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RESEARCH ESTABLISHMENT

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THE STATISTICAL DISTRIBUTION FUNCTIONS FOR PRODUCT RATIOS AND SUMS OF PRODUCT RATIOS

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E. K. ROSE

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ABSTRACT

The statistical distributions for product ratios and sums of product ratios are discussed and formulae are given for the computation of special moments, which have to be defined because the usual definition of moments diverges for such distributions.

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DIFFERENTIAL EQUATIONS; DISTRIBUTION FUNCTIONS; GAUSSIAN PROCESSES; INTEGRALS; MELLIN TRANSFORM; QUANTITY RATIO

CONTENTS

1. INTRODUCTION

In a previous report (Cook et al. 1974) the distribution functions that are obtained for the products of statistical variates drawn from a Gaussian distribution with zero mean were investigated.

Series expansions for small and large values of the product variate were derived which permitted the functions to be evaluated with ease. In this paper we explore, firstly, the distributions obtained for ratios of products of Gaussian variates and, secondly, the distributions for two-term sums of such variates.

2. THE DISTRIBUTIONS FOR RATIOS OF PRODUCTS

We consider a set of n characteristic probability distribution functions for a set of n independent variates x_i of the form

$$P_i(x_i) = \frac{2}{\sqrt{\pi}} e^{-x_i^2}, \quad 0 \leqslant x_i < \infty , \tag{1}$$

and a second set of similar probability distributions, m in number, of the form (1) with m independent variates y_j.

Let the resultant variate be defined as

$$u = \prod_{i=1}^{n} x_i \Big/ \prod_{j=1}^{m} y_j \quad . \tag{2}$$

We shall derive the characteristic probability distribution function for u.

By forming the joint probability distribution function as the product of terms set out in (1) we obtain the distribution of the product ratio as

$$R_{n/m}(u) = \left(\frac{2}{\sqrt{\pi}}\right)^{n+m} \int_0^\infty \cdots \int_0^\infty dx_1 \ldots dx_n \, dy_1 \ldots dy_{m-1} \quad \times$$

$$\times \exp\left\{-\sum_{i=1}^{n-1} x_i^2 - \sum_{j=1}^{m-1} y_j^2 - \prod_{i=1}^{n} x_i^2 \Big/ \left(\prod_{j=1}^{m-1} y_j^2 \, u^2\right)\right\} \frac{\prod_{i=1}^{n} x_i}{u^2 \prod_{j=1}^{m-1} y_j} \quad . \tag{3}$$

This form is not particularly amenable to numerical computation as errors accumulate when performing the (n + m) integrations.

Let

$$X = \prod_{i=1}^{n} x_i \quad ; \quad Y = \prod_{j=1}^{m} y_j \quad ; \quad U = X/Y \quad . \tag{4}$$

As shown in a previous report (Cook et al.1974), X is distributed as the function $C_n(X)$ and Y as $C_m(Y)$ where

$$C_n(X) = \left(\frac{2}{\sqrt{\pi}}\right)^n \int_0^\infty \cdots \int_0^\infty dx_1 \ldots dx_n \quad \times \quad \prod_{i=1}^{n-1} x_i \quad \times$$

$$\times \exp\left\{-\sum_{i=1}^{n-1} x_i^2 - X^2 \Big/ \prod_{i=1}^{n-1} x_i^2\right\} . \tag{5}$$

The product ratio distribution is, therefore, given by

$$R_{n/m}(U) = \int_0^\infty C_n(X)\, C_m(XU)\; X\, dX/U^2 \quad . \qquad (6)$$

Only a few of these functions can be evaluated in terms of known functions and the formula (3) is useful in this respect. With the aid of the evaluation techniques reported earlier (Cook et al. 1974), it also provides a rapid means of computing the $R_{m/n}$ functions by standard integration methods. With the aid of Equation (6) and the definitions

$$\text{(i)} \quad C_1(X) = \frac{2}{\sqrt{\pi}}\, e^{-X^2}$$

$$\text{(ii)} \quad C_2(X) = \frac{4}{\pi}\, K_0(2X) \quad , \qquad (7)$$

where $K_0(X)$ is the associated Bessel function, we find the particular cases

$$\text{(i)} \quad R_{1/1}(u) = \frac{2}{\pi}\; \frac{1}{1+u^2} \quad ,$$

$$\text{(ii)} \quad R_{1/2}(u) = \frac{2}{\pi\sqrt{\pi}} \cdot \frac{1}{u^2} \cdot e^{\frac{1}{u^2}}\; E_1\left(\frac{1}{u^2}\right) \quad ,$$

$$\text{(iii)} \quad R_{2/1}(u) = \frac{2}{\pi\sqrt{\pi}}\; e^{u^2}\, E_1(u^2) \quad , \qquad (8)$$

where $E_1(z) = \int_z^\infty e^{-t}\, \frac{dt}{t} \quad (\,|\arg z| < \pi\,)$,

$$\text{(iv)} \quad R_{2/2}(u) = \frac{2}{\pi^2}\; \frac{\ell n\, u^2}{u^2 - 1}$$

and $$\text{(v)} \quad R_{3/3}(u) = \frac{1}{\pi^3} \cdot \frac{1}{1+u^2}\; [\,\pi^2 + (2\,\ell n\, u)^2\,] \; .$$

These are the only cases we could express in terms of relatively common functions. A method for evaluating the function $E_1(z)$ can be found rapidly in Abramowitz and Stegun (1965).

Should we take the derivation of formula (3) and eliminate the last variate of the x_i rather than the y_i, then compare the resulting integral with formula (3), where m and n have been exchanged, we find an inversion theorem

$$R_{n/m}(u) = R_{m/n}\left(\frac{1}{u}\right) \Big/ u^2 \quad , \qquad (9)$$

which helps to reduce the class of functions to be evaluated from those having all integer values of n and m to those for which $n \geqslant m$.

3. EXPANSIONS AND MOMENTS

Following the derivation given in formula (1) we find that the Mellin transform (3) defined as

$$\bar{R}_{n/m}(s) = \int_0^\infty R_{n/m}(u)\; u^{s-1}\, du \qquad (10)$$

is given by

$$\bar{R}_{n/m}(s) = \pi^{-\frac{n+m}{2}}\; [\,\Gamma(\tfrac{s}{2})\,]^n \cdot [\,\Gamma(1-\tfrac{s}{2})\,]^m$$

$$0 < s < 2 \quad . \qquad (11)$$

Thus the function $R_{n/m}$ is expressible in terms of the inverse transform 3.

$$R_{n/m}(u) = \frac{\pi^{-\frac{n+m}{2}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} [\,\Gamma(\tfrac{s}{2})\,]^n \cdot [\,\Gamma(1-\tfrac{s}{2})\,]^m\, u^{-s}\, ds \; . \qquad (12)$$

This formula enables us to recognise the $R_{n/m}$ as a special case of the Meijer (1936) G-function as it has the form for Meijer's definition of the function in terms of a Mellin-Barnes (Erdelyi et al. 1953) integral:

$$G^{mn}_{pq}\left(u \,\middle|\, \begin{matrix} a_1, \dots a_p \\ b_1, \dots b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}\; u^s\, ds \qquad (13)$$

which yields the equality

$$2\,\pi^{\frac{n+m}{2}}\; G^{m+1,n+1}_{m+1,n+1}\left(u^2 \,\middle|\, \begin{matrix} 1, \dots 1 \\ 1, \dots 1 \end{matrix}\right) = R_{n/m}(u) \quad . \qquad (14)$$

This relation allows us to write down the special case of the Meijer differential equation which is obeyed by the $R_{m/n}$. The equation is

$$\left\{ \left(\frac{1}{2}\, u\, \frac{d}{du} - 1 \right)^{n+1} + (-1)^m\; u^2 \left(\frac{1}{2}\, u\, \frac{d}{du} \right)^{m+1} \right\} R_{m/n}(u) = 0 \, . \qquad (15)$$

Unfortunately, the above equation did not prove nearly so useful as the special differential equation derived for the functions $C_n(x)$, since the derivations of series expansions from it are prohibitively difficult.

By the direct evaluation of the contour integral (12), using the Cauchy residue theorem, we obtain the series

$$\text{(i)}\quad R_{m/n}(u) = \frac{\pi^{-\frac{(m+n)}{2}}}{(m-1)!} \frac{d^{m-1}}{d\left\{\frac{1}{\Gamma(\frac{s}{2})}\right\}^{m-1}} \left\{ \frac{\Gamma^n(1-\frac{s}{2})\, u^{-s}}{\frac{d}{ds}\left\{\Gamma(\frac{s}{2})\right\}^{-1}} \right\}_{\Gamma(\frac{s}{2})\to\infty}$$

for small u ;

$$\text{(ii)}\quad R_{m/n}(u) = \frac{(-1)^{n-1}}{(n-1)!} \pi^{-\frac{(m+n)}{2}} \frac{d^{n-1}}{d\left\{\frac{1}{\Gamma(1-\frac{s}{2})}\right\}^{n-1}} \times$$

$$\times \left\{ \frac{\Gamma(\frac{s}{2})\, u^{-s}}{\frac{d}{ds}\left\{\Gamma(1-\frac{s}{2})\right\}^{-1}} \right\}_{\Gamma(1-\frac{s}{2})\to\infty} \qquad (16)$$

for large u.

Again, these series do not yield simple forms when fully evaluated, except for the special cases listed in Equations (8). These forms are

$$\text{(i)}\quad R_{m/n}(u) = \frac{(-1)^{m-1}}{(m-1)!} \pi^{-\frac{(m+n)}{2}} \sum_{k=0}^{\infty} a^{(m-1)} \sum_{r=1}^{m-1} C_r D^{(m-r)} (A_{n,\nu})_{\nu\to\infty} \qquad (17)$$

where

$$A_{n,\nu} = \frac{-2\pi\Gamma^n(k+\nu+1)\, u^{2(k+\nu)}}{\left(\frac{d}{d\nu}(\Gamma(k+\nu+1)\sin\pi(k+\nu+1) + \pi\Gamma(k+\nu+1)\cos\pi(k+\nu+1))\right)},$$

$$C_1 = 1, \quad C_2 = \frac{(m-1)(m-2)}{2} \frac{a'_k}{a_k}, \quad \text{etc.,}$$

and, for example

$$a_k = (-1)^k\, \Gamma(k+\nu+1),$$

$$a'_k = 2\pi\Gamma'(k+\nu+1)/\Gamma(k+\nu+1),$$

$$a''_k = \pi(3\Gamma\Gamma'' - 4(\Gamma')^2 - \pi^2\Gamma^2)/\Gamma^2 \;;\; \Gamma^{(n)} = \frac{d^n\Gamma(k+\nu+1)}{d\nu^r}.$$

$$\text{(ii)}\quad R_{n/m}(u) = \frac{(-1)^{n-1}\pi^{-\frac{(n+m)}{2}}}{(n-1)!} \sum_{k=0}^{\infty} a_k^{(n-1)} \sum_{r=1}^{n-1} C_r D^{n-r} (B_m)_{\nu\to 0}$$

where

$$B_m = -2\pi\Gamma^m(k+\nu+1)\, u^{-2(k+\nu+1)} \Big/ (\Gamma' \sin\pi(k+\nu+1) + \pi\Gamma\cos\pi(k+\nu+1))$$

and the other coefficients are as above.

By the computation of the general expressions (17) (i) and (ii) on an IBM360/50 computer, we found the first to be useful in the range $0 \leq x \leq 0.1$ and the second gives four digit accuracy in the range $5.0 \leq x < \infty$. It appears that one or other of the series is always divergent for m or n greater than one, but the divergent one can still be useful as an asymptotic series. The lack of utility of this approach in the tabulation of the $R_{n/m}$ led us to decide that a comprehensive tabulation of the series coefficients would not be of value to those interested in computing these error distributions.

An important practical difficulty, when one is seeking to derive variances and higher moments from product ratios, can be seen from the Mellin transform (10). The p[th] moment of a distribution function about the origin is (Cramer 1945):

$$M_P(f(x)) = \int_0^\infty x^P f(x)\, dx \qquad (18)$$

and we see immediately, from the Mellin transform (10), that

$$M_P(R_{n/m}(x)) = \bar{R}_{n/m}(P+1) = \pi^{\frac{(n+m)}{2}} \left[\Gamma(\tfrac{1+P}{2})\right]^n \left[\Gamma(\tfrac{1-P}{2})\right]^m . \qquad (19)$$

$$(-1 < P < 1)$$

Therefore, we cannot choose P for the usual positive integers to define an infinite set of moments. For such product ratio distributions, we propose instead the definition for unit normalisation

$$M_Q(f(x)) = \int_0^\infty x^{\frac{1}{Q}} f(x)\, dx \qquad (20)$$

where $\quad Q = 2, 3, 4, \ldots \quad$ i.e. $Q \geq 2$,

and M_Q approaches unity as Q tends to infinity. All of the moments become

$$M_Q(R_{n/m}(x)) = \pi^{-\frac{(n+m)}{2}} \left[\Gamma(\tfrac{1}{2}+\tfrac{1}{2Q})\right]^n \left[\Gamma(\tfrac{1}{2}-\tfrac{1}{2Q})\right]^m , \qquad (21)$$

and can be estimated from experimental measurements by forming the sums

$$M_Q \approx \left(\sum_{i=1}^{N} (|u_i|)^{1/Q}\right) \Big/ N \qquad (22)$$

where N is the number of derived values of u.

We also used Equation (21) to provide a range of sum checks for our calculated values of $R_{m/n}$.

In determining errors, the moment chosen to represent best the analogue of variance is that with $Q = 2$, the quantity equivalent to a standard deviation being $N(M_2)^2$. Since the functions $R_{n/m}$ extend readily to a variate range from $-\infty$ to ∞, where the distributions become $R_{n/m}(|r|)$, we need only consider this usual application and take Q to be an even number. The mean of the distribution is then zero.

4. NUMERICAL COMPUTATION OF PRODUCT RATIO DISTRIBUTIONS

We have found that the most convenient method for computing the product ratio distribution is either to use the convolution (6), the series expansions for the $C_n(x)$ and a fine mesh trapezoidal rule integration, or to make use of integral recurrence relations similar to that for the C_n functions. These are derived readily by applying the product rule to the distributions for one less variate in either the numerator or denominator of u, and the distribution of this variate. This yields:

(i) $$R_{n/m}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2/u^2} \,.\, R_{n/m-1}(x)\, x \,\frac{dx}{u^2} \;;$$

(ii) $$R_{n/m}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2 u^2} \,.\, R_{n-1/m}\left(\frac{1}{x}\right) \frac{dx}{x} \quad . \qquad (23)$$

We can begin the integrations from

(i) $R_{n/o}(u) = C_n(u)$,

or

(ii) $R_{o/m}(u) = C_m\left(\frac{1}{u}\right)/u^2$ (24)

and generate the values for $0 \leqslant m \leqslant n$ in the first case, for example, by using Equation (23 (ii)) repeatedly. This can be checked by using Equation (23 (ii)) and then repeatedly applying Equation (23 (i)) for $0 \leqslant n \leqslant m$. The two tabulations can then be checked using the inversion theorem (9).

The trapezoidal rule integration was increased in speed, without loss of accuracy, by applying the conversion

$$\int_0^\infty g(x)\, f(xy)\, x\, dx = \int_0^1 \left\{ g(x)\, f(xy) + \frac{1}{x^2}\, g\left(\frac{1}{x}\right) f\left(\frac{y}{x}\right) \right\} x dx \qquad (25)$$

which converts the integration to one over a finite range. From the above method we obtained the values for $R_{n/m}(u)$ quoted to three significant digits in Table 1. The Table is structured so that linear interpolation $\ell n\, u$ to intermediate values is satisfactory. We have quoted values for $1 \leqslant n \leqslant 4$ and $1 \leqslant m \leqslant 4$ as the most likely to occur in physical measurements.

We note that the generalisations of Equations (23 (i)) and (23 (ii)) are

(i) $$R_{p+m/q+n}(u) = \int_0^\infty R_{n/m}\left(\frac{x}{u}\right) R_{p/q}(x)\, \frac{x dx}{u^2} \;;$$

(ii) $$R_{p+n/q+m}(u) = \int_0^\infty R_{n/m}(xu)\, R_{p/q}\left(\frac{1}{x}\right) \frac{dx}{x} \quad . \qquad (26)$$

5. SUMS OF PRODUCTS AND PRODUCT RATIOS

In an extension of the work described in previous sections, we also considered distributions formed from sums of products and product ratios of variates to a small degree. The ultimate goal here is to establish the rules for determining the error distribution for any rational function. Let us first discuss the distributions for sums of products. For this, we define the variate

$$w = \prod_{i=1}^{m} x_i + \prod_{j=1}^{n} y_j \;\{-\infty < x_i < \infty\,,\; -\infty < y_j < \infty\} \qquad (27)$$

where x_i and y_j are variates with Gaussian distributions of zero mean.

Let

$$X = \prod_{i=1}^{m} x_i \,,\quad Y = \prod_{j=1}^{n} y_j \,, \qquad (28)$$

then X is distributed as $\frac{1}{2} C_m(|X|)$ and Y as $\frac{1}{2} C_n(|Y|)$. Using the product rule for the probability distributions, we find that w is distributed as

$$S_{m/o,n/o}(w) = \frac{1}{4} \int_{-\infty}^{\infty} C_m(|x|)\, C_n(|w-x|)\, dx \quad . \qquad (29)$$

With this definition, the S-functions have a normalisation of unity. Some special cases can be evaluated in terms of known functions.

For example

(i) $S_{1/o,1/o}(w) = \sqrt{\frac{2}{\pi}}\, e^{-w^2/2}$;

(ii) $S_{2/o,2/o}(w) = e^{-2|w|}$. (30)

The second result can be proved by using the integral representation

$$\int_0^\infty e^{-t^2 - u^2/t^2} \frac{dt}{t} = K_o(2u) \qquad (31)$$

for the associated Bessel functions. The other simple cases appear to yield integrals that cannot be expressed in terms of known functions.

For example

$$S_{1/o,2/o}(w) = \left(\frac{2}{\pi}\right)^{3/2} \int_0^\infty e^{-t^2 - w^2/(1+t^2)} \,.\, \frac{dt}{\sqrt{1+t^2}} \quad . \qquad (32)$$

We note that these functions have the symmetry property

$$S_{m/p,n/q}(w) = S_{n/q,m/p}(w) \quad . \qquad (33)$$

Preliminary efforts to tabulate these functions were frustrated by the large errors encountered using the usual integration procedures. These errors arose from the rapid variation of the integrand in Equation (29) when X is in the neighbourhood of w, and we could not find an alternative integral representation that eliminated this difficulty.

The two-term sum of product ratios has a probability distribution of

$$S_{n/p,n/q}(w) = \frac{1}{4} \int_{-\infty}^{\infty} R_{m/p}(|x|)\, R_{n/q}(|w-x|)\, dx \qquad (34)$$

where

$$w = \frac{\prod_{i=1}^{m} x_i}{\prod_{k=1}^{p} y_k} + \frac{\prod_{j=1}^{n} z_j}{\prod_{\ell=1}^{q} t_\ell}$$

and the (x_i, y_k, z_j, t_ℓ) have Gaussian distributions with zero mean and variance of one half. The only results we have on these distributions for $p \neq 0$ and $q \neq 0$ are

$$\text{(i)} \quad S_{1/1,1/1}(w) = \frac{2}{\pi} \; \frac{1}{w^2+4} \quad ;$$

$$\text{(ii)} \quad S_{1/0,1/1}(w) = \psi(w, \tfrac{1}{4}) \tag{35}$$

where $\psi(w, \theta)$ is the Voigt profile function (Cook and Elliott 1960)

$$\psi(w,\theta) = \frac{1}{2\sqrt{\pi\theta}} \int_{-\infty}^{\infty} e^{-(x-w)^2/4\theta} \frac{dx}{1+x^2} \quad .$$

We hope to devise an integration scheme to evaluate Equation (34) directly.

To find the general expressions for the moments of the S-functions, we define the Mellin transform

$$\bar{S}_{m/p,n/q}(s) = \int_0^{\infty} S_{m/p,n/q}(w) \cdot w^{s-1}\, dw, \quad (0 < s < 2) \;, \tag{36}$$

and note that

$$S_{m/p,n/q}(w) = \int_0^{\infty} dx \cdot R_{m/p}(x) \,\{R_{n/q}(|w-x|) + R_{n/q}(|w+x|)\} \;. \tag{37}$$

After some manipulation we can show, with the aid of the binomial theorem, that

$$\bar{S}_{m/p,n/q}(s) = 2\,\Gamma(s) \sum_{r=0}^{\infty} \frac{\bar{R}_{m/p}(s-2r) \cdot \bar{R}_{n/q}(2r+1)}{\Gamma(s-2r)\,\Gamma(2r+1)} \tag{38}$$

which, from Equation (11), yields

$$\bar{S}_{m/p,n/q}(s) = \pi^{-\ell} \sum_{r=0}^{\infty} \frac{\Gamma^m(\frac{s}{2}-r)\,\Gamma^p(1+r-\frac{s}{2})\,\Gamma^n(r+\frac{1}{2})\,\Gamma^q(\frac{1}{2}-r)}{\Gamma(s-2r)\,\Gamma(2r+1)}$$

$$\ell = (m+n+p+q)/2 \quad . \tag{39}$$

Once again we must take S to be non-integer and preferably of the form $S = 1 + 1/Q$, where Q is an integer. The sum (39) is then convergent and gives all moments.

6. CASES OF NON-ZERO MEANS

The distributions for the product ratio of variates with normal distributions and non-zero means can be dealt with as the product distribution was treated in Equation (1). We write the derived variate as

$$w = \prod_{i=1}^{n} x_i \cdot \prod_{j=1}^{m} \frac{1}{y_j} \quad , \tag{40}$$

define

$$\bar{x}_i = \text{average value of } x_i$$

$$1/\bar{y}_j = \text{average value of } 1/y_j \;,$$

and put

$$w = \prod_{i=1}^{n} (x-\bar{x}_i) \prod_{j=1}^{m} \left(\frac{1}{y_j} - \frac{1}{\bar{y}_j}\right) + \ldots\ldots\ldots +$$

$$+ \sum_{k=1}^{n} \bar{x}_k \prod_{i \neq k}^{n} (x_i - \bar{x}_i) \prod_{j=1}^{m} \left(\frac{1}{y_j} - \frac{1}{\bar{y}_j}\right) + \frac{\prod_{i=1}^{n} \bar{x}_i}{\prod_{j=1}^{m} \bar{y}_j} \quad , \tag{41}$$

so that we have a sum of terms each of which has an R-distribution derived from variates with zero means. The resulting distribution for w is a convolution of R-functions. The general result is too unwieldly to quote but, for cases up to order four, the integral representations for each distribution function are not too difficult to evaluate.

7. ACKNOWLEDGEMENTS

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TABLE 1

TABLES OF $R_{m/n}(w)$; $w = \prod_{i=1}^{m} x_i \Big/ \prod_{j=1}^{n} y_i$

w	$R_{1/1}$	$R_{1/2}$	$R_{1/3}$	$R_{1/4}$	$R_{2/2}$	$R_{2/3}$	$R_{2/4}$	$R_{3/3}$	$R_{3/4}$	$R_{4/4}$
0.001	0.637	0.359	0.203	0.114	2.800	1.645	0.965	6.474	3.977	10.880
0.005	0.637	0.359	0.203	0.114	2.147	1.277	0.758	3.940	2.480	5.497
0.01	0.637	0.359	0.203	0.114	1.867	1.119	0.668	3.054	1.951	3.812
0.05	0.635	0.358	0.202	0.114	1.217	0.753	0.462	1.472	0.992	1.546
0.1	0.630	0.356	0.201	0.113	0.943	0.597	0.374	0.992	0.692	0.963
0.2	0.612	0.346	0.196	0.111	0.679	0.448	0.289	0.627	0.458	0.569
0.3	0.584	0.331	0.188	0.107	0.536	0.365	0.242	0.464	0.350	0.408
0.4	0.549	0.315	0.180	0.103	0.442	0.310	0.210	0.368	0.285	0.318
0.5	0.509	0.296	0.172	0.100	0.375	0.270	0.186	0.304	0.242	0.260
0.6	0.468	0.278	0.164	0.0959	0.323	0.239	0.168	0.259	0.210	0.220
0.7	0.427	0.261	0.156	0.0924	0.283	0.215	0.153	0.225	0.185	0.190
0.8	0.388	0.244	0.149	0.0890	0.251	0.195	0.141	0.198	0.166	0.167
0.9	0.352	0.229	0.142	0.0858	0.225	0.178	0.131	0.177	0.150	0.149
1.0	0.318	0.214	0.135	0.0827	0.203	0.164	0.122	0.159	0.137	0.134
1.5	0.196	0.158	0.109	0.0700	0.131	0.116	0.0909	0.104	0.0951	0.0880
2.0	0.127	0.120	0.0896	0.0605	0.0936	0.0882	0.0725	0.0761	0.0722	0.0644
2.5	0.0878	0.0950	0.0756	0.0531	0.0707	0.0705	0.0601	0.0588	0.0578	0.0502
3.0	0.0637	0.0771	0.0649	0.0472	0.0557	0.0582	0.0512	0.0474	0.0479	0.0408
3.5	0.0480	0.0639	0.0565	0.0424	0.0451	0.0492	0.0445	0.0393	0.0407	0.0341
4.0	0.0374	0.0539	0.0499	0.0385	0.0375	0.0424	0.0392	0.0333	0.0353	0.0291
4.5	0.0300	0.0462	0.0444	0.0352	0.0317	0.0370	0.0350	0.0287	0.0310	0.0253
5.0	0.0245	0.0401	0.0399	0.0324	0.0272	0.0327	0.0316	0.0251	0.0275	0.0222
5.5	0.0204	0.0352	0.0361	0.0299	0.0236	0.0292	0.0287	0.0222	0.0247	0.0198
6.0	0.0172	0.0311	0.0329	0.0278	0.0207	0.0263	0.0263	0.0198	0.0224	0.0177
7.0	0.0127	0.0249	0.0278	0.0243	0.0154	0.0217	0.0224	0.0161	0.0187	0.0146
8.0	0.00979	0.0205	0.0239	0.0215	0.0134	0.0184	0.0194	0.0135	0.0160	0.0123
9.0	0.00776	0.0172	0.0208	0.0193	0.0111	0.0158	0.0171	0.0115	0.0139	0.0106
10.0	0.00630	0.0146	0.0183	0.0174	0.00943	0.0137	0.0152	0.00924	0.0122	0.00920
20.0	0.00159	0.00488	0.00755	0.00841	0.00304	0.00532	0.00672	0.00368	0.00504	0.00357

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