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ON THE NON-INVARIANCE OF DISTRIBUTIONS OF REACTION MATRIX PARAMETERS UNDER CHANGES IN BOUNDARY CONDITIONS

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ABSTRACT

Contrary to current opinion, the statistical distributions of level spacings and reduced widths when applied to the reaction matrix, are not invariant under changes in the boundary condition matrix or the matching radius. General arguments are given, together with specific examples which violate the invariance requirements. We conclude that it is the parameters of the collision matrix which should be analysed and considered as the invariant parameters.

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ANALYTICAL SOLUTION; BOUNDARY CONDITIONS; EIGENVALUES; INVARIANCE PRINCIPLES; LAPLACE TRANSFORMATION; LEVEL WIDTHS; NUCLEAR RADII; NUCLEAR REACTIONS; PORTER–THOMAS DISTRIBUTION; R MATRIX; S MATRIX; SPATIAL DISTRIBUTION; STATISTICS; WIGNER DISTRIBUTION; WIGNER–EISENBUD THEORY

CONTENTS

Note in Proof

Equation (72) is valid only for slowly – varying $\langle \gamma^2 \rangle$. The exact result is obtained by substituting (77) into (71) to give

$$\langle D(a) \rangle = \langle D(a_0) \rangle / \exp\,[f(a) - f(a_0)]$$

where

$$f(a) = \frac{a}{C + 4ma^3} - \frac{\alpha}{6C}\,\ell n\left[\frac{(a+\alpha)^2}{a^2 + \alpha a + \alpha^2}\right] - $$

$$- \frac{1}{\sqrt{3}}\,\frac{\alpha}{C}\left[\tan^{-1}\left(\frac{a\sqrt{3}}{2\,\alpha - a}\right)\right]$$

$$\alpha = \sqrt[3]{\frac{C}{4m}} .$$

Choosing $a_0 = 1$, $C = 1$, $4m = 1$, $\langle \gamma^2 \rangle = \frac{1}{2}$

We get the values;

a	1.0	1.25	1.5	2.0	3.0	5.0	∞
$\langle D \rangle$	1.0	0.830	0.706	0.563	0.453	0.393	0.347

1. INTRODUCTION

The parameters which occur in statistical theories of nuclear reactions usually depend on the statistical properties of the parameters in the reaction matrix, R. For this reason, the statistics of R-matrix parameters have been widely investigated (Lynn, 1968). Distribution laws for level spacings and reduced widths have been derived by Wigner (1956) and Porter and Thomas (1956) respectively.

Though no formal proof has ever been given, it is usually assumed that the statistical distributions of R-matrix parameters are independent of the boundary conditions used in the definition of the R-matrix. Lane and Thomas (1958) claimed this on physical grounds and supported their view with a plausibility argument. Teichmann and Wigner (1952), by considering the effect of changes in the boundary conditions, suggested the existence of correlations between the level spacings and the reduced widths. On the other hand Lane and Thomas dismiss this argument as being based on mere speculation.

More recently, Moldauer (1964) investigated some of the statistical properties of the R-matrix numerically and concluded that they were not independent of the boundary conditions. He discovered that by fixing the distributions of the reaction matrix parameters, and varying the boundary conditions, the distributions of parameters of the collision matrix varied; this should not occur. Unfortunately, changes in statistical distributions due to changes in boundary conditions are not investigated easily by computer experiments. The main difficulty is the restriction to a finite number of levels. Initially, an R-matrix many be constructed such that the distributions of the level spacings and reduced widths are independent of the sampling range. However, after changing the boundary conditions the resulting R-matrix no longer has this property. Distributions obtained by sampling the poles near the extremities of the range are generally different from those obtained by sampling near the centre of the range.

In this paper, the problem of whether or not the reaction matrix parameters vary with boundary conditions is investigated analytically for a single channel R-matrix by considering the moment generating functions of the distributions. The statistics of R-matrix parameters are generally not invariant under changes in boundary conditions. This is demonstrated by considering first and second order variations in the moments and distributions due to variations in boundary conditions.

2. FORMULATION OF THE PROBLEM

In the Wigner-Eisenbud (1947) R-matrix formalism, the single channel scattering matrix is expressed in terms of the R-function

$$R(E) = \sum_{\lambda} \frac{\gamma_\lambda^2}{E_\lambda - E} + R\infty \qquad \ldots(1)$$

where γ_λ^2 = reduced widths,

E_λ = resonance poles,

$R\infty$ = residual constant.

The eigenvalues E_λ of the nuclear Hamiltonian are defined by imposing on the eigenfunctions at a chosen radius a, boundary conditions which are characterized by a real constant B,

$$\left[\frac{d\psi_\lambda(r)}{dr} \right]_{r=a} = \frac{B}{a} \psi_\lambda(a) \qquad \ldots(2)$$

where $\psi_\lambda(E_\lambda,r) = \psi(E_\lambda,r)$ with the E_λ as eigenfunctions of (2),

$\psi(E,r)$ = the wave function in the region of interaction.

The reduced width amplitudes are defined as

$$\gamma_\lambda^2 = \frac{\hbar^2}{2Ma} \psi_\lambda^2(a) \qquad \ldots(3)$$

where M = the reduced mass of the system.

If R_o is an R-function corresponding to the boundary condition B_o, then the R-function for a different boundary condition B is given by the relation (Lane and Thomas)

$$R = R_o(1 - (B-B_o)R_o)^{-1} \qquad \ldots(4)$$

The collision function S is given by

$$S = \Omega^2(1 - (L-B)R)^{-1} (1 -(L^*-B)R), \qquad \ldots(5)$$

where $\Omega = e^{i\phi}$,

ϕ = hard sphere phase shift,

$L = S + iP$,

S = level shift,

P = penetration factor.

The function (5) is invariant under the transformation (4). It should also be invariant with respect to changes in the arbitrarily chosen radius, a. We can express these invariances in the equations

$$\frac{\partial S}{\partial B} = 0 \qquad \text{...(6a)}$$

$$\frac{\partial S}{\partial a} = 0. \qquad \text{...(6b)}$$

The positions E'_λ of the poles in R are given by the solutions of the equation

$$1 - (B-B_o)R_o = 0 \qquad \text{...(7)}$$

and the residues are

$$\gamma'^2_\lambda = \left[(B-B_o)^2 \frac{dR}{dE} \right]^{-1}_{E=E'_\lambda} . \qquad \text{...(8)}$$

Equation (4) was shown by Teichmann (1950) to be equivalent to a differential form

$$\frac{\partial R}{\partial B} = R^2 . \qquad \text{...(9)}$$

By equating residues at the poles on either side of equation (9) we find

$$\frac{\partial E_\lambda}{\partial B} = -\gamma^2_\lambda , \qquad \text{...(10a)}$$

$$\frac{\partial \gamma^2_\lambda}{\partial B} = 2\gamma^2_\lambda \sum_{\mu \neq \lambda} \frac{\gamma^2_\mu}{E_\mu - E_\lambda} . \qquad \text{...(10b)}$$

These relations also follow from (6a) and (5).

In order to study the statistics of R-matrix parameters, Wigner (1956) introduced the concept of the statistical R-function in which the reduced widths γ^2_λ and level spacings D_λ, where

$$D_\lambda = E_{\lambda+1} - E_\lambda \qquad \text{...(11)}$$

have definite distributions which are independent of the sampling range. We shall consider the statistical R-function defined by equation (1) in which the summation over λ extends from $-\infty$ to $+\infty$. We suppose there is some prescribed order in which positive and negative terms in equation (1) are to be summed.

The effects of changes in the boundary conditions on the statistics of the parameters is most conveniently investigated by means of moment generating functions. The moment generating function (Cramer, 1946) of a distribution P(x) is the laplace transform

$$\overline{P}(S) = \int_0^\infty e^{-Sx} P(x)\, dx \qquad \ldots(12)$$

so that when $\overline{P}(S)$ is expanded in ascending powers of S

$$\overline{P}(S) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M_k S^k \qquad \ldots(13)$$

where M_k is the k th moment of the distribution. If the series (13) is absolutely convergent for some S>0, the distribution is uniquely determined by its moments. Therefore the dependence of the distributions of level spacings and widths on the boundary conditions is completely determined by the dependence of the moments upon B and a.

Some common examples of the above which are used in calculations are

a) <u>Wigner distribution</u>

$$P(D)\, dD = \frac{\Pi D}{2\overline{D}^2} e^{-\Pi D^2/(4\overline{D})^2} \qquad \ldots(14)$$

$$\overline{P}_D(S) = 1 - \overline{D}S\, e^{(\overline{D}^2 S^2)/\Pi} \operatorname{erfc}\left(\frac{\overline{D}S}{\sqrt{\Pi}}\right) \qquad \ldots(15)$$

$$\text{where } \operatorname{erfc}(x) = \frac{2}{\sqrt{\Pi}} \int_x^\infty e^{-t^2}\, dt\ .$$

The Taylor expansion of (15) gives all of the moments of the distribution in terms of $\overline{D}$ alone.

$$P_D(S) \approx 1 - \overline{D}S + \frac{2}{\Pi} \overline{D}^2 S^2 + \ldots \qquad \ldots(16a)$$

$$M_o(D) = 1,\ M_1(D) = \overline{D},\ M_2(D) = \frac{4\overline{D}^2}{\Pi}\ . \qquad \ldots(16b)$$

b) <u>Repulsive exponential level distribution</u>

$$P(D)\, dD = \frac{4}{\overline{D}^2} D\, e^{-2D/\overline{D}}\ , \qquad \ldots(17)$$

$$\bar{P}_D(S) = (1 + \frac{S\bar{D}}{2})^{-2} \qquad \text{...(18)}$$

$$\approx 1 - \bar{D}S + \frac{3}{4}\bar{D}^2S^2 \; ....$$

giving

$$M_o\,(D) = 1,\; M_1\,(D) = \bar{D}$$

$$M_2\,(D) = \langle D^2 \rangle$$

$$= \frac{3}{2}\bar{D}^2\,,\quad \text{from (13)}\;. \qquad \text{...(19)}$$

c) Porter-Thomas distribution of reduced widths

$$P(\gamma^2)\; d\gamma^2 = (2\gamma^2\Pi\,\overline{\gamma^2}\,)^{-\frac{1}{2}}\; e^{-\gamma^2/(2\,\overline{\gamma^2})}\; d\gamma^2. \qquad \text{...(20)}$$

$$P_\gamma(S) = (1 + 2S\,\overline{\gamma^2})^{-1} \qquad \text{...(21)}$$

$$\approx 1 - \overline{\gamma^2}\,S + \frac{3}{2}\,(\overline{\gamma^2})^2\,S^2 - \ldots$$

which yields

$$M_o(\gamma^2) = 1,\quad M_1(\gamma^2) = \overline{\gamma^2}\,,\quad M_2(\gamma^2) = 3(\overline{\gamma^2})^2\,, \qquad \text{...(22)}$$

d) Exponential distribution of reduced widths

$$P(\gamma^2)\; d\gamma^2 = \frac{1}{\overline{\gamma^2}}\;.\; e^{-\gamma^2/\overline{\gamma^2}}\; d\gamma^2\,, \qquad \text{...(23)}$$

$$\bar{P}_\gamma(S) = (1 + S\,\overline{\gamma^2})^{-1} = 1 - \overline{\gamma^2}S + (\overline{\gamma^2})^2S^2 + \ldots \qquad \text{...(24)}$$

from which we find

$$M_o(\gamma^2) = 1,\quad M_1(\gamma^2) = \overline{\gamma^2}\,,\; M_2(\gamma^2) = 2(\overline{\gamma^2})^2\,. \qquad \text{...(25)}$$

3. THE LEVEL SPACING DISTRIBUTION

Suppose that when $B = B_o$, D_λ and γ^2_λ are uncorrelated. The dependence of the level spacings on B is obtained from equation (10a) as

$$\frac{\partial D_\lambda}{\partial B} = \gamma^2_\lambda - \gamma^2_{\lambda+1}\;. \qquad \text{...(26)}$$

Using the fact that

$$\left\langle \frac{\partial D_\lambda}{\partial B} \right\rangle = \frac{\partial < D>}{\partial B} , \qquad \ldots(27)$$

where < > denotes the average over λ, we find

$$\frac{\partial < D>}{\partial B} = < \gamma_\lambda^2 - \gamma_{\lambda+1}^2 > = < \gamma^2 > - < \gamma^2 > = 0. \qquad \ldots(28)$$

It can be shown that the operation < > commutes with $\frac{\partial}{\partial B}$. Similarly, by repeated differentiation of (28) we can show that for all n

$$\frac{\partial^n < D>}{\partial B^n} = \left\langle \frac{\partial^{n-1} \gamma_\lambda^2}{\partial B^{n-1}} - \frac{\partial^{n-1} \gamma_{\lambda+1}^2}{\partial B^{n-1}} \right\rangle = 0 . \qquad \ldots(29)$$

This result reflects the well known fact that the average density of levels is independent of B (Lane and Thomas). However, it is not necessarily independent of a. Furthermore, we can show that the first derivatives of all higher moments vanish. Thus

$$\begin{aligned} \left\langle \frac{\partial D_\lambda^n}{\partial B} \right\rangle &= \left\langle n\, D_\lambda^{n-1} \, (\gamma_\lambda^2 - \gamma_{\lambda+1}^2) \right\rangle \\ &= n \left\langle D_\lambda^{n-1} > < \gamma_\lambda^2 - \gamma_{\lambda+1}^2 \right\rangle \\ &= 0 , \end{aligned} \qquad \ldots(30)$$

since D_λ and γ_λ^2 are uncorrelated.

Needless to say, if the moments are to be truly independent of B, then all higher derivatives of all moments must be identically zero,

that is $$\frac{\partial^n M_k}{\partial B^n} = 0 \qquad \ldots(31)$$

for all (n,k). This is certainly not the case in general as can be seen by considering the second derivative of the second moment M_2 (D).

From

$$\frac{\partial^2 D_\lambda^2}{\partial B^2} = 2(\gamma_\lambda^2 - \gamma_{\lambda+1}^2)^2 + 2D_\lambda \left(\frac{\partial \gamma_\lambda^2}{\partial B} - \frac{\partial \gamma_{\lambda+1}^2}{\partial B} \right) \qquad \ldots(32)$$

we obtain

$$\frac{\partial^2 M_2}{\partial B^2} = \frac{\partial^2 < D^2 >}{\partial B^2} = 2 \left\langle (\gamma_\lambda^2 - \gamma_{\lambda+1}^2)^2 \right\rangle + $$

$$+ 2 \left\langle D_\lambda \left(\frac{\partial \gamma_\lambda^2}{\partial B} - \frac{\partial \gamma_{\lambda+1}^2}{\partial B} \right) \right\rangle \qquad \ldots(33)$$

The first term on the right in equation (33) is non-zero and its value depends only on the distribution of the reduced widths. In fact

$$2 < (\gamma_\lambda^2 - \gamma_{\lambda+1}^2)^2 > = 4 \ (< \gamma^4 > - < \gamma^2 >^2) \quad . \qquad \ldots(34)$$

In general the second term will not cancel the first term as its value depends not only on the distribution of widths but also on the distribution of level spacings. The value of the second term can be determined. Using the fact that at B_o, widths and spacings are uncorrelated, we obtain from (10b)

$$\left\langle D_\lambda \frac{\partial \gamma_\lambda^2}{\partial B} \right\rangle = < \gamma_\lambda^2 >^2 \left\langle D_\lambda \sum_{\mu \neq \lambda} \frac{1}{E_\mu - E_\lambda} \right\rangle \quad . \qquad \ldots(35)$$

For those terms in (19) for which $\mu = \lambda - k$, $k > 0$, we have

$$\left\langle \frac{D_\lambda}{E_{\lambda-k} - E_\lambda} \right\rangle = - \left\langle \frac{D_\lambda}{\sum_{i=1}^{k} D_{\lambda-i}} \right\rangle = - < D > . \left\langle \frac{1}{\sum_{i=1}^{k} D_{\lambda-i}} \right\rangle$$

$$\ldots(36)$$

Writing $$\frac{1}{\sum_{i=1}^{k} D_{\lambda-i}} = \int_o^\infty \prod_{i=1}^{k} e^{-D_{\lambda-i} S} \, dS \qquad \ldots(37)$$

we find $$\left\langle \frac{D_\lambda}{E_{\lambda-k} - E_\lambda} \right\rangle = - < D > \int_o^\infty \left[\overline{P}_D(S) \right]^k dS \qquad \ldots(38)$$

where $\overline{P}_D(S)$ is the moment generating function of the distribution of level spacings.

Similarly, those terms in (35) which have $\mu = \lambda + k$, $k > 0$, can be written as

$$\left\langle \frac{D_\lambda}{E_{\lambda+k} - E_\lambda} \right\rangle = \left\langle \frac{D_\lambda}{\sum\limits_{i=o}^{k-1} D_{\lambda+i}} \right\rangle = \frac{1}{k} , \qquad \ldots(39)$$

which may also be expressed as an integral

$$< \frac{D_\lambda}{E_{\lambda+k} - E_\lambda} > = < D > \int_o^\infty e^{-k< D> S} \, dS . \qquad \ldots(40)$$

Therefore, combining (40) and (38), (35) becomes

$$\left\langle D_\lambda \frac{\partial \gamma_\lambda^2}{\partial B} \right\rangle = < \gamma^2 >^2 < D> \left\{ \sum_{k=1}^{\infty} \int_o^\infty dS \left(e^{-k< D> S} - (\overline{P}_D(S))^k \right) \right\}$$

$$\ldots(41)$$

The two summations in (40) must of course be carried out simultaneously in the same order as used to define the summation in equation (1). To avoid difficulties arising from the lower limits of the integrations, we replace them by a small positive ϵ and consider the limit as ϵ tends to zero. Since, for all $S>\epsilon$ the two series in equation (41) are absolutely convergent, the summations may be taken inside the integrals. Summing the series separately we obtain when $\epsilon \to 0$

$$\left\langle D_\lambda \frac{\partial \gamma_\lambda^2}{\partial B} \right\rangle = < \gamma^2 >^2 < D> \int_o^\infty \left\{ \frac{e^{-< D> S}}{1 - e^{-< D> S}} - \frac{\overline{P}_D(S)}{1 - \overline{P}_D(S)} \right\} .$$

$$\ldots(42)$$

In terms of averages we obtain finally

$$\frac{\partial^2 < D^2 >}{\partial B^2} = 4 \left(< \gamma^4 > - < \gamma^2 >^2 \right) + 4 < \gamma^2 >^2 \; C(< D >) \; ; \qquad \ldots(43)$$

$$C(< D>) = < D> \int_o^\infty \left\{ \frac{e^{-< D> S}}{1 - e^{-< D> S}} - \frac{\overline{P}_D(S)}{1 - \overline{P}_D(S)} \right\} ds \; . \qquad \ldots(44)$$

We note that the integral on the right of (44) exists only if $\overline{P}_D(S)$ tends to zero faster than 1/S, as $S \to \infty$. For large S, we may write

$$\overline{P}_D(S) = \int_0^\infty e^{-SD} \left[P(0) + D \left(\frac{dP}{dD}\right)_{D=0} + \dots \right] dD$$

$$= \frac{P(0)}{S} + \frac{1}{S^2} \left(\frac{dP}{dD}\right)_{D=0} + \dots \qquad \dots(45)$$

Therefore, the above analysis holds only if

$$\lim_{D \to 0} P(D) = 0 \qquad \dots(46)$$

which is an expression of the fact that levels repel each other.

As an example, consider the repulsive exponential distribution (17). Using (18) in (44) we find

$$\frac{\partial^2 < D^2 >}{\partial B^2} = 4\,(<\gamma^4> - <\gamma^2>^2) - 4\,\ln(4)\,<\gamma^2>^2$$

$$= 4 <\gamma^4> - 9.55 <\gamma^2>^2 \quad . \qquad \dots(47)$$

If, furthermore, we use the exponential reduced width distribution (24), we find

$$<\gamma^4> = <(\gamma^2)^2> = 2(\overline{\gamma^2})^2$$

and
$$\frac{\partial^2 < D^2 >}{\partial B^2} = -1.55\ (\gamma^2)^2 \quad . \qquad \dots(48)$$

We note that often $\overline{P}_D(S)$ has the form $\overline{P}_D(\overline{D}S)$ and therefore we obtain

$$C(< D >) = \int_0^\infty \left\{ \frac{e^{-u}}{1 - e^{-u}} - \frac{\overline{P}_D(u)}{1 - \overline{P}_D(u)} \right\} du \qquad \dots(49)$$

which is independent of $<D>$, and is therefore constant. Similarly, if $P(\gamma^2)$ can be written as a function $P(\gamma^2/\overline{\gamma^2})$ then $\overline{P}_\gamma(S)$ has the form $\overline{P}_\gamma(\overline{\gamma^2}\,S)$ and the second moment of the distribution is a constant times $(\overline{\gamma^2})^2$. It follows that

$$\frac{\partial^2 < D^2 >}{\partial B^2} = (\text{constant}) \times (\overline{\gamma^2})^2 \quad . \qquad \dots(50)$$

As a more realistic example, we take the Wigner distribution of spacings (14) and the Porter-Thomas distribution of widths (20) to obtain

$$\frac{\partial^2 \langle D^2 \rangle}{\partial B^2} = (8-4C) \;(\overline{\gamma^2})^2 \qquad \ldots(51)$$

where
$$C = \int_0^\infty \left\{ \frac{1}{1 - \overline{P}_D(u)} - \frac{1}{1 - e^{-u}} \right\} du$$
$$= \int_0^\infty \left\{ \frac{e^{-u^2/\pi}}{u \; \mathrm{erfc}(u/\sqrt{\pi})} - \frac{1}{1 - e^{-u}} \right\} \qquad \ldots(52)$$
$$\sim 0.49 \quad .$$

4. DISTRIBUTION OF REDUCED WIDTHS

If we consider equations (7) and (8), then since $R_o'(E_\lambda')$ is finite for every $B \neq B_o$, it follows that all γ_λ^2 tend to zero when $(B - B_o)$ tends to infinity. Thus the average reduced width $\langle \gamma^2 \rangle$ is certainly not independent of B. We therefore examine the effect of the transformation (4) on the distribution of the quantities $\gamma_\lambda^2 / \langle \gamma^2 \rangle$. It is not difficult to see from equation (10b) that

$$\frac{\partial \langle \gamma^2 \rangle}{\partial B} = 0 \qquad \ldots(53)$$

though in general higher derivatives do not vanish. Using the fact that γ_λ^2 and D are uncorrelated, we obtain for $n > 1$

$$\frac{\partial^2}{\partial B^2} \left\langle \left(\frac{\gamma_\lambda^2}{\langle \gamma^2 \rangle} \right) \right\rangle = 4n(n-1) \frac{\langle \gamma^{2n} \rangle \; \langle \gamma^2 \rangle^2}{\langle \gamma^2 \rangle^n} \; A \; +$$
$$+ \left[2n(2n-1) \frac{\langle \gamma^{2n} \rangle \langle \gamma^4 \rangle}{\langle \gamma^2 \rangle^n} - 2n \frac{\langle \gamma^{2n+2} \rangle \; \langle \gamma^2 \rangle}{\langle \gamma^2 \rangle^n} \right] F \; , \qquad \ldots(54)$$

where
$$A = \left\langle \sum_{\mu \neq \lambda} \sum_{\nu \neq \mu} \frac{1}{(E_\mu - E_\lambda)} \cdot \frac{1}{(E_\nu - E_\lambda)} \right\rangle$$
$$F = \left\langle \sum_{\mu \neq \lambda} (E_\mu - E_\lambda)^{-2} \right\rangle \quad .$$

The conditions on A and F under which the second derivatives of all moments vanish can be represented by a set of simultaneous equations

$$a_n A + f_n F = 0 \quad , \quad n = 2,3,4\ldots \qquad \ldots(55)$$

where (a_n, f_n) are the coefficients. Since F is always non-zero, (55) can only be satisfied if

(a) $A = 0$ and $f_n = 0$, or

(b) a_n/f_n = (a constant independent of n).

Clearly, neither of these conditions are satisfied in general, and therefore the distributions of reduced widths change when the boundary conditions value of B is altered.

5. LEVEL SPACING DISTRIBUTION AFTER A SMALL CHANGE IN BOUNDARY CONDITIONS

Suppose that initially we have a set of level spacings $\{D_1, D_2, D_3, \ldots D_N\}$ and a set of reduced widths $\{Y_1, Y_2, \ldots Y_N\}$ such that the probability that D_i has a value between D and D + dD is P(D)dD and the probability that Y_i has a value between Y and Y + dY is Q(Y)dY. Let us consider the effect of the transformation (4) on a particular D_λ, say D_1. We write the transformed level spacing as

$$\mathcal{D}_1 = f(D_1 , \ldots D_N ; Y_1 , \ldots Y_N) . \qquad \ldots(56)$$

The inverse transformation may be written as

$$D_1 = F(\mathcal{D}_1, D_2, \ldots D_N ; Y_1, Y_2 \ldots Y_N) \qquad \ldots(57)$$

Then the probability that $\mathcal{D}_1$ has a value between $\mathcal{D}$ and $\mathcal{D} + d\mathcal{D}$ is given by $\mathcal{P}(\mathcal{D})d\mathcal{D}$ where

$$\mathcal{P}(\mathcal{D}) = \int\!\!\cdot\!\!\int P\Big[F(\mathcal{D}, D_2 \ldots D_N, Y_1 \ldots Y_N)\Big]\Big[P(D_2) \ldots P(D_N)\Big] \times$$

$$\times \Big[Q(Y_1) \ldots Q(Y_N)\Big] \frac{\partial F}{\partial \mathcal{D}} \cdot dD_2\, dD_3 \ldots dD_N\, dY_1 \ldots dY_N . \qquad \ldots(58)$$

This can be written as

$$\mathcal{P}(\mathcal{D}) = \left\langle P\Big[F(\mathcal{D}, D_2 \ldots D_N ; Y_1 \ldots Y_N)\Big] \frac{\partial F}{\partial \mathcal{D}} \right\rangle \qquad \ldots(59)$$

where the average is taken over all variables except .

In particular, when $B = B_o + \delta B$, we can write to second order in δB

$$\mathcal{D}_i = D_i + (\gamma_i^2 - \gamma_{i+1}^2)\,\delta B + \left[\gamma_i^2 \sum_{\mu\neq i} \frac{\gamma_\mu^2}{E_\mu - E_i} - \gamma_{i+1}^2 \sum_{\mu\neq i+1} \frac{\gamma_\mu^2}{E_\mu - E_{i+1}} \right] (\delta B)^2 \qquad \ldots(60)$$

Using the same argument as that used to derive equations (38) and (39) we can obtain the inverse transformation in order $(\delta B)^2$, as

$$F(\mathcal{D}, D_2 \ldots D_N ; \gamma_1^2 \ldots \gamma_N^2)$$

$$= \mathcal{D} - (\gamma_i^2 - \gamma_{i+1}^2)\,\delta B - \left\{ \gamma_i^2 \left[\sum_{\mu>i} \frac{\gamma_\mu^2}{\mathcal{D} + \sum_k D_{i+k}} - \sum_{\mu<i} \frac{\gamma_\mu^2}{\sum_k D_{i-k}} \right] - \gamma_{i+1}^2 \left[\sum_{\mu>i+1} \frac{\gamma_\mu^2}{\sum_k D_{i+k}} - \sum_{\mu<i+1} \frac{\gamma_\mu^2}{\mathcal{D} + \sum_k D_{i-k}} \right] \right\} (\delta B)^2 . \qquad \ldots(61)$$

Substituting (61) into (59) and neglecting terms of order higher than $(\delta B)^2$ eventually yields

$$\mathcal{P}(\mathcal{D}) = P(\mathcal{D}) + \left\{ G(\mathcal{D})\, P(\mathcal{D}) + H(\mathcal{D}) \frac{dP(\mathcal{D})}{d\mathcal{D}} + K \frac{d^2P(\mathcal{D})}{d\mathcal{D}^2} \right\} (\delta B)^2 , \qquad \ldots(62)$$

where

$$G(\mathcal{D}) = \left\langle \gamma_i^2 \left[\sum_{\mu>i} \frac{\gamma_\mu^2}{(\mathcal{D} + \Sigma D_{i+k})^2} + \gamma_{i+1}^2 \sum_{\mu<i+1} \frac{\gamma_\mu^2}{(\mathcal{D} + \sum_k D_{i-k})^2} \right\rangle$$

$$= 2 < \gamma^2 >^2 \int_0^\infty e^{-\mathcal{D}S}\, S \left[1 - \overline{P}_D(S) \right]^{-1} dS , \qquad \ldots(63)$$

$$H(\mathcal{D}) = \left\langle \gamma_i^2 \left[\sum_{\mu>i} \frac{\gamma_\mu^2}{\mathcal{D} + \sum_k D_{i+k}} \quad \sum_{\mu<i} \frac{\gamma_\mu^2}{\sum_k D_{i-k}} \right] - \gamma_{i+1}^2 \left[\sum_{\mu>i+1} \frac{\gamma_\mu^2}{\Sigma D_{i+k}} \quad \sum_{\mu<i+1} \frac{\gamma_\mu^2}{\mathcal{D} + \sum_k D_{i-k}} \right] \right\rangle$$

$$= \quad 2 < \gamma^2>^2 \int_0^\infty (e^{-\mathfrak{D}S} - 1)\ (1 - \overline{P}_D(S))^{-1}\ dS \qquad \ldots(64)$$

$$K \quad = \quad < \gamma^4> \qquad < \gamma^2>^2 \quad . \qquad \ldots(65)$$

It is quite clear from equation (62) that the distribution function has changed from the initial distribution P(D) to one which depends upon $< \gamma^2>$ as well as $<D>$. Therefore, we expect that the moments M_k (D) are strongly correlated with the moments $M_j(\gamma^2)$ except for a special 'physical' value of $B = B_o$ where we can assume that the correlations vanish.

6. CHANGE IN PARAMETERS WITH CHANGE IN RADIUS

Let us consider the physical requirement that the collision function S given by equation (5) should be independent of the arbitrary radius a. Then from (6b) and (5) we obtain

$$\frac{\partial R}{\partial a} = i\,\frac{d\Omega}{da}\ \frac{1}{P\Omega}\ [1 + (B^2+P^2)R^2] \ - \frac{1}{P}\ \frac{dP}{da}\ R[1+BR] \ . \qquad \ldots(66)$$

We shall consider s-waves only where (Preston, 1965)

$$P \ = \ ka \ = \sqrt{2ME}\ .\ a \ , \quad S = 0 \ ,$$

$$\Omega \ = \ \exp(-2ika)$$

from which we obtain

$$\frac{\partial R}{\partial a} = \frac{2}{a} + \left(\frac{2B^2}{a} + 4\ Ma\ E - \frac{B}{a} \right) R^2 - \frac{1}{a}\ R\ . \qquad \ldots(67)$$

However, using equation (1) we get

$$\frac{\partial R}{\partial a} = \sum_\lambda \frac{\partial \gamma_\lambda^2}{\partial a} \cdot \frac{1}{E_\lambda - E} - \sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E}\ \frac{\partial E_\lambda}{\partial a} + \frac{\partial R_\infty}{\partial a} \qquad \ldots(68a)$$

Substituting equation (1) into (67) and equating the residues of poles of each order in $(E_\lambda - E)$, we obtain the three equations

$$\frac{\partial E_\lambda}{\partial a} = -\ \gamma_\lambda^4 \left[4\ Ma\ E_\lambda + \frac{2B(2B-1)}{a} \right] \qquad \ldots(68b)$$

$$\frac{\partial \gamma_\lambda^2}{\partial a} = \gamma_\lambda^2 \left[\frac{2B(2B-1)}{a} F_\lambda - 4 \, Ma \, (\gamma_\lambda^2 - 2E_\lambda F_\lambda) - \frac{1}{a} \right] \quad \text{...(68c)}$$

$$\frac{\partial R\infty}{\partial a} = \frac{2}{a} \quad \text{...(68d)}$$

where $$F_\lambda = \sum_{\mu \neq \lambda} \frac{\gamma_\mu^2}{E_\lambda - E_\mu} \quad , \quad \sum_\lambda F_\lambda = 0. \quad \text{...(69)}$$

The most common value for B chosen in s-states is B=0, for which we obtain

$$\frac{\partial E_\lambda}{\partial a} = 4 \, Ma \, E_\lambda \, \gamma_\lambda^4 \quad \text{...(70a)}$$

$$\frac{\partial \gamma_\lambda^2}{\partial a} = 4 \, Ma \, \gamma_\lambda^2 \, [\, 2E_\lambda F_\lambda - \gamma_\lambda^2 \,] - \frac{\gamma_\lambda^2}{a} \quad \text{...(70b)}$$

$$\frac{\partial R\infty}{\partial a} = \frac{2}{a} \quad . \quad \text{...(70c)}$$

Equation (70a) gives

$$\frac{\partial D_\lambda}{\partial a} = 4 \, Ma \, [\, E_\lambda \gamma_\lambda^4 - E_{\lambda+1} \gamma_{\lambda+1}^4 \,] \, ,$$

that is

$$\frac{\partial \langle D \rangle}{\partial a} = - 4 \, Ma \langle \gamma^4 \rangle \langle D \rangle \quad . \quad \text{...(71)}$$

This equation may be integrated immediately to yield

$$\langle D(a) \rangle = \langle D(a_o) \rangle \, . \, \exp\left(-2M \langle \gamma^4 \rangle (a^2 - a_o^2)\right) \quad . \quad \text{...(72)}$$

For a Porter-Thomas distribution at $(a_o, B_o = 0)$ we obtain a dependence of the average level spacing on a and $\overline{\gamma^2}$ as

$$\overline{D}(a) = \overline{D}(a_o) \quad . \quad \exp\left(-6M \, (\overline{\gamma^2})^2 (a^2 - a_o^2)\right) \quad . \quad \text{...(73)}$$

Clearly the minimum acceptable radius at which the maximum spacing, and hence lowest pole density is achieved, is a choice of a_o just at the radius where the nuclear interaction cuts off. At all larger radii, the pole density increases monotonically with a.

Next consider

$$\left\langle \frac{\partial \gamma^2}{\partial a} \right\rangle = \frac{\partial < \gamma^2 >}{\partial a}$$

$$= \left\langle - 4 \text{ Ma } \gamma_\lambda^4 + 8 \text{ Ma } \gamma_\lambda^2 E_\lambda F_\lambda - \gamma_\lambda^2 /a \right\rangle . \qquad ...(74)$$

The term containing E_λ can be shown to have a zero average and therefore we have by the methods of the previous section

$$\frac{\partial < \gamma^2 >}{\partial a} = - 4 \text{ Ma} < \gamma^4 > - < \gamma^2 >/a . \qquad ...(75)$$

For a Porter-Thomas distribution, equation (75) becomes

$$\frac{\partial < \gamma^2 >}{\partial a} = - 12 \text{ Ma} < \gamma^2 >^2 - < \gamma^2 >/a . \qquad ...(76)$$

Equation (76) has a unique solution

$$< \gamma^2 > = \overline{\gamma^2} = a(C + 4 \text{ Ma}^3)^{-1} \qquad ...(77)$$

where C is a constant of integration which is independent of a. Equation (77) gives explicitly the dependence of the mean distribution of reduced widths on the matching radius a.

In terms of the width $\overline{\gamma^2}(a_o)$ at the nuclear radius, we have

$$\overline{\gamma^2}(a) = \overline{\gamma^2}(a_o) \cdot \frac{a}{a_o} \cdot \frac{(C + 4 \text{ Ma}_o^3)}{(C + 4 \text{ Ma}^3)} . \qquad ...(78)$$

Finally, the equation (70c) can be integrated to give

$$R\infty = \ln \left(\frac{a}{a_o} \right)^2 \qquad ...(79)$$

where the boundary condition matrix, B, and the matching radius, a, are chosen such that the constant term in equation (1) is zero at the nuclear radius.

Since the mean values $< D>$ and $< \gamma^2 >$ are explicit functions of a, then to first order for any change from a to a + δa, the distribution functions alter in the first moments, and the new distributions are given approximately by the functional forms of the old distributions, with means altered according to (72) and (75).

7. CONCLUSION

It has been illustrated by several methods that contrary to current opinion, the statistical distributions of reaction matrix parameters do depend on both the choice of the boundary condition matrix B, and the matching radius. Explicit expressions for the change in moments and their distributions were derived and particular important examples of experimentally verified distributions were considered, and shown to be non-invariant under the relevant transformations. We conclude therefore that the statistical sampling of reaction matrix parameters from such distributions in order to generate cross sections is valid only for a particular value of B and a special radius. We infer that the probable required choices are $B = -\ell$, where ℓ = the orbital angular momentum and $a = a_o$, where a_o is the nuclear radius.

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