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THE SYNTHETIC KERNELS OF NEUTRON SLOWING-DOWN THEORY AND A RELATED EXPANSION

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ABSTRACT

Various approximate solutions of the integral equation for neutron moderation are obtained by truncating an asymptotic expansion for the collision density, a process which is shown to be equivalent to replacing the scattering kernel by an alternative synthetic kernel which permits "exact" solution of the integral equation. Truncation after one or two terms, and approximation of a constant in one case, yield the classical kernels.

It is shown that the appropriate kernel to be used depends on the particular quantity under investigation. In particular, an illustration is given showing why the Wigner kernel is superior to that of Goertzel and Greuling for calculating resonance absorption.

Difficulties may arise when using the entire non-truncated series. It is shown that, for the sources of practical interest in reactor theory, these difficulties may be removed by splitting the source into two "components", one rapidly decaying and the other not.

When the source is not known sufficiently accurately for this splitting to be done, the series must, in general, be truncated to avoid divergence.

The method of obtaining a kernel by truncation is consistent with the matching of moments method.

CONTENTS

1. INTRODUCTION

Since they were first explicitly noted, apparently by Soodak (1955), the classical synthetic kernels of slowing-down theory have been used quite frequently. Their major application appears to be in obtaining relatively crude first approximations to guide further investigation. They are of some practical use when any basic functions or quantities involved in the problem are not known very accurately. Although referred to as "approximations", each kernel is perhaps better regarded as a simplifying model; the true form of the scattering kernel is replaced by an artificial one, which is often chosen so as to reproduce formally the result of paramount interest. For instance, use of the Wigner kernel is equivalent to use of the Narrow Resonance formula. Generally, the results given by the classical kernels are of an asymptotic nature.

Keane (1961) gave an expansion of the kernel which shows that the classical kernels are special cases of a more general approach. Difficulties can arise in applying this expansion and the major purpose of this paper is to show how these difficulties may be overcome. However, in Section 2, an attempt is made at a systematic discussion to outline the advantages and disadvantages of the classical kernels. In this section, a result concerning the Fermi kernel is established, which is needed in the later work. The following sections deal with the expansion itself, particularly how the source should be modified and the effect of truncation of the series.

2. THE CLASSICAL KERNELS

2.1 Derivation of the Classical Kernels from the Asymptotic Expansion

The collision density $F(u)$ per unit lethargy per c.c. per sec for neutrons slowing down from a distributed source $S(u)$ in a non-absorbing medium satisfies the integral equation:

$$F(u) = S(u) + \int_0^u f(u-u') \, F(u') \, du' \quad , \tag{1}$$

where $f(u)$ is the slowing down kernel. For spherically symmetric scattering in the centre of mass system by nuclei of mass A the kernel is given by:

$$f(u) = \begin{cases} \dfrac{e^{-u}}{1-\alpha} \quad , & 0 < u < \ell n \dfrac{1}{\alpha} \\[2ex] 0 \quad , & u > \ell n \dfrac{1}{\alpha} \quad , \end{cases} \tag{2}$$

where $\alpha = \left(\dfrac{A-1}{A+1}\right)^2$.

Keane (1961) obtained an asymptotic series for $F(u)$ by expanding its Laplace Transform $\overline{F}(p)$ as follows:

$$\overline{F}(p) = \overline{S}(p) \; (1-\overline{f}(p))^{-1} = \overline{S}(p)/(p\xi) + \sum_{k=0}^{\infty} \overline{S}(p) \; \gamma_k \, p^k \tag{3}$$

Here, ξ is the average lethargy gained per collision, $\overline{S}(p)$ is the transform of the source $S(u)$, and $\overline{f}(p)$ is the transform of the kernel. The classical kernels may be obtained by taking $S(u) = \delta(u)$, where $\delta(u)$ is the Dirac delta function, truncating the series (3) after the first one or two terms, solving the resulting equation for $\overline{f}(p)$, and then inverting.

Truncation after one term and solution for $f(u)$ yields:

$$f(u) = \delta(u) - \xi \, \delta'(u). \tag{4}$$

This is the Fermi kernel, which reproduces the results of Fermi Age theory. Truncation after two terms gives:

$$f(u) = \left(1 - \frac{1}{\gamma_0}\right) \delta(u) + (\gamma_0^2 \, \xi)^{-1} \exp\left(-u/\gamma_0 \, \xi\right) \quad , \tag{5}$$

which reproduces the Goertzel– Greuling approximation. The constant γ_0 may be shown to equal $S/(2\xi^2)$ where S is the average squared lethargy gained per collision. The Wigner kernel is obtained by approximating γ_0 and setting it equal to 1 in equation (5).

It is to be noted, as Goertzel and Greuling (1960) have pointed out, that not all these kernels are probability functions; the kernel (5) gives a "negative probability" of zero lethargy gain. Synthetic kernels are not probability functions in general, and should not be referred to or interpreted as such.

2.2 Applicability of the Fermi and Goertzel–Greuling Approximations

The three classical kernels may be arranged in a hierarchy, according to the complexity of the situations for which each is suitable and for which it yields sensible results. If the source has a finite range, then below the (normalised) source, each gives the correct asymptotic value $1/\xi$ of the collision density. If the source is ultimately smoothly decreasing to zero, but not as quickly as F(u) attains its asymptotic form when the source is monoenergetic, then the Goertzel–Greuling kernel is more suitable than the Fermi kernel. This may be demonstrated as follows:

The collision density at u is given by

$$F(u) = \int_0^u S(u')\, P(u-u')du' + S(u) \quad , \tag{6}$$

where P(u) is the Placzek function, expressed as a function of lethargy, that is, (Teichmann 1960),

$$P(u) = \sum_0^\infty \left(-\frac{\alpha}{1-\alpha}\right)^{n+1} (n!)^{-1} \Big[(u+\overline{n+1}\,\ell n\,\alpha)^n \exp\left\{\frac{\alpha(u+\overline{n+1}\,\ell n\,\alpha)}{1-\alpha}\right\} H(u+\overline{n+1}\,\ell n\,\alpha)$$

$$- \alpha^{-1}(u+n\ell n\,\alpha)^n \exp\left\{\frac{\alpha(u+n\ell n\,\alpha)}{1-\alpha}\right\} H(u+n\ell n\,\alpha)\Big] \quad , \quad u > 0 \ , \tag{7}$$

where H(x) is the Heaviside unit function.

Since P(u) assumes its asymptotic value $1/\xi$ for values of u greater than about $-3\ \ell n\ \alpha$, then roughly:

$$F(u) \sim \frac{1}{\xi} \int_0^u S(u')du' + \int_{u+3\ell n\,\alpha}^u S(u')\left(P(u-u') - 1/\xi\right)du' + S(u) \ . \tag{8}$$

In view of the assumed behaviour of S(u), and using a well–known renewal theoretical result, the second term is about:

$$S(u) \int_{u+3\ell n\,\alpha}^u \left(P(u-u') - 1/\xi\right) du' = S(u)(\gamma_0 - 1) \quad , \tag{9}$$

for values of u in the higher–lethargy tail of the source. Hence, ultimately,

$$F(u) \sim \frac{1}{\xi}\int_0^u S(u')du' + \gamma_0\, S(u) \ . \tag{10}$$

This is the expression for F(u) given by the Goertzel–Greuling approach, while the Fermi kernel gives:

$$F(u) \sim \frac{1}{\xi} \int_0^u S(u')du' \ . \tag{11}$$

If S(u) tends to zero very rapidly, that is, faster than P(u) attains its asymptotic value, then both Equations 10 and 11 agree, since the S(u) term in (10) would be negligible. If S(u) decays less rapidly, Equation 10 is the more accurate.

The approximation (10) will obviously be better, the heavier the species in the moderator, since the heavier the moderating species is, the more quickly does P(u) approach $1/\xi$ and the more slowly does S(u) vary by comparison.

2.3 Applicability of the Wigner Kernel

In order to investigate the power of the Wigner kernel, and to illustrate the problems associated with the use of synthetic kernels, a hypothetical badly-behaved source is here introduced. Consider the "source" defined by:

$$S(u) = \begin{cases} 2^{n-1} , & n < u < n + 4^{-n} , \quad n = 0,1,2,\ldots \\ 0 , & \text{otherwise.} \end{cases} \tag{12}$$

This source is a true probability density, since it is always non-negative, and has unit area. However, although its integral converges, S(u) itself does not; in fact, S(u) oscillates with ever-increasing amplitude as u goes to infinity.

Since F(u) is still given by Equation 6, it follows that F(u) will oscillate with S(u). If $n < u < n + 4^{-n}$, and n is very large, then S(u) will be very large also. In relation to S(u), the bounded integral term:

$$\int_0^u S(u') P(u-u')du' \quad ,$$

will be negligible. Hence for such a u:

$$F(u) \sim S(u) \ . \tag{13}$$

Now consider the expressions (10), (11) and the expression for F(u) yielded by the Wigner kernel, namely:

$$F(u) = \frac{1}{\xi}\int_0^u S(u')du' + S(u) \ . \tag{14}$$

There will of course, be an error in Equation 14, but obviously the Fermi, Goertzel-Greuling, and Wigner expressions agree where S(u) is zero, while where S(u) is not zero, only the Wigner kernel will give a reasonable asymptotic approximation the the true value of F(u).

2.4 General Conclusions about the Classical Kernels and the Collision Density

The results of the preceding two subsections may be summarised thus:

S(u) rapidly decaying	:	the Fermi kernel is appropriate;
S(u) decaying, but slowly	:	the Goertzel-Greuling kernel is appropriate;
S(u) not decaying	:	the Wigner kernel is appropriate.

If the source is actually a "negative" one, caused by absorption of neutrons, then the above considerations and examples are seen to be in agreement with the well-known facts that Fermi Age methods work best when absorption is zero, that the Goertzel-Greuling kernel works best for slowly varying absorption, and that the Wigner kernel is best for use inside a resonance, where the "source" is rapidly varying over small intervals.

2.5 The Classical Kernels and the Integral of the Collision Density

It is interesting to note that though the Goertzel-Greuling kernel appears to be inferior to the Wigner one when estimating F(u) for the source (12), it is superior for estimating the integral of F(u). This emphasises the fact that care is needed when choosing a kernel for a particular investigation.

The integral of F(u) may be written:

$$\int_0^u F(u')du' = \left(\frac{u}{\xi}+\gamma_o\right)\int_0^u S(u')du' - \frac{1}{\xi}\int_0^u u'S(u')du' + \int_0^u S(u')\left[\int_{u'}^u P(u''-u')du'' + 1 - \frac{u-u'}{\xi} - \gamma_o\right]du' . \tag{15}$$

Since (by the same renewal result as used earlier):

$$\lim_{u-u'\to\infty}\left[\int_{u'}^u P(u''-u')du'' + 1 - \frac{u-u'}{\xi} - \gamma_o\right] = 0 , \tag{16}$$

then for any given value of ϵ, we can choose a value of k so large (but fixed) that for u > k, we have:

$$\left|\int_0^u S(u')\left[\int_{u'}^u P(u''-u')du'' + 1 - \frac{u-u'}{\xi} - \gamma_o\right]du'\right| = \left|\left(\int_0^{u-k} + \int_{u-k}^u\right)S(u')\left[\int_{u'}^u P(u''-u')du'' + 1 - \frac{u-u'}{\xi} - \gamma_o\right]du'\right| < \epsilon\int_0^{u-k} S(u')du' + B\int_{u-k}^u S(u')du' , \tag{17}$$

where B is an upper bound of the factor of the integrand in square brackets. Since the integral of S(u) converges, (even though S(u) itself does not) we finally see that for large enough values of u, the integral of F(u) has an arbitrarily small difference from:

$$\left(\frac{u}{\xi} + \gamma_o\right)\int_0^u S(u')du' - \frac{1}{\xi}\int_0^u u'S(u')du' . \tag{18}$$

The last expression is precisely the Goertzel-Greuling estimate of the integral, namely:

$$\int_0^u \left(\frac{1}{\xi}\int_0^{u'} S(u'')du'' + \gamma_o S(u')\right)du' .$$

The corresponding Wigner estimate is obtained by putting γ_o equal to 1, and hence has an asymptotic error of about:

$$(1-\gamma_o)\int_0^u S(u')du' .$$

It follows that the Goertzel-Greuling kernel is better for estimating the asymptotic integral F(u) than is the Wigner kernel, at least for positive sources.

The point is that though S(u) as in Equation 12 is oscillating, the integral of S(u) is not. In fact, for large u, the function:

$$\int_0^u S(u')du'$$

varies only slowly, since the increments:

$$\int_{n}^{n+4^{-n}} S(u)du$$

are asymptotically small. Hence one would expect the Goertzel–Greuling kernel to be better here than the Wigner kernel.

The integral of F(u) gives the average number of collisions in (0,u). In general, the moments of the number of collisions in (0,u) will be less sensitive to S(u) than F(u), as they depend on integrals. The reader interested in pursuing such questions from a less physical and more abstract and mathematically rigorous standpoint, is referred to the renewal theory literature. Here we are interested primarily in synthetic kernels.

3. THE SERIES EXPANSION OF THE COLLISION DENSITY

In this section, only sources with a finite range, or which are ultimately exponentially decreasing, will be considered. All the various positive sources considered in reactor theory appear to be of such forms. Also, the constants γ_i will be given their true values, so that the Wigner kernel and the one discussed by Wilkins (1966) will not be considered here.

3.1 Convergence of the Series

Before proceeding further, the convergence of the series (3) will be considered.

Let r denote the radius of convergence of the Laurent series for $\left(1-\bar{f}(p)\right)^{-1}$(expanded about the origin), and consider the source:

$$S(u) = k_1 e^{-k_1 u} \quad , \tag{19}$$

where k_1 is greater than r. Then if the series (3) is formally inverted term by term, we obtain:

$$\frac{1}{\xi}\int_0^u k_1 e^{-k_1 u'} du' + k_1 e^{-k_1 u} \sum_0^\infty (-k_1)^j \gamma_j \, , \quad u > 0 \, , \tag{20}$$

which must diverge. (Of course, the source (19) is so rapidly decaying that the Fermi kernel would suffice for many problems, that is, F(u) would attain its asymptotic value $1/\xi$ fairly rapidly). The point is that if just one pole of $\bar{S}(p)$, say p_1, is such that $|p_1| > r$, the series will diverge. This is illustrated again by the source:

$$S(u) = \frac{1}{2}k_2 e^{-k_2 u} + \frac{1}{2}k_1 e^{-k_1 u} \quad , \tag{21}$$

where k_1 is as before, but k_2 is less than r. In special cases, the series may converge, as is shown by the source with the same form as (19), but with k_2 substituted for k_1.

The above examples show that the series (3) may be expected to diverge in many cases. Nevertheless, it will be demonstrated below that the whole series can be taken, and yields useful results, provided suitable modifications are made.

3.2 The Expansion for a Particular Type of Source

To generalise slightly, and for later purposes, consider the source:

$$S(u) = k^{n+1}(n!)^{-1} u^n e^{-ku} \quad , \tag{22}$$

where n is non-negative and integral. Formal substitution of this source into Equation 5 and inversion term by term, yields:

$$F(u) = k^{n+1}(n!)^{-1}\left\{\frac{1}{\xi}\int_0^u u'^n\, e^{-ku'}\,du'\right.$$

$$+ \gamma_0\, u^n\, e^{-uk}$$

$$+ \gamma_1\, u^n(-k)e^{-uk} + \gamma_1\, n u^{n-1}\, e^{-uk}$$

$$+ \gamma_2\, u^n(-k)^2\, e^{-uk} + \gamma_2\, 2nu^{n-1}(-k)e^{-uk} + \gamma_2 n(n-1)u^{n-2}e^{-uk}$$

$$\vdots$$

$$+ \gamma_n\, u^n(-k)^n e^{-uk} + \gamma_n \binom{n}{1} nu^{n-1}(-k)^{n-1} e^{-uk} + \gamma_n \binom{n}{2} n(n-1)u^{n-2}(-k)^{n-2} e^{-uk} +$$

$$\ldots + \gamma_n \binom{n}{n} n!\, u^0\, e^{-uk}$$

$$+ \gamma_{n+1}\, u^n(-k)^{n+1} e^{-uk} + \gamma_{n+1}\binom{n+1}{1} nu^{n-1}(-k)^n\, e^{-uk} + \gamma_{n+1}\binom{n+1}{2} n(n-1)u^{n-2}$$

$$\cdot\, (-k)^{n-1} e^{-uk} + \ldots + \gamma_{n+1}\binom{n+1}{n} n!\, u^0(-k)e^{-uk}$$

$$\vdots$$

$$\left. + \ldots \right\},\quad u > 0\ . \tag{23}$$

By the ratio test or otherwise, this series, and the series obtained from each column, converges absolutely if $k < r$ and diverges if $k > r$. Summing by columns gives:

$$F(u) = k^{n+1}(n!)^{-1}\left\{\frac{1}{\xi}\int_0^u u'^n\, e^{-ku'}\,du' + u^n\, e^{-uk}\left[\bar{P}_o(-k) + \frac{1}{k\xi}\right] + u^{n-1}\, e^{-uk}\left[\bar{P}_o'(-k) + \frac{1!}{k^2\xi}\right]\binom{n}{1} + \right.$$

$$\left. u^{n-2}\, e^{-uk}\left[\bar{P}_o''(-k) + \frac{2!}{k^3\xi}\right]\binom{n}{2} + \ldots + e^{-uk}\left[\bar{P}_o^{(n)}(-k) + \frac{n!}{k^{n+1}\xi}\right]\right\}$$

$$= \frac{1}{\xi} + k^{n+1}(n!)^{-1} \sum_0^n u^{n-j}\, e^{-uk}\, \bar{P}_o^{(j)}(-k)\binom{n}{j}$$

$$= \frac{1}{\xi} + k^{n+1}(n!)^{-1}\left\{\frac{d^n}{dp^n}\, e^{pu}\left(1 - \bar{f}(p)\right)^{-1}\right\}_{p=-k}\ , \tag{24}$$

where $P_o(u)$ is the sum of $P(u)$ and $\delta(u)$.

3.3 Singularities of the Transforms of the Sources in Slowing-Down Theory

It appears safe to assume that the transforms of the sources used in neutron slowing-down theory are everywhere analytic or have only real poles for singularities. The transform is everywhere analytic when the source has a finite range, since then the transform:

$$\bar{S}(p) = \int_0^\infty e^{-pu}\, S(u)\,du\ , \tag{25}$$

converges for all p, and the result follows (Keane 1965). The source of greatest importance, that of Cranberg et al. (1956):

$$S(u) = Ce^{-u}\, e^{-ae^{-u}} \sinh(be^{-u/2})\ , \tag{26}$$

is an example of a source with only real poles for singularities of its transform, for the latter is given by:

$$\bar{S}(p) = C \sum_{n=0}^{\infty} (n!)^{-1} \sum_{\substack{j=0 \\ (j \neq n-2k)}}^{n} \binom{n}{j} (-a)^j \, b^{n-j} \left(p + 1 + \frac{n+j}{2}\right)^{-1} . \tag{27}$$

3.4 Modification of the Series to Obtain Asymptotically Valid Results

The expansion (3) is an expansion about $p = 0$, so the Tauberian theory indicates that any approximations it yields will generally be asymptotic ones.

Suppose S(p) has poles at $p_1, \ldots p_s$, inside the circle $|p| < r$, with the residue at: $p_j \, (1 \leqslant j \leqslant s)$ of $(2\pi i)^{-1} e^{pu} \bar{S}(p)$ being equal to $K_j \, u^{n_j} e^{p_j u}$.

Write $S(u) = \sum_{1}^{s} K_j \, u^{n_j} e^{p_j u} + Q(u) = R(u) + Q(u)$. (28)

Some of the K_j could, conceivably, be negative, while Q(u) is not necessarily always positive. Nevertheless, we can think of each term in (28) as a source, perhaps of both "positive" and "negative" neutrons. The collision density set up by S(u) will then be the algebraic sum of the densities set up by the positive and negative parts. For R(u), we have asymptotically:

$$F(u) = \frac{1}{\xi} \int_0^{\infty} R(u) du + \sum_{1}^{s} K_j \left\{ \frac{d^{n_j}}{dp^{n_j}} \, e^{pu} \left(1 - \bar{f}(p)\right)^{-1} \right\}_{p = p_j} , \tag{29}$$

by taking the residues at the poles $p = 0$, $p = p_j$, $1 \leqslant j \leqslant s$, of $\frac{1}{2\pi i} e^{pu} L\left(R(u)\right) \left(1 - \bar{f}(p)\right)^{-1}$; according to the Tauberian theory. By the previous work (Equation 24) and by the definition of R(u) as a linear combination of a finite number of terms of the form (22), the right hand side is seen to be:

$$\frac{1}{\xi} \int_0^u R(u') du' + \sum_{k=0}^{\infty} \gamma_k \, R^{(k)}(u) . \tag{30}$$

Now consider Q(u). Any poles of $L\left(Q(u)\right)$ lie outside the circle $|p| < r$, so Q(u) is relatively rapidly decaying. Hence for Q(u), we use the Fermi result and put:

$$F(u) = \frac{1}{\xi} \int_0^u Q(u') du' . \tag{31}$$

Addition of (30) and (31) gives the asymptotic expression:

$$F(u) = \frac{1}{\xi} \int_0^u S(u') du' + \sum_{0}^{\infty} \gamma_j \, R^j(u) . \tag{32}$$

Equation 32 differs from the asymptotic expression derived from Laplace transform theory, namely:

$$F(u) = \frac{1}{\xi} \int_0^{\infty} S(u') du' + \sum_{1}^{s} K_j \left\{ \frac{d^{n_j}}{dp^{n_j}} \, e^{pu} \left(1 - \bar{f}(p)\right)^{-1} \right\}_{p = p_j} \tag{33}$$

by $\frac{1}{\xi} \int_u^{\infty} Q(u') du'$. Because of the comparatively rapid decay of Q(u), this difference is negligible.

Thus, provided some care is exercised, the expansion (3), yields worthwhile higher order approximations to F(u). The expression (32) is obviously more accurate, the larger the value of r .

If the source is finite, then R(u) does not appear in Equation 32. However, the finite sources of interest to reactor theory all appear to have extremely short ranges, so that F(u) assumes its asymptotic form $\frac{1}{\xi}$ extremely rapidly, and Equation 32 (with R(u) = 0) again applies.

3.4.1 A summary of the method in the preceding sub-section

Briefly, the source S(u) is split into two parts, R(u) and Q(u). The function R(u) is obtained by summing the residues of $(2\pi i)^{-1} e^{pu}\bar{S}(p)$ inside the circle $|p| < r$, and $Q(u) = S(u) - R(u)$. Thus R(u) generally represents a "component" of the source which is more slowly decaying than $P(u) - 1/\xi$, and the expansion (3) may be validly applied to it. The function Q(u) represents the remaining "component", and it is decaying so rapidly that we can justifiably use the Fermi kernel to obtain its contribution to the asymptotic value of F(u).

4. TRUNCATION OF THE SERIES

If the source is known only roughly, there will be no certainty about the poles of its transform. Then it will not be possible to determine R(u) and Q(u), and there will be no assurance that the series (3) converges – and in general, it won't. In such a case, we can nevertheless attempt to obtain higher order approximations to F(u) by taking just a finite number of terms of the series (3). In terms of R(u) and Q(u), this gives:

$$F(u) = \frac{1}{\xi} \int_0^u S(u')\,du' + \sum_0^n \gamma_j\, R^{(j)}(u) + \sum_0^n \gamma_j\, Q^{(j)}(u) \quad . \tag{34}$$

Because of the more rapid decay of the terms arising from Q(u), Equation 34 is in asymptotic agreement with the expression obtained from the first n + 2 terms on the right hand side of Equation 32. Thus use of the finite approximation (34) is asymptotically equivalent to truncation of the series (32), and does away with the problem of determining R(u). The magnitude of the error caused by such truncation will vary with the species. If the moderating species is very heavy the series will converge fairly rapidly, and the error will not be important in relation to, for instance, experimental errors.

Retaining only the first n + 2 terms is formally and effectively equivalent to choosing a kernel with transform given by:

$$\left(1 - \bar{f}(p)\right)^{-1} = \frac{1}{p\xi} + \sum_0^n \gamma_j\ p^j \ . \tag{35}$$

Such a kernel reproduces the correct asymptotic value of

$$\int_0^u \int_0^{u_1} \int_0^{u_2} \int_0^{u_n} F(u_{n+1})\ du_{n+1}\, du_n \ldots du_2\, du_1 \ , \tag{36}$$

for a monoenergetic source. For the transform of the functions,(36) is:

$$\frac{1}{p^{n+2}\,\xi} + \frac{\gamma_0}{p^{n+1}} + \ldots + \frac{\gamma_n}{p} + \gamma_{n+1} + \gamma_{n+2}\, p + \ldots , \tag{37}$$

and the Tauberian theory indicates that the asymptotic value of (36) may be obtained by formally inverting the terms in (37) with negative powers of p. Thus, for instance, as seen above, the Goertzel–Greuling kernel gives the correct asymptotic value of:

$$\int_0^u F(u')\ du' \ . \tag{38}$$

Choosing a kernel by this method is consistent with Soodak's "matching of moments" method, for γ_n is a function of the first n + 2 moments of the lethargy gain, so that the kernel determined by the transform (35) gives the correct first n + 2 moments. (The constants γ_n may also be expressed as certain integrals involving P(u) – see the appendix).

5. CONCLUDING REMARKS

It is implicit in the series (3) as it stands, that only a finite number of terms should be taken, otherwise the series will not in general converge. By truncating the series, one obtains corrections to the usual formulae. Taking a finite number of terms is formally equivalent to adoption of a synthetic kernel. For the positive sources of practical importance in reactor theory, such an approach always yields correct limiting results.

Although, in general, it is not valid to retain all terms of the series (3), one may do so if S(u) is split into the sum of the dominant term R(u) and the remainder Q(u). Then if R(u) is used, instead of S(u), in all but a finite number of terms (which terms must include any involving the integral of the source) a valid asymptotic approximation is obtained. Use of a synthetic kernel is equivalent to truncation of the corresponding series.

A similar analysis obviously applies to other functions besides the collision density. As an example, it is easy to see that analogous results apply to the average number of collisions in (0,u), namely, the integral from zero to u of the collision density.

The analysis was dependent upon the Laplace transform, though, as shown in the appendix, the general truncated formula may be obtained in other ways. It is to be noted that there is a resultant loss of information owing to a source such as that of Cranberg et al. being regarded as having finite range; the phrase "below the source" occurs frequently in the literature. Although it may be physically valid to regard the source as having just a finite range, treating it as so in any analysis is equivalent to ignoring all the poles of its transform. This will affect the mathematical formalism in any analytic investigation.

The analysis may also be applied to mixtures, provided all the cross sections are similarly varying, so that Laplace transform techniques may be applied.

6. REFERENCES

Cranberg, L., Frye, G., Nereson, N., and Rosen, L.. (1956). - Physics Review, 103:662.

Glasstone, S. and Edlund, M. (1942). - The Elements of Nuclear Reactor Theory. Van Nostrand, New York.

Goertzel, G. and Greuling, E. (1960). - Nucl. Sci. Eng. 7:69.

Keane, A. (1961). - Nucl. Sci. Eng. 10: 117.

Keane, A. (1965). - Integral Transforms. Science Press, Sydney.

Soodak, H. (1955). - Reactor Physics Handbook, p. 367. McGraw-Hill.

Teichmann, T. (1960). - Nucl. Sci. Eng. 7: 292.

Wilkins, C.A. (1966). - Nucl. Sci. Eng. 24: 89.

APPENDIX 1

AN ALTERNATIVE APPROACH

The basic equation for F(u) is Equation 1, for which an exact solution is given by Equation 8. It is instructive to attack the latter equation by an approach similar to that used by Goertzel and Greuling (see Glasstone and Edlund 1942). Take k large enough so that for $u > k$, the difference from zero of:

$$u^j \left(P(u) - \frac{1}{\xi}\right), \quad j = 0, 1, \ldots n, \qquad \text{A}^1$$

may be safely ignored. Then assume that over the interval (u–k,u), S(u) is varying sufficiently slowly for it to be approximated by:

$$S(u') = S(u) + (u' - u)\, S'(u) + \ldots (u' - u)^n\, S^{(n)}(u)/n! \qquad \text{A}^2$$

Equation 8 becomes:

$$F(u) \sim \frac{1}{\xi} \int_0^u S(u')\,du' + S(u) + \sum_{j=0}^{n} S^{(j)}(u) \int_{u-k}^{u} (u' - u)^j \left(P(u-u') - \frac{1}{\xi}\right) du'/j! \;,$$

$$= \frac{1}{\xi} \int_0^u S(u')\,du' + \gamma_0 S(u) + \sum_{1}^{n} S^{(j)}(u)\,(-1)^j \int_0^k y^{(j)} \left(P(y) - \frac{1}{\xi}\right) dy/j! \;. \qquad \text{A}^3$$

Because of the choice of k, each of the integrals involving P(y) will effectively have its asymptotic value. Now:

$$L\left(\int_0^u (-1)^j y^j \left(P(y) - \frac{1}{\xi}\right) dy\right)/j! = \frac{1}{p} \frac{d^j}{dp^j} \left[L\left(P(y)\right) - \frac{1}{p\xi}\right]/j! \;. \qquad \text{A}^4$$

For small values of p, the right–hand term is:

$$\frac{1}{p}\left(\frac{(-1)^j}{p^{j+1}\xi} + \gamma_j + \gamma_{j+1}(j+1)p + \ldots - \frac{(-1)^j}{p^{j+1}\xi}\right), \qquad \text{A}^5$$

so, asymptotically:

$$(-1)^j \int_0^u y^j \left(P(y) - \frac{1}{\xi}\right) dy = \gamma_j \;. \qquad \text{A}^6$$

Hence, approximately:

$$F(u) = \frac{1}{\xi} \int_0^u S(u')\,du' + \sum_{0}^{n} S^{(j)}(u)\,\gamma_j \;, \qquad \text{A}^7$$

which is precisely the same as the expression obtained by inverting the first n +2 terms on the righthand side of Equation 3. Note that only a finite number of terms have been taken. In general, unless it is truncated or unless S(u) and its derivatives are modified, the series (3) will diverge.

This approach illustrates the fact that if S(u) is known only experimentally, then the number of terms to be taken in the truncated series may be judged from a determination of the derivatives, if the first few of these can be determined reasonably accurately.