UNCLASSIFIED

AAEC/E 32

AUSTRALIAN ATOMIC ENERGY COMMISSION

A NOTE ON THERMO–ELASTIC STRESS IN AXIALLY SYMMETRIC ANISOTROPIC CYLINDERS

by

J. J. Thompson

Summary

The equations of elasticity are used to determine the axially symmetric thermo–elastic stresses and displacements for an anisotropic cylinder with elastic symmetry under conditions of plane strain.

August, 1958

1. INTRODUCTION

1.1 To estimate elastic stresses due to non–uniform temperature distribution in nuclear reactor fuel elements, the condition of plane strain is assumed and the conventional formulae involving the quantity $\alpha E/(1-\sigma)$ are applied. Here α is the coefficient of linear expansion, E is Young's Modulus and σ is Poisson's Ratio. However, these formulae are correct only for isotropic material, and it is to be expected that many fuel element materials will exhibit different elastic and thermal expansion properties in the axial and radial directions. The object of this note is to derive the correct combination of constants for use with such materials.

2. THEORY

2.1 In the absence of body forces the equation of equilibrium in terms of the radial and tangential stress components rr and $\theta\theta$ is:–

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0$$

It follows that this is satisfied if the stresses are derived from a stress function ϕ i.e.,

$$\widehat{rr} = \phi/r \;;\quad \widehat{\theta\theta} = \frac{\partial \phi}{\partial r}$$

2.2 If the radial displacement at radius r is u, then the strains are:–

$$e_{rr} = \frac{\partial u}{\partial r} \;;\quad e_{\theta\theta} = \frac{u}{r}$$

which implies that the strains must satisfy the compatibility equation:–

$$\frac{\partial e_{\theta\theta}}{\partial r} + \frac{e_{\theta\theta} - e_{rr}}{r} = 0$$

If now the strins can be expressed in terms of the stresses, the compatibility equation will lead to a differential equation for ϕ whose solution, satisfying the correct boundary conditions, will yield the stress components $\widehat{rr}$ and $\widehat{\theta\theta}$.

2.3 The generalized Hooke's Law involves 21 elastic constants. For anisotropic material with an axis of symmetry the number reduces to 5. For an axially symmetric stress distribution independent of z only 4 of these are required as the shear stresses and strains in cylindrical co–ordinates are zero. Love (1) has given the form of the Strain Energy function W under these conditions, referred to Cartesian co–ordinates. Transforming to cylindrical co–ordinates

$$2W = A(e_{rr}^2 + e_{\theta\theta}^2) + C(e_{zz}^2) + 2Fe_{zz}(e_{rr} + e_{\theta\theta})$$

$$+ 2(A-2N)\, e_{rr}\, e_{\theta\theta}$$

As the stress components are the derivatives of W with respect to the corresponding strain components, the stress–strain relations are:–

$$\begin{Bmatrix} \widehat{rr} \\ \widehat{\theta\theta} \\ \widehat{zz} \end{Bmatrix} = \begin{bmatrix} A & A-2N & F \\ A-2N & A & F \\ F & F & C \end{bmatrix} \begin{Bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \end{Bmatrix}$$

Solving the system of equations gives:–

$$\begin{Bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \end{Bmatrix} = \frac{1}{4N(AC-NC-F^2)} \begin{bmatrix} AC-F^2 & -(AC-2NC-F^2) & -2NF \\ -(AC-2NC-F^2) & AC-F^2 & -2NF \\ -2NF & -2NF & 4N(A-N) \end{bmatrix} \begin{Bmatrix} \widehat{rr} \\ \widehat{\theta\theta} \\ \widehat{zz} \end{Bmatrix}$$

2.4 The radial and axial Young's Moduli and Poisson's Ratios can now be interpreted as follows. Considering only the plane normal to the z axis, it is obvious that

$$E_r = 4N(AC-NC-F^2)\;/\;(AC-F^2)$$

$$\therefore \sigma_r/E_r = (AC-2NC-F^2)\;/\;4N(AC-NC-F^2)$$

$$\therefore \sigma_r = (AC-2NC-F^2)\;/\;(AC-F^2)$$

It is also abvious that $E_z = (AC-NC-F^2)\;/\;(A-N)$

Since it would be expected that e_{zz} would contain a term $-\sigma_z\,(\widehat{rr}+\widehat{\theta\theta})/E_r$, then σ_z can be interpreted as:–

$$\sigma_z = 2NF\;/\;(AC-F^2)$$

From the symmetry of the stress–strain relations e_{rr} and $e_{\theta\theta}$ must both contain terms $-\sigma_z\ \widehat{zz}\;/\;E_r$ representing lateral contraction due to axial stress. This point should be noted as it seems natural to assume that the correct expression would be $-\sigma_r\ \widehat{zz}\;/\;E_z$

2.5 The stress–strain relations can now be summarized as:–

$$\begin{Bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \end{Bmatrix} = \begin{bmatrix} 1/E_r & -\sigma_r/E_r & -\sigma_z/E_r \\ -\sigma_r/E_r & 1/E_r & -\sigma_z/E_r \\ -\sigma_z/E_r & -\sigma_z/E_r & 1/E_z \end{bmatrix} \begin{Bmatrix} \widehat{rr} \\ \widehat{\theta\theta} \\ \widehat{zz} \end{Bmatrix}$$

The total strains due to heating, with the temperature T as a function only of the radius, can now be written in full.

$$e_{rr} = (\widehat{rr} - \sigma_r\widehat{\theta\theta} - \sigma_z\ \widehat{zz})/E_r + \alpha_r T$$

$$e_{\theta\theta} = (\widehat{\theta\theta} - \sigma_r\widehat{rr} - \sigma_z\ \widehat{zz})/E_r + \alpha_r T$$

$$e_{zz} = (\widehat{zz} - \sigma_z(\widehat{rr} + \widehat{\theta\theta})\,E_z/E_r)/E_z + \alpha_z T = \text{const.}$$

where α_r and α_z are the radial and axial coefficients of expansion. Thus,

$$e_{rr} = T(\alpha_r + \alpha_z\,\sigma_z\,E_z/E_r) - \sigma_z\ e_{zz}\ E_z/E_r + \widehat{rr}\,(1-\sigma_z{}^2E_z/E_r)E_r + \widehat{\theta\theta}(\sigma_r + \sigma_z{}^2E_z/E_r)E_r$$

$$e_{\theta\theta} = T(\alpha_r + \alpha_z\,\sigma_z\,E_z/E_r) - \sigma_z\ e_{zz}\ E_z/E_r + \widehat{rr}\,(\sigma_r + \sigma_z{}^2E_z/E_r)/E_r + \widehat{\theta\theta}(1-\sigma_z{}^2\,E_z/E_r)/E_r$$

Substituting into the compatibility equation and using the stress function ϕ gives the equation:-

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\phi)\right] + \left(\frac{E_r\,\alpha_r + \sigma_z E_z\,\alpha_z}{1 - \sigma_z{}^2\,E_z/E_r}\right)\frac{\partial T}{\partial r} = 0$$

The quantity in brackets, denoted by K reduces to $\alpha E/(1-\sigma)$ for isotropic materials.

2.6 The stresses, therefore, take the usual form:-

$$\widehat{rr} = C_1 - C_2/r^2 - (K/r^2)\int_0^r T\,r\,dr$$

$$\widehat{\theta\theta} = C_1 + C_2/r^2 + (K/r^2)\int_0^r T\,r\,dr - KT$$

the effect of anistropy being incorporated in the quantity K.

2.7 To complete the analysis it is desirable to have explicit formulae for the displacement u as this function will be required for satisfying boundary conditions in the analysis of concentric tubes. From the equations already given, strains can be expressed in terms of stresses and the elastic constants.

$$(\gamma-\delta)\widehat{rr} = E_r(\gamma e_{rr} + \delta e_{\theta\theta})/(1+\sigma_r) + E_z\sigma_z\,e_{zz} - (E_r\,\alpha_r + \sigma_z E_z\,\alpha_z)\,T$$

$$(\gamma-\delta)\widehat{\theta\theta} = E_r(\delta e_{rr} + \gamma e_{\theta\theta})/(1+\sigma_r) + E_z\sigma_z\,e_{zz} - (E_r\,\alpha_r + \sigma_z E_z\,\alpha_z)\,T$$

$$(\gamma-\delta)\widehat{zz} = E_z\sigma_z\,(e_{rr} + e_{\theta\theta}) + E_z\,e_{zz}(1-\sigma_r) - E_z\,(2\,\alpha_r\sigma_z + \alpha_z(1-\sigma_r))\,T$$

where $\gamma = 1-\sigma_z{}^2\,E_z/E_r$

$\delta = \sigma_r + \sigma_z{}^2\,E_z/E_r$

Expressing strains in terms of u and using the equation of equilibrium, gives:-

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right] = \frac{1+\sigma_r}{E_r}\,K\,\frac{\partial T}{\partial r}$$

Thus,

$$u = K\,\frac{1+\sigma_r}{E_r}\,\frac{1}{r}\int_0^r Tr\,dr + \frac{D_1\,r}{2} + \frac{D_2}{r}$$

2.8 The constants C_1 C_2 D_1 and D_2 are not independent, the stress-strain relations showing that:-

$$(\gamma-\delta)\,C_1 = D_1\,E_r/2 + E_z\,\sigma_z\,e_{zz}$$

$$(1+\sigma_r)C_2 = D_2\,E_r$$

The constant strain e_{zz} is determined by the conditions of constraint in the axial direction. For zero constraint, i.e., zero resultant axial load,

$$\left(e_{zz} + \frac{2C_1\,\sigma_z}{E_r}\right)\int_A r^2 dr = \frac{\alpha_z + \sigma_z\,\alpha_r}{1-\sigma_z{}^2\,E_z/E_r}\int_A T\,r^2\,dr$$

3. CONCLUSION

3.1 Formulae for thermal stress in long isotropic cylinders can be applied to anistropic material by replacing $E\alpha/(1-\sigma)$ by $(E_r \alpha_r + \sigma_z E_z \alpha_z)/(1-\sigma_z^2 E_z/E_r)$. The interpretation of the radial and axial Poisson's Ratios is not straightforward, but in practice, this can be overcome by working in terms of the undefined quantities which are equal to σ_r/E_r and σ_z/E_r.

4. REFERENCE

1. Love Mathematical Theory of Elasticity, Ch.6,
4th Ed., Camb. Uni. Press.