.37557E-02	2.75992E-02	1.23370E-02	5.62794E-03	2.60368E-03
0.9800	6.20931E-02	2.68558E-02	1.19983E-02	5.47148E-03	2.53065E-03
0.9900	6.04817E-02	2.61368E-02	1.16712E-02	5.32051E-03	2.46023E-03
1.0000	5.89195E-02	2.54414E-02	1.13552E-02	5.17481E-03	2.39230E-03
1.0500	5.17884E-02	2.22867E-02	9.92694E-03	4.51776E-03	2.08652E-03
1.1000	4.56520E-02	1.95982E-02	8.71659E-03	3.96303E-03	1.82903E-03
1.1500	4.03507E-02	1.72953E-02	7.68496E-03	3.49173E-03	1.61076E-03
1.2000	3.57541E-02	1.53133E-02	6.80093E-03	3.08901E-03	1.42462E-03
1.2500	3.17550E-02	1.36001E-02	6.03964E-03	2.74305E-03	1.26500E-03
1.3000	2.82649E-02	1.21132E-02	5.38105E-03	2.44439E-03	1.12740E-03
1.3500	2.52100E-02	1.08178E-02	4.80867E-03	2.18538E-03	1.00822E-03
1.4000	2.25288E-02	9.68533E-03	4.30978E-03	1.95980E-03	9.04538E-04
1.4500	2.01695E-02	8.69198E-03	3.87283E-03	1.76255E-03	8.13954E-04
1.5000	1.80864E-02	7.81796E-03	3.48896E-03	1.58943E-03	7.34508E-04
1.5500	1.62485E-02	7.04670E-03	3.15060E-03	1.43696E-03	6.64575E-04
1.6000	1.46185E-02	6.36425E-03	2.85145E-03	1.30223E-03	6.02805E-04
1.6500	1.31715E-02	5.75881E-03	2.58618E-03	1.18280E-03	5.48068E-04
1.7000	1.18843E-02	5.22037E-03	2.35033E-03	1.07663E-03	4.99414E-04
1.7500	1.07374E-02	4.74041E-03	2.14007E-03	9.81994E-04	4.56044E-04
1.8000	9.71356E-03	4.31162E-03	1.95217E-03	8.97404E-04	4.17276E-04
1.8500	8.79816E-03	3.92774E-03	1.78386E-03	8.21610E-04	3.82534E-04
1.9000	7.97839E-03	3.58338E-03	1.63276E-03	7.53537E-04	3.51322E-04
1.9500	7.24317E-03	3.27389E-03	1.49683E-03	6.92260E-04	3.23216E-04
2.0000	6.58282E-03	2.99522E-03	1.37428E-03	6.36982E-04	2.97851E-04
2.0500	5.93890E-03	2.74386E-03	1.26359E-03	5.87014E-04	2.74910E-04
2.1000	5.45403E-03	2.51677E-03	1.16344E-03	5.41757E-04	2.54120E-04
2.1500	4.97172E-03	2.31127E-03	1.07264E-03	5.00689E-04	2.35242E-04
2.2000	4.53628E-03	2.12501E-03	9.90198E-04	4.63358E-04	2.18068E-04
2.2500	4.14269E-03	1.95596E-03	9.15215E-04	4.29363E-04	2.02418E-04
2.3000	3.78653E-03	1.80231E-03	8.46911E-04	3.98357E-04	1.88131E-04
2.3500	3.46389E-03	1.66245E-03	7.84596E-04	3.70031E-04	1.75068E-04
2.4000	3.17131E-03	1.53499E-03	7.27665E-04	3.44114E-04	1.63104E-04
2.4500	2.90573E-03	1.41868E-03	6.75579E-04	3.20367E-04	1.52131E-04
2.5000	2.66440E-03	1.31242E-03	6.27863E-04	2.98578E-04	1.42052E-04
2.5500	2.44492E-03	1.21522E-03	5.84093E-04	2.78557E-04	1.32781E-04
2.6000	2.24513E-03	1.12620E-03	5.43893E-04	2.60138E-04	1.24242E-04
2.6500	2.06309E-03	1.04460E-03	5.06927E-04	2.43171E-04	1.16367E-04
2.7000	1.89708E-03	9.69709E-04	4.72896E-04	2.27521E-04	1.09095E-04
2.7500	1.74558E-03	9.00909E-04	4.41532E-04	2.13071E-04	1.02372E-04
2.8000	1.60720E-03	8.37641E-04	4.12593E-04	1.99712E-04	9.61484E-05
2.8500	1.48070E-03	7.79404E-04	3.85865E-04	1.87349E-04	9.03815E-05
2.9000	1.36499E-03	7.25747E-04	3.61153E-04	1.75896E-04	8.50316E-05
2.9500	1.25905E-03	6.76266E-04	3.38283E-04	1.65273E-04	8.00633E-05
3.0000	1.16200E-03	6.30595E-04	3.17096E-04	1.55412E-04	7.54446E-05

X	N = 3	N = 4	N = 5	N = 6	N = 7
3.0500	1.07299E-03	5.88405E-04	2.97452E-04	1.46249E-04	7.11465E-05
3.1000	9.91375E-04	5.49398E-04	2.79220E-04	1.37726E-04	6.71431E-05
3.1500	9.16451E-04	5.13304E-04	2.62286E-04	1.29791E-04	6.34106E-05
3.2000	8.47628E-04	4.79881E-04	2.46542E-04	1.22397E-04	5.99274E-05
3.2500	7.84369E-04	4.48906E-04	2.31893E-04	1.15502E-04	5.66741E-05
3.3000	7.26169E-04	4.20179E-04	2.18252E-04	1.09066E-04	5.36328E-05
3.3500	6.72648E-04	3.93516E-04	2.05540E-04	1.03054E-04	5.07874E-05
3.4000	6.23348E-04	3.68754E-04	1.93685E-04	9.74332E-05	4.81231E-05
3.4500	5.77928E-04	3.45739E-04	1.82619E-04	9.21745E-05	4.56264E-05
3.5000	5.36058E-04	3.24335E-04	1.72284E-04	8.72506E-05	4.32850E-05
3.5500	4.97441E-04	3.04416E-04	1.62625E-04	8.26369E-05	4.10874E-05
3.6000	4.61804E-04	2.85866E-04	1.53590E-04	7.83107E-05	3.90235E-05
3.6500	4.28902E-04	2.68581E-04	1.45134E-04	7.42514E-05	3.70837E-05
3.7000	3.98509E-04	2.52465E-04	1.37214E-04	7.04399E-05	3.52592E-05
3.7500	3.70420E-04	2.37429E-04	1.29792E-04	6.68586E-05	3.35421E-05
3.8000	3.44447E-04	2.23394E-04	1.22833E-04	6.34915E-05	3.19249E-05
3.8500	3.20420E-04	2.10284E-04	1.16303E-04	6.03238E-05	3.04009E-05
3.9000	2.98182E-04	1.98033E-04	1.10172E-04	5.73418E-05	2.89638E-05
3.9500	2.77592E-04	1.86578E-04	1.04412E-04	5.45330E-05	2.76078E-05
4.0000	2.58519E-04	1.75861E-04	9.89983E-05	5.18857E-05	2.63276E-05
4.0500	2.40843E-04	1.65828E-04	9.39069E-05	4.93892E-05	2.51181E-05
4.1000	2.24454E-04	1.56434E-04	8.91160E-05	4.70337E-05	2.39750E-05
4.1500	2.09254E-04	1.47631E-04	8.46054E-05	4.48098E-05	2.28938E-05
4.2000	1.95149E-04	1.39380E-04	8.03565E-05	4.27092E-05	2.18707E-05
4.2500	1.82056E-04	1.31640E-04	7.63522E-05	4.07240E-05	2.09021E-05
4.3000	1.69898E-04	1.24378E-04	7.25764E-05	3.88468E-05	1.99846E-05
4.3500	1.58602E-04	1.17561E-04	6.90143E-05	3.70709E-05	1.91150E-05
4.4000	1.48105E-04	1.11159E-04	6.56523E-05	3.53900E-05	1.82904E-05
4.4500	1.38345E-04	1.05143E-04	6.24776E-05	3.37983E-05	1.75082E-05
4.5000	1.29269E-04	9.94876E-05	5.94784E-05	3.22902E-05	1.67657E-05
4.5500	1.20824E-04	9.41695E-05	5.66437E-05	3.08608E-05	1.60606E-05
4.6000	1.12964E-04	8.91663E-05	5.39633E-05	2.95053E-05	1.53907E-05
4.6500	1.05647E-04	8.44573E-05	5.14277E-05	2.82193E-05	1.47541E-05
4.7000	9.88320E-05	8.00234E-05	4.90280E-05	2.69987E-05	1.41487E-05
4.7500	9.24828E-05	7.58470E-05	4.67561E-05	2.58397E-05	1.35727E-05
4.8000	8.65655E-05	7.19115E-05	4.46042E-05	2.47388E-05	1.30246E-05
4.8500	8.10491E-05	6.82016E-05	4.25652E-05	2.36925E-05	1.25028E-05
4.9000	7.59047E-05	6.47031E-05	4.06324E-05	2.26978E-05	1.20057E-05
4.9500	7.11058E-05	6.14027E-05	3.87995E-05	2.17518E-05	1.15321E-05
5.0000	6.66276E-05	5.82880E-05	3.70607E-05	2.08517E-05	1.10806E-05
5.0500	6.24477E-05	5.53477E-05	3.54112E-05	1.99949E-05	1.06500E-05
5.1000	5.85448E-05	5.25709E-05	3.38446E-05	1.91791E-05	1.02392E-05
5.1500	5.48996E-05	4.99476E-05	3.23568E-05	1.84020E-05	9.84724E-06
5.2000	5.14941E-05	4.74685E-05	3.09433E-05	1.76616E-05	9.47299E-06
5.2500	4.83115E-05	4.51249E-05	2.96000E-05	1.69558E-05	9.11557E-06
5.3000	4.53366E-05	4.29087E-05	2.83229E-05	1.62827E-05	8.77410E-06
5.3500	4.25549E-05	4.08123E-05	2.71084E-05	1.56407E-05	8.44775E-06
5.4000	3.99532E-05	3.88285E-05	2.59530E-05	1.50281E-05	8.13575E-06
5.4500	3.75192E-05	3.69508E-05	2.48534E-05	1.44434E-05	7.83736E-06
5.5000	3.52415E-05	3.51728E-05	2.38067E-05	1.38850E-05	7.55189E-06

X	N = 3	N = 4	N = 5	N = 6	N = 7
5.5500	3.31094E-05	3.34889E-05	2.28099E-05	1.33517E-05	7.27871E-06
5.6000	3.11132E-05	3.18935E-05	2.18604E-05	1.28422E-05	7.01719E-06
5.6500	2.92438E-05	3.03815E-05	2.09557E-05	1.23552E-05	6.76676E-06
5.7000	2.74926E-05	2.89482E-05	2.00934E-05	1.18896E-05	6.52687E-06
5.7500	2.58517E-05	2.75892E-05	1.92713E-05	1.14443E-05	6.29702E-06
5.8000	2.43139E-05	2.63000E-05	1.84872E-05	1.10183E-05	6.07671E-06
5.8500	2.28723E-05	2.50770E-05	1.77392E-05	1.06107E-05	5.86548E-06
5.9000	2.15206E-05	2.39162E-05	1.70254E-05	1.02206E-05	5.66291E-06
5.9500	2.02528E-05	2.28144E-05	1.63442E-05	9.84707E-06	5.46858E-06
6.0000	1.90635E-05	2.17681E-05	1.56937E-05	9.48933E-06	5.28210E-06
6.0500	1.79476E-05	2.07744E-05	1.50724E-05	9.15030E-06	5.10312E-06
6.1000	1.69003E-05	1.98303E-05	1.44790E-05	8.82173E-06	4.93127E-06
6.1500	1.59172E-05	1.89332E-05	1.39119E-05	8.50680E-06	4.76623E-06
6.2000	1.49941E-05	1.80804E-05	1.33699E-05	8.20486E-06	4.60769E-06
6.2500	1.41272E-05	1.72697E-05	1.28517E-05	7.91530E-06	4.45535E-06
6.3000	1.33129E-05	1.64987E-05	1.23562E-05	7.63754E-06	4.30894E-06
6.3500	1.25478E-05	1.57653E-05	1.18822E-05	7.37104E-06	4.16818E-06
6.4000	1.18288E-05	1.50676E-05	1.14287E-05	7.11527E-06	4.03283E-06
6.4500	1.11530E-05	1.44036E-05	1.09947E-05	6.86974E-06	3.90264E-06
6.5000	1.05177E-05	1.37715E-05	1.05793E-05	6.63399E-06	3.77739E-06
6.5500	9.92033E-06	1.31697E-05	1.01815E-05	6.40756E-06	3.65687E-06
6.6000	9.35850E-06	1.25966E-05	9.80062E-06	6.19005E-06	3.54085E-06
6.6500	8.82999E-06	1.20507E-05	9.43576E-06	5.98105E-06	3.42916E-06
6.7000	8.33274E-06	1.15306E-05	9.08618E-06	5.78018E-06	3.32160E-06
6.7500	7.86481E-06	1.10350E-05	8.75118E-06	5.58708E-06	3.21800E-06
6.8000	7.42439E-06	1.05626E-05	8.43006E-06	5.40141E-06	3.11819E-06
6.8500	7.00977E-06	1.01122E-05	8.12219E-06	5.22285E-06	3.02201E-06
6.9000	6.61939E-06	9.68276E-06	7.82696E-06	5.05108E-06	2.92931E-06
6.9500	6.25174E-06	9.27317E-06	7.54379E-06	4.88581E-06	2.83995E-06
7.0000	5.90545E-06	8.88246E-06	7.27213E-06	4.72677E-06	2.75377E-06
7.0500	5.57922E-06	8.50966E-06	7.01146E-06	4.57368E-06	2.67227E-06
7.1000	5.27183E-06	8.15389E-06	6.76128E-06	4.42629E-06	2.59204E-06
7.1500	4.98214E-06	7.81430E-06	6.52112E-06	4.28436E-06	2.51463E-06
7.2000	4.70909E-06	7.49009E-06	6.29054E-06	4.14767E-06	2.43993E-06
7.2500	4.45167E-06	7.18052E-06	6.06910E-06	4.01599E-06	2.36783E-06
7.3000	4.20895E-06	6.88485E-06	5.85640E-06	3.88910E-06	2.29822E-06
7.3500	3.98005E-06	6.60241E-06	5.65207E-06	3.76683E-06	2.23100E-06
7.4000	3.76415E-06	6.33256E-06	5.45572E-06	3.64897E-06	2.16609E-06
7.4500	3.56048E-06	6.07469E-06	5.26703E-06	3.53534E-06	2.10338E-06
7.5000	3.36831E-06	5.82823E-06	5.08564E-06	3.42577E-06	2.04279E-06
7.5500	3.18696E-06	5.59262E-06	4.91126E-06	3.32010E-06	1.98425E-06
7.6000	3.01581E-06	5.36736E-06	4.74357E-06	3.21817E-06	1.92767E-06
7.6500	2.85424E-06	5.15194E-06	4.58229E-06	3.11983E-06	1.87297E-06
7.7000	2.70170E-06	4.94590E-06	4.42715E-06	3.02494E-06	1.82009E-06
7.7500	2.55766E-06	4.74880E-06	4.27789E-06	2.93335E-06	1.76895E-06
7.8000	2.42162E-06	4.56021E-06	4.13426E-06	2.84495E-06	1.71949E-06
7.8500	2.29313E-06	4.37975E-06	3.99602E-06	2.75959E-06	1.67164E-06
7.9000	2.17175E-06	4.20703E-06	3.86296E-06	2.67717E-06	1.62535E-06
7.9500	2.05706E-06	4.04169E-06	3.73484E-06	2.59757E-06	1.58055E-06
8.0000	1.94868E-06	3.88339E-06	3.61148E-06	2.52068E-06	1.53719E-06

X	N = 3	N = 4	N = 5	N = 6	N = 7
8.0500	1.84625E-06	3.73181E-06	3.49267E-06	2.44640E-06	1.49523E-06
8.1000	1.74943E-06	3.58663E-06	3.37823E-06	2.37462E-06	1.45459E-06
8.1500	1.65790E-06	3.44757E-06	3.26798E-06	2.30525E-06	1.41525E-06
8.2000	1.57135E-06	3.31435E-06	3.16175E-06	2.23820E-06	1.37715E-06
8.2500	1.48951E-06	3.18670E-06	3.05937E-06	2.17338E-06	1.34024E-06
8.3000	1.41210E-06	3.06437E-06	2.96069E-06	2.11071E-06	1.30449E-06
8.3500	1.33886E-06	2.94712E-06	2.86557E-06	2.05011E-06	1.26984E-06
8.4000	1.26961E-06	2.83472E-06	2.77386E-06	1.99149E-06	1.23627E-06
8.4500	1.20407E-06	2.72696E-06	2.68542E-06	1.93480E-06	1.20374E-06
8.5000	1.14205E-06	2.62363E-06	2.60013E-06	1.87994E-06	1.17220E-06
8.5500	1.08335E-06	2.52453E-06	2.51785E-06	1.82686E-06	1.14162E-06
8.6000	1.02779E-06	2.42947E-06	2.43848E-06	1.77550E-06	1.11197E-06
8.6500	9.75190E-07	2.33828E-06	2.36190E-06	1.72578E-06	1.08322E-06
8.7000	9.25391E-07	2.25079E-06	2.28800E-06	1.67765E-06	1.05533E-06
8.7500	8.78235E-07	2.16684E-06	2.21668E-06	1.63106E-06	1.02828E-06
8.8000	8.33576E-07	2.08627E-06	2.14783E-06	1.58594E-06	1.00204E-06
8.8500	7.91277E-07	2.00893E-06	2.08137E-06	1.54225E-06	9.76570E-07
8.9000	7.51209E-07	1.93468E-06	2.01719E-06	1.49993E-06	9.51857E-07
8.9500	7.13248E-07	1.86340E-06	1.95522E-06	1.45893E-06	9.27871E-07
9.0000	6.77279E-07	1.79496E-06	1.89537E-06	1.41922E-06	9.04588E-07
9.0500	6.43195E-07	1.72922E-06	1.83755E-06	1.38073E-06	8.81983E-07
9.1000	6.10892E-07	1.66608E-06	1.78170E-06	1.34344E-06	8.60035E-07
9.1500	5.80273E-07	1.60543E-06	1.72774E-06	1.30700E-06	8.38722E-07
9.2000	5.51248E-07	1.54716E-06	1.67560E-06	1.27197E-06	8.18023E-07
9.2500	5.23729E-07	1.49117E-06	1.62520E-06	1.23802E-06	7.97916E-07
9.3000	4.97636E-07	1.43736E-06	1.57649E-06	1.20509E-06	7.78384E-07
9.3500	4.72893E-07	1.38564E-06	1.52941E-06	1.17316E-06	7.59407E-07
9.4000	4.49426E-07	1.33593E-06	1.48388E-06	1.14220E-06	7.40967E-07
9.4500	4.27167E-07	1.28814E-06	1.43986E-06	1.11216E-06	7.23047E-07
9.5000	4.06051E-07	1.24220E-06	1.39730E-06	1.08303E-06	7.05630E-07
9.5500	3.86019E-07	1.19801E-06	1.35613E-06	1.05476E-06	6.88699E-07
9.6000	3.67011E-07	1.15552E-06	1.31631E-06	1.02734E-06	6.72240E-07
9.6500	3.48974E-07	1.11465E-06	1.27778E-06	1.00072E-06	6.56237E-07
9.7000	3.31856E-07	1.07534E-06	1.24051E-06	9.74891E-07	6.40676E-07
9.7500	3.15608E-07	1.03752E-06	1.20445E-06	9.49819E-07	6.25543E-07
9.8000	3.00185E-07	1.00113E-06	1.16954E-06	9.25480E-07	6.10824E-07
9.8500	2.85543E-07	9.66116E-07	1.13577E-06	9.01851E-07	5.96507E-07
9.9000	2.71641E-07	9.32418E-07	1.10307E-06	8.78907E-07	5.82579E-07
9.9500	2.58441E-07	8.99983E-07	1.07142E-06	8.56627E-07	5.69027E-07
10.0000	2.45905E-07	8.68762E-07	1.04077E-06	8.34988E-07	5.55841E-07