AAEC/E217

AAEC/E217

AUSTRALIAN ATOMIC ENERGY COMMISSION
RESEARCH ESTABLISHMENT
LUCAS HEIGHTS

ANALYSIS OF FLOW STABILITY IN BOILING SYSTEMS WITH THE TOSCLE CODE

by

N.SPINKS

May 1971

ISBN 0 642 99417 X

AUSTRALIAN ATOMIC ENERGY COMMISSION

RESEARCH ESTABLISHMENT

LUCAS HEIGHTS

ANALYSIS OF FLOW STABILITY IN BOILING SYSTEMS WITH THE TOSCLE CODE

by

N. SPINKS

ABSTRACT

Two-phase-flow stability can be analysed with the TOSCLE code. TOSCLE uses equations derived by linearising and taking Laplace transformations of the mass, energy and momentum conservation equations and solving the resultant one-space-dimension equations analytically along a straight, uniform cross-section pipe with uniform heat addition or removal per unit length. Separate solutions are obtained for single-phase regions and two-phase regions of the pipe. Single-phase regions can be either subcooled liquid or superheated vapour. Complicated circuits can be analysed by subdividing the components until they conform to the "straight, uniform cross-section, uniform heat" description.

The code calculates steady-state pressure drop for given inlet velocity, then perturbed pressure drop for given inlet velocity perturbation. The latter calculation is repeated at a number of frequencies to permit the drawing of a Nyquist plot of the inlet velocity to pressure drop transfer function from which stability is assessed.

A simple slip correlation is incorporated as well as a two-phase-friction multiplier, which is taken as a quadratic in quality. Localised restrictions are permitted.

The deficiencies of the analysis are a neglect of subcooled boiling and of variations in fluid saturation properties with pressure. The assumption of uniform power requires more subdivision of the circuit than would otherwise be necessary and the simple slip correlation is inaccurate near unit void fraction.

National Library of Australia card number and ISBN 0 642 99417 X

CONTENTS

1. INTRODUCTION

In a continuous-flow boiler reliance is usually placed on a stable flow of coolant. Without this the boiler could be damaged when heat-transfer characteristics are impaired during periods of low flow. Methods for analysing flow stability are obviously needed.

The methods used for analysing the stability of two-phase flow systems are discussed by Neal and Zivi (1965). They divide the methods into two classes; direct solution in the time domain and solution in the Laplace transform domain. The latter method is made possible by the linearising assumption of small perturbations from the steady state. The linearised methods have limited application, being suitable only for analysis of stability. The time-domain methods have no such restriction but are less economical.

Neal and Zivi subdivide the linearised methods into two categories, distributed parameter methods and lumped parameter methods. In the latter, assumptions are made regarding the spatial distributions of the quantities of interest which allow the space dependences to be eliminated. Shotkin (1967) assumes that steady-state distributions hold for all time and so averages out the spatial dependences. In the distributed parameter method (see for example Jones 1961) the spatial dependences are averaged out over each of a large number of spatial elements. The correct solution is approached as the number of spatial elements is increased. Such methods might be better classified as numerical to distinguish them from the more recently published method of Davies and Potter (1966). Here the Laplace transformed differential equations in the spatial co-ordinate are solved analytically. However Davies and Potter make several assumptions which are not needed in the other methods. Slip between phases is neglected and a simple form $dP/dZ = K\rho V^2$ is used for the pressure gradient due to two-phase friction, ρ being the density of the two-phase mixture and V the mixture velocity. K is constant. Apart from these assumptions the analytical method is most attractive, as it combines the economy of the lumped parameter method and the accuracy of the numerical method.

The analytical method developed below and incorporated in the computer code TOSCLE was used (Spinks 1968) to show the complicated variation of the stability-threshold power when subcooling is varied at the inlet of the test section of a natural circulation loop. This complicated behaviour had remained undiscovered by users of numerical methods possibly because of the expense of calculating a large number of points or because the user of a numerical method expects to find some numerical inaccuracies and would tend to draw a smooth curve through the points. In fact discontinuities in the gradient of the threshold line were discovered using the analytical technique. Shotkin (1967) used the lumped parameter method to investigate the same problem but it failed to show the fine structure of the threshold line presumably because the fine structure was 'averaged-out' by the space-averaged model.

2. ANALYTICAL METHOD

2.1 Advantages

The analytical technique is pursued here to improve the assumptions of Davies and Potter (1966). Slip between the phases is allowed by the correlation of Zuber and Findlay (1965),

$$U_V = c\,J + v \qquad (2.1)$$

where U_V is the vapour velocity, J the volume average velocity, v the constant drift velocity and c the reciprocal of the constant K due to Bankoff (1960). Two-phase friction is accounted for by the Martinelli-Nelson (1948) form

$$\frac{dP}{dZ} = -\chi(x,P)\,\frac{2f}{D_e}\,\frac{G^2}{\rho_L}\;, \qquad (2.2)$$

where G is the total mass velocity. The relation of Colebrook (1938) is used for the Fanning friction factor f and the multiplier χ is dependent on quality x and pressure P. The multiplier is approximated by the quadratic

$$\chi(x,P) = 1 + a(P)x + b(P)x^2\;, \qquad (2.3)$$

a form which, when fitted to data in the full quality range, is accurate to within ± 5 per cent except at qualities in the region 1 to 5 per cent where the quadratic gives a multiplier which is low by up to 20 per cent. This is probably a sub-cooled boiling effect. The equations are so derived as to allow results to be generalised to a higher order polynomial.

A feature of the presentation is the generality of pipe configurations. Any number of pipe lengths are allowed in series, the orientation, flow area and the heating or cooling rate being constant through each length. Any arrangement of preheaters, subcoolers, boilers, superheaters, condensers and connecting pipes is permitted. The treatment is suited to the analysis of instability experiments where the heating rate is usually constant through the test section and where a variety of configurations exist.

Pressure drop over the total length is assumed constant, which permits study of the following configurations:

Parallel Channels. If flow is assumed constant in the circuit external to the parallel section, pressure drop will be constant over the parallel section.

Open Circuit Flow. Pressures at each end are constant so that pressure drop is constant.

Closed Circuit Flow. If a closed circuit contains an element such as a steam drum where mass flow and fluid enthalpy into the circuit from the element are weakly dependent on mass flow and fluid enthalpy into the element from the circuit then the closed circuit can be treated as an open circuit which is started and ended at the element.

2.2 Assumptions and Weaknesses

The analysis has the following weaknesses:

Uniform Power. A non-uniform power density must be approximated by subdividing the element length until power is effectively constant in each subdivision. No other subdivision of any length of straight uniform pipe is necessary.

Constant Fluid Properties. The variations of saturated densities and enthalpies, as pressure varies along an element, are neglected. Also the two-phase-friction multiplier is evaluated at the system mean pressure. While it is difficult to incorporate variable fluid properties in a numerical treatment it appears impossible to do so in an analytical treatment such as this. Even so, the gross effect of variable fluid properties can be assessed provided an accurate determination of these effects, in steady-state, is made. A most important single quantity determining pressure drop in a boiling channel is steam quality at exit. The true steam quality will depend on the variation of fluid properties along the length as well as on the heat input. If the properties are not permitted to vary in the calculational model, the heat input can be adjusted so that the true steady-state quality is obtained.

Simple Slip Correlation. As noted by Jones and Dight (1962), the K in Bankoff's slip correlation must be unity at unit void fraction and cannot be truly constant with variable void fraction. Similarly in the slip correlation of Equation 2.1, c should go to unity and v to zero at unit void fraction. With c and v kept constant, as in this analysis, slip is overestimated at high void fraction. Slip gives small vapour propagation time so frequencies of flow oscillation are too high when slip is overestimated. At high frequency, the stabilising single-phase-momentum component of the perturbed pressure drop is high so slip is stabilising. To analyse a system using TOSCLE, a pessimistic estimate of the stability threshold power can be obtained by removing slip (set $c = 1$ and $v = 0$). A second run using estimates of c and v obtained from low and intermediate quality data will give a somewhat optimistic estimate of the threshold power.

No Subcooled Boiling. The assumption of thermodynamic equilibrium means that subcooled boiling is ignored. Subcooled boiling could be important in low-quality systems, but flow stability is of less concern in such systems.

Single Boiling and Condensing Regimes. In calculating void fraction and two-phase friction, no consideration is given to different flow regimes. This level of treatment is a reflection of the current state of the art.

Simple Treatment of Localised Pressure Changes; Pressure changes at flow restrictions and at changes in flow area are represented simply by $\chi KG^2/\rho_L$ where K is constant and the form of the multiplier χ is taken to be the same as for distributed friction, Equation 2.3. This approach seems reasonable when the localised pressure changes arise from friction but may not apply to the reversible component at changes in flow area. D. Beattie (private communication) points out that with a homogeneous flow model, the homogeneous multiplier is the correct multiplier for this reversible component. This suggests the use of the inhomogeneous multiplier for inhomogeneous flow. Again, this level of treatment reflects the current state of the art

Heat Source Independent of Flow Velocity: Only time-independent heat inputs are considered. In fact if heat sources are of low thermal capacity the heat input could depend significantly on flow. It is expected that the analysis could be fairly readily extended to include such a flow dependence.

Laminar Flow Not Considered. Fluid flow is assumed to be turbulent rather than laminar. Laminar flow could be treated but such a model would have limited application.

Mechanical Energy Ignored. Mechanical energy is assumed to be small in comparison with other sources of heat input; an assumption which is usually quite valid in boiling systems. The resulting simplification of the energy equation makes the analytical treatment possible.

Steady-State Correlations. Void-fraction and frictional pressure drop correlations, derived from experiments in steady-state situations, are used in the dynamic situation. No better method is yet available.

2.3 Procedure

A length of pipe with single-phase flow and a length of pipe with two-phase flow are considered in turn. The velocity and enthalpy distributions along the single-phase element are determined in Section 4. The phase-velocity and void-fraction distributions along the two-phase element are determined in Section 5. These results are used to evaluate pressure drop along single-phase and two-phase elements. Other useful quantities for two-phase pressure-drop evaluation are the distributions of mass flow and quality, which are determined by the methods of Sections 6 and 7.

The relationships between quantities at outlet from an element and at inlet to the downstream element are discussed in Section 8.

If a single-phase/two-phase boundary occurs in a pipe element, the length is divided at the position of the steady-state boundary. In Sections 9 and 10 the conditions to be applied at a single-phase/two-phase interface are determined.

Single-phase and two-phase pressure drops are derived in Sections 11 and 12. The total pressure drop is calculated by summation over all elements. In each of Sections 4 to 12 the steady-state quantity (velocity, enthalpy, voidage, etc.) is first derived and then the dynamic behaviour of the quantity is evaluated. The variation of a quantity about the steady-state value will depend on the variation of the inlet velocity about the steady-state. Such variations about steady-state are assumed to be small. This assumption produces linear equations in the perturbed quantities allowing Laplace transforms to be taken.

2.4 The Transfer Function

The analysis produces the transfer function F(s) in the equation

$$-\overline{\Delta P} = F(s).\,\overline{U}_{in} \qquad (2.4)$$

$\overline{U}_{in}$ is the Laplace transformed velocity perturbation at inlet to the first element and $\overline{\Delta P}$ is the Laplace transformed total pressure drop perturbation. We require knowledge of the zeros of F(s), since F(s) = 0 permits a non zero inlet velocity perturbation with no associated change in pressure drop. Applying the inverse Laplace transform, the inlet velocity perturbation U^*_{in} resulting from an applied overall pressure drop perturbation ΔP^* is

$$U^*_{in}(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\,\frac{\overline{\Delta P}(s)}{F(s)}\,ds \; . \qquad (2.5)$$

If $F(s) = 0$ at $s = x + i\omega$ where x is positive, then from the Cauchy residue theorem, U^*_{in} will diverge as exp(xt).

By examining the individual components of $\overline{\Delta P}$, given in Sections 11 and 12, it can be shown that F(s) has no poles. Similarly it can be shown that

$$\text{Limit}_{s \to \infty} F(s) = KS \tag{2.6}$$

where K is a positive constant. It follows that there are no difficulties in applying Nyquist's criterion to locate the zeros of $F(x + i\omega)$, $x > 0$. $F(i\omega)$ is evaluated as ω increases from zero and equation 2.6 ensures that the F plane plot closes at infinity in the right-hand half plane.

3. THE TOSCLE CODE

Hand calculation of the complex transfer function would be laborious so a computer code, TOSCLE, has been written to implement the theory. The code has been in use for three years now in the analysis of experiments and for the evaluation of flow stability in nuclear power reactors. It is written in Fortran IV and contains sufficient comment cards for preparation of input data.

The first step in any TOSCLE calculation is to obtain the steady-state solution. The code proceeds section by section to evaluate the total pressure drop for a given inlet-velocity or the inlet velocity is determined by iteration for given overall pressure drop or for given exit quality. The code prints the steady-state velocities, mass flows, enthalpies, voidages and qualities at inlet to and outlet from each section. Single-phase/two-phase interface positions are printed as well as a table of pressure drops in each section subdivided into gravity, momentum, friction and restriction components. Restriction components are not further subdivided into momentum and friction components as the user specifies a single K factor to cover both components of a localised pressure drop.

The second step is to calculate and print changes in the above steady-state pressure-drop components arising from a small (1%) change in inlet velocity. This calculation, which requires a repeat steady-state calculation, serves as a check on the zero-frequency components of the transfer function, which are subsequently printed. Thus TOSCLE is self checking.

The third step of TOSCLE is to evaluate quantities which remain constant, i.e. independent of s, in the transfer function calculation. For example, the quantity

$$\rho_L \; \frac{e^{ncq\tau_2} - 1}{ncq} \; - D \; \frac{e^{(n-1)cq\tau_2} - 1}{(n-1)\,cq}$$

of equation 12.25 need not be recalculated for every new value of s.

The final step is to evaluate the transfer function F(s) which the code does by setting $\overline{U}_{in} = 1$ and $\overline{h}_{in} = 0$ at inlet to the first section and proceeding to find the pressure-drop perturbation for each section in turn. F(s) is then the sum of pressure-drop perturbations. The real part of s is specified in the input data but is usually zero so that a Nyquist plot will reveal the number of zeros of F(s) in the right-hand half plane. The program automatically adjusts increments in the imaginary part of s so that, with each increment, a rotation of about 20° occurs about the origin of the complex F plane.

The system under consideration will be on the verge of instability if $F(i\omega) = 0$ for a particular value of ω. The amount by which $F(i\omega)$ avoids the origin is thus a measure of the margin to an instability threshold. The code examines the absolute values of F when ω is such that F is purely real. The minimum absolute value (there may be several purely real values of F) is taken as the measure of stability, and the corresponding ω is interpolated as F swings through the real axis during its 20° rotation. A final F calculation is made at this interpolated frequency.

The zero-frequency transfer function components are printed in a table similar to the steady-state pressure-drop table, i.e. subdivided by section and by type of pressure drop. The table is printed twice more giving the real and then the imaginary parts of the minimum absolute purely real F.

TOSCLE is able to find an instability threshold. Any single component in the input-data set can be varied until the instability threshold is reached, that is, until the minimum absolute purely real F is zero. A vector plot of the components of F = 0 will reveal the instability mechanism.

The instability threshold becomes a line if two components of the input-data set are allowed to vary. TOSCLE can march along such a line provided a point on the line has been found in the previous calculation. Let Q and H be the two components (for example, power in one section and inlet-subcooling). Then, noting that a step along the line will entail a change in frequency ω,

$$dF = \frac{\partial F}{\partial \omega} d\omega + \frac{\partial F}{\partial Q} dQ + \frac{\partial F}{\partial H} dH \tag{3.1}$$

or

$$dF = F_\omega \, d\omega + F_Q \, dQ + F_H \, dH \tag{3.2}$$

where

$$\frac{\partial F}{\partial \omega} = F_\omega \quad , \quad \text{etc.} \tag{3.3}$$

The distance stepped along the line can be fixed by specifying $d\omega$. The increments dQ and dH which make dF = 0 are required. From equation 3.2

$$F_Q \, dQ + F_H \, dH = - F_\omega \, d\omega \quad . \tag{3.4}$$

Since F is a complex quantity,

$$F = R + i A \tag{3.5}$$

and

$$F_\omega = R_\omega + i A_\omega \quad , \quad \text{etc.} \tag{3.6}$$

Separating the real and imaginary parts of equation 3.4 and solving for dQ and dH;

$$dQ = \frac{A_\omega R_H - A_H R_\omega}{A_H R_Q - A_Q R_H} d\omega \tag{3.7}$$

and

$$dH = \frac{A_Q R_\omega - A_\omega R_Q}{A_H R_Q - A_Q R_H} d\omega \quad . \tag{3.8}$$

TOSCLE evaluates the partial derivatives numerically, calculates dQ and dH from the above equations and then recalculates F at the new ω, Q and H. Numerical errors or too big a step in ω will make F depart from zero. The error dF can be corrected by subtracting dQ from the new Q and dH from the new H where

$$dF = F_H \, dH + F_Q \, dQ \quad .$$

Equating real and imaginary parts

$$dQ = \frac{A_H \, dR - R_H \, dA}{A_H R_Q - A_Q R_H} \tag{3.9}$$

$$dH = \frac{R_Q \, dA - A_Q \, dR}{A_H R_Q - A_Q R_H} \quad . \tag{3.10}$$

Repeated application of equations 3.9 and 3.10 should reduce dF to an acceptably small value. The partial derivatives need not be re-evaluated during this process.

Occasionally $d\omega$ is too large and F does not converge to zero. TOSCLE then halves the frequency increment and starts again from equations 3.7 and 3.8.

Running Time. To calculate the steady-state velocity by iteration followed by a typical transfer function calculation, TOSCLE takes about 3 seconds per uniform zone of the problem on an IBM 360/50 computer. An instability threshold requires 4 or 5 such calculations, say about 15 seconds per zone, and in a march along a threshold line this reduces to about a second per zone per threshold point.

4. VELOCITY AND ENTHALPY DISTRIBUTIONS IN SINGLE-PHASE FLOW

Consider a length of pipe with uniform cross-section and uniform heat addition or subtraction. At inlet the steady-state velocity is U°_{in} and is perturbed by the amount U^{*}_{in} . The steady-state enthalpy at inlet is h°_{in} with perturbation h^{*}_{in}. The density is assumed constant. The velocity and enthalpy distributions are determined by solution of the equations of mass and energy conservation. Ignoring mechanical energy, these are,

Mass

$$\frac{\partial}{\partial Z}(\rho U) + \frac{\partial}{\partial t}(\rho) = 0 \qquad (4.1)$$

Energy

$$\frac{\partial}{\partial Z}(\rho Uh) + \frac{\partial}{\partial t}(\rho h) = Q \qquad (4.2)$$

Velocity Distributions

Steady-State

$$\frac{d}{dZ}(\rho U^{\circ}) = 0$$

but ρ is constant, therefore

$$U^{\circ} = \text{constant} = U^{\circ}_{in} \qquad (4.3)$$

Perturbation

$$\frac{d}{dZ}(U^{\circ} + U^{*}) = 0$$

therefore

$$U^{*} = \text{constant} = U^{*}_{in} \ .$$

Take Laplace transforms

$$\overline{U} = \overline{U}_{in} \qquad (4.4)$$

Enthalpy Distributions

Steady-State

$$\frac{d}{dZ}(\rho U^{\circ}h^{\circ}) = Q \ .$$

With Q independent of Z

$$h^{\circ} = h^{\circ}_{in} + \frac{QZ}{\rho U^{\circ}_{in}} \qquad (4.5)$$

Perturbation

$$\frac{\partial}{\partial Z}(U^\circ h^* + U^* h^\circ) + \frac{\partial}{\partial t}(h^*) = 0 .$$

Take Laplace transforms and rearrange

$$\frac{d\bar{h}}{dZ} + \frac{s\bar{h}}{U^\circ} = -\frac{Q\bar{U}}{\rho U^\circ U^\circ}$$

which solves to

$$\bar{h} = \bar{h}_{in}\, e^{-s\tau_1} - \frac{\bar{U}}{U^\circ}\,\frac{Q}{\rho}\,\frac{1-e^{-s\tau_1}}{s} \tag{4.6}$$

where

$$\tau_1 = \int_0^Z \frac{dZ}{U^\circ} \quad . \tag{4.7}$$

5. VELOCITY AND VOID-FRACTION DISTRIBUTIONS IN TWO-PHASE FLOW

Consider a length of pipe with uniform cross-section and uniform heat addition or subtraction. At inlet the steady-state volume-average velocity is J°_{in} perturbed by the amount J^*_{in} . The steady-state void fraction at inlet is α°_{in} with perturbation α^*_{in} . The saturation values of densities and enthalpies are assumed constant. The velocity and void fraction distributions are found by solving the equations of mass and energy conservation. Ignoring mechanical energy, these equations are

Mass

$$\frac{\partial}{\partial Z}[\rho_L(1-\alpha)U_L + \rho_v \alpha U_v] + \frac{\partial}{\partial t}[\rho_L(1-\alpha) + \rho_v\alpha] = 0 \tag{5.1}$$

Energy

$$\frac{\partial}{\partial Z}[\rho_L(1-\alpha)U_L h_L + \rho_v\alpha U_v h_v] + \frac{\partial}{\partial t}[\rho_L(1-\alpha)h_L + \rho_v\alpha h_v] = Q . \tag{5.2}$$

Multiplying the mass equation by h_L and subtracting the result from the energy equation gives

$$\frac{\partial}{\partial Z}(\alpha U_v) + \frac{\partial}{\partial t}(\alpha) = \frac{q\rho_L}{\Delta\rho} \quad , \tag{5.3}$$

where

$$q = \frac{\Delta\rho}{\rho_L\,\rho_v}\cdot\frac{Q}{(h_v - h_L)} \quad , \tag{5.4}$$

and

$$\Delta\rho = \rho_L - \rho_v \quad . \tag{5.5}$$

Multiplying the mass equation by h_v and subtracting the result from the energy equation gives

$$\frac{\partial}{\partial Z}\Big[(1-\alpha)U_L\Big] + \frac{\partial}{\partial t}(1-\alpha) = -\frac{q\,\rho_v}{\Delta\rho} \quad . \tag{5.6}$$

Velocity Distributions

The volume-average velocity J is defined as

$$J = (1-\alpha)U_L + \alpha U_v \quad . \tag{5.7}$$

From equations 5.3 and 5.6

$$\frac{\partial J}{\partial Z} = q \quad , \tag{5.8}$$

giving

$$J = J_{in} + qZ \quad . \tag{5.9}$$

The vapour velocity is, from the correlation due to Zuber and Finlay (1965),

$$U_v = c\,J + v \,. \tag{5.10}$$

From equation 5.9

$$U_v = c\,J_{in} + v + cqZ \,. \tag{5.11}$$

From equations 5.7 and 5.10 the liquid velocity is

$$\begin{aligned} U_L &= \frac{J(1-c\alpha) - \alpha v}{1-\alpha} \\ &= \frac{(J_{in} + qZ)(1 - c\alpha) - \alpha v}{1-\alpha} \,. \end{aligned} \tag{5.12}$$

Steady-State Velocities

From equation 5.9

$$J^\circ = J_{in}^\circ + qZ \,. \tag{5.13}$$

From equation 5.11

$$U_v^\circ = c\,J_{in}^\circ + v + cqZ = U_{v_{in}}^\circ + cqZ \,. \tag{5.14}$$

From equation 5.12

$$U_L^\circ = \frac{(J_{in}^\circ + qZ)(1 - c\,\alpha^\circ) - \alpha^\circ v}{1 - \alpha^\circ} \,. \tag{5.15}$$

The steady-state void fraction α° is as yet undetermined.

Perturbed Velocities

From equation 5.9

$$J^* = J_{in}^* \,.$$

Taking Laplace transforms,

$$\bar{J} = \bar{J}_{in} \,. \tag{5.16}$$

From equation 5.11

$$U_v^* = c\,J^*$$

therefore $\bar{U}_v = c\,\bar{J}_{in}$. (5.17)

From equation 5.7

$$J^* = (1 - \alpha^\circ)\,U_L^* - \alpha^* U_L^\circ + \alpha^\circ U_v^* + \alpha^* U_v^\circ \,.$$

Take Laplace transforms; use equation 5.17 and rearrange;

$$\bar{U}_L = \frac{1}{1-\alpha^\circ}\,[\,\bar{J}\,(1 - c\alpha^\circ) + \bar{\alpha}\,(U_L^\circ - U_v^\circ)\,] \,. \tag{5.18}$$

The void distributions α° and $\bar{\alpha}$ are as yet undetermined.

Steady-State Void Fraction

From equation 5.3

$$\frac{d}{dZ}(\alpha^\circ U_v^\circ) = \frac{q\,\rho_L}{\Delta\rho} \quad . \qquad (5.19)$$

Using equation 5.14 the solution is

$$\alpha^\circ = \frac{\alpha_{in}^\circ U_{v_{in}}^\circ + q\rho_L Z/\Delta\rho}{U_{v_{in}}^\circ + cqZ} \quad , \qquad (5.20)$$

however it will be convenient to express α° in terms of the variable τ_2 defined as

$$\tau_2 = \int_0^Z \frac{dZ}{U_v^\circ} \quad . \qquad (5.21)$$

Using equation 5.14

$$\tau_2 = \frac{1}{cq} \mathrm{Ln} \frac{U_v^\circ}{U_{v_{in}}^\circ} \qquad (5.22)$$

or

$$U_v^\circ = U_{v_{in}}^\circ e^{cq\tau_2} \quad . \qquad (5.23)$$

In terms of τ_2 equation 5.19 becomes

$$\frac{1}{U_v^\circ} \quad \frac{d}{d\tau_2}(\alpha^\circ U_v^\circ) = \frac{q\,\rho_L}{\Delta\rho} \quad .$$

Using equation 5.23, the solution is

$$(\rho_L - c\Delta\rho\,\alpha^\circ) = (\rho_L - c\Delta\rho\;\alpha_{in}^\circ)\, e^{-cq\tau_2} \quad . \qquad (5.24)$$

Let

$$D = \rho_L - c\Delta\rho\;\alpha_{in}^\circ \quad , \qquad (5.25)$$

then

$$\rho_L - c\Delta\rho\;\alpha^\circ = D\;e^{-cq\tau_2} \qquad (5.26)$$

or

$$\alpha^\circ = \frac{1}{c\Delta\rho}\;(\rho_L - D\,e^{-cq\tau_2}) \quad . \qquad (5.27)$$

Perturbed Void Fraction

From equation 5.3

$$\frac{\partial}{\partial Z}\;(\alpha^\circ U_v^* + \alpha^* U_v^\circ) + \frac{\partial}{\partial t}(\alpha^*) = 0 \quad .$$

Take Laplace transforms and rearrange;

$$\frac{d}{dZ}(U_v^\circ \bar{\alpha}) + s\,\bar{\alpha} = -\,\bar{U}_v\,\frac{d\alpha^\circ}{dZ} \quad .$$

In terms of the variable τ_2 defined in equation 5.21

$$\frac{d}{d\tau_2}\;(U_v^\circ\,\bar{\alpha}) + U_v^\circ\,s\,\bar{\alpha} = -\,\bar{U}_v\;\frac{d\alpha^\circ}{d\tau_2} \quad ,$$

therefore $\frac{d}{d\tau_2}(U_v^\circ \bar{\alpha}\, e^{s\tau_2}) = -\bar{U}_v \frac{d\alpha^\circ}{d\tau_2} e^{s\tau_2}$. (5.28)

From equation 5.27

$$\frac{d\alpha^\circ}{d\tau_2} = \frac{q}{\Delta\rho} D\, e^{-cq\tau_2} . \tag{5.29}$$

Substitute into equation 5.28 and solve;

$$U_v^\circ \bar{\alpha} = U_{v_{in}}^\circ \bar{\alpha}_{in}\, e^{-s\tau_2} - \frac{cq}{\Delta\rho} \bar{J}_{in}\, D \frac{e^{-cq\tau_2} - e^{-s\tau_2}}{s - cq} . \tag{5.30}$$

6. MASS-FLOW DISTRIBUTIONS IN TWO-PHASE FLOW

Expressions for the mass flow of the individual phases and for the total mass flow are useful in determining pressure drop. Mass flow definitions are;

Liquid: $G_L = \rho_L (1-\alpha) U_L$ (6.1)

Vapour: $G_v = \rho_v\, \alpha\, U_v$ (6.2)

Total: $G = \rho_L (1-\alpha) U_L + \rho_v\, \alpha\, U_v$. (6.3)

From the slip correlation, equation 5.10, and the definition of volume-average velocity, equation 5.7, U_L can be expressed in terms of U_v ;

$$(1-\alpha) U_L = \frac{U_v - v}{c} - \alpha\, U_v .$$

Substituting in equation 6.3;

$$G = \frac{\rho_L}{c}(U_v - v) - \Delta\rho\, \alpha U_v . \tag{6.4}$$

Steady-State

$$G_L^\circ = \rho_L (1-\alpha^\circ) U_L^\circ \tag{6.5}$$

$$G_v^\circ = \rho_v\, \alpha^\circ U_v^\circ . \tag{6.6}$$

Perturbations

$$\bar{G}_L = \rho_L (1-\alpha^\circ) \bar{U}_L - \rho_L\, \bar{\alpha}\, U_L^\circ \tag{6.7}$$

$$\bar{G}_v = \rho_v \alpha^\circ \bar{U}_v + \rho_v\, \bar{\alpha}\, U_v^\circ \tag{6.8}$$

$$\bar{G} = \frac{\rho_L}{c} \bar{U}_v - \Delta\rho\, \alpha^\circ \bar{U}_v - \Delta\rho\, \bar{\alpha}\, U_v^\circ . \tag{6.9}$$

Develop this expression for G by substituting from equation 5.17 for $\bar{U}_v$ and from equation 5.30 for $\bar{\alpha}$;

$$\bar{G} = (\rho_L - \Delta\rho c\, \alpha^\circ)\bar{J}_{in} - \Delta\rho\, U_{v_{in}}^\circ \bar{\alpha}_{in} e^{-s\tau_2}$$

$$+ cq\, \bar{J}_{in} (\rho_L - \Delta\rho\, c\, \alpha_{in}^\circ) \frac{e^{-cq\tau_2} - e^{-s\tau_2}}{s - cq}$$

Using equations 5.25 and 5.26;

$$\bar{G} = D e^{-cq\tau_2} \bar{J}_{in} - \Delta\rho U^{\circ}_{v_{in}} \bar{\alpha}_{in} e^{-s\tau_2} + cq \bar{J}_{in} D \frac{e^{-cq\tau_2} - e^{-s\tau_2}}{s - cq}$$

$$= D \bar{J}_{in} e^{-cq\tau_2} \frac{s}{s - cq} - e^{-s\tau_2}\left[\frac{cq D}{s - cq} \bar{J}_{in} + \Delta\rho U^{\circ}_{v_{in}} \bar{\alpha}_{in}\right] . \tag{6.10}$$

7. QUALITY DISTRIBUTION IN TWO-PHASE FLOW

Expressions for vapour quality are used to determine frictional pressure drop. Quality is defined as

$$x = \frac{G_v}{G} . \tag{7.1}$$

From the definition of G_v, equation 6.2;

$$x = \frac{\rho_v \alpha U_v}{G} . \tag{7.2}$$

Steady-State

$$x^{\circ} = \frac{\rho_v \alpha^{\circ} U^{\circ}_v}{G^{\circ}} .$$

Using equation 5.27 for α° and 5.23 for U°_v;

$$x^{\circ} = \frac{\rho_v U^{\circ}_{v_{in}}}{G^{\circ} c\Delta\rho} (\rho_L e^{cq\tau_2} - D) . \tag{7.3}$$

Perturbation

From equation 7.2

$$\bar{x} = \frac{\rho_v}{(G^{\circ})^2} (G^{\circ} \alpha^{\circ} \bar{U}_v + G^{\circ} \bar{\alpha} U^{\circ}_v - \alpha^{\circ} U^{\circ}_v \bar{G}) . \tag{7.4}$$

Substitute from equations 5.27, 5.17, 5.30, 5.23 and 6.10 for α°, $\bar{U}_v$, $\bar{\alpha} U^{\circ}_v$, U°_v and $\bar{G}$ respectively;

$$\bar{x} = \frac{\rho_v}{(G^{\circ})^2} \left\{ \frac{G^{\circ} \bar{J}_{in}}{\Delta\rho} \left(\rho_L - D e^{-cq\tau_2}\right) + \right.$$

$$+ G^{\circ}\left(U^{\circ}_{v_{in}} \bar{\alpha}_{in} e^{-s\tau_2} - \frac{qDc\bar{J}_{in}}{\Delta\rho} \frac{e^{-s\tau_2} - e^{-cq\tau_2}}{cq - s}\right) -$$

$$- \frac{U^{\circ}_{v_{in}}}{c\Delta\rho} \left[\left(\frac{qDc}{cq - s} \bar{J}_{in} - \Delta\rho U^{\circ}_{v_{in}} \bar{\alpha}_{in}\right)\left(\rho_L e^{(cq-s)\tau_2} - D e^{-s\tau_2}\right) - \right.$$

$$\left.\left. - D \bar{J}_{in} \frac{s}{cq - s} \left(\rho_L - D e^{-cq\tau_2}\right)\right]\right\} . \tag{7.5}$$

8. COUPLING BETWEEN ELEMENTS

From conservation of mass and energy, the fluid condition at inlet to a pipe element of constant flow area and uniform power per unit length is determined by the fluid condition at exit from the upstream element. Conditions at an interface between a single-phase region and a two-phase region are considered separately in following sections.

Single Phase Flow

Between elements n–1 and n, conservation of mass and energy gives the steady-state relations

$$(A\,U^\circ_{in})_n = (A\,U^\circ_{out})_{n-1} \tag{8.1}$$

and

$$(h^\circ_{in})_n = (h^\circ_{out})_{n-1}\ . \tag{8.2}$$

A is the flow area, and mechanical energy is ignored.

Relations for perturbations in these quantities are

$$(A\,\bar{U}_{in})_n = (A\,\bar{U}_{out})_{n-1} \tag{8.3}$$

and

$$(\bar{h}_{in})_n = (\bar{h}_{out})_{n-1} \tag{8.4}$$

Two-Phase Flow

Between elements n–1 and n, conservation of mass and energy gives the steady-state relations

$$(A\,G^\circ_{in})_n = (A\,G^\circ_{out})_{n-1} \tag{8.5}$$

$$(x^\circ_{in})_n = (x^\circ_{out})_{n-1}\ . \tag{8.6}$$

Again, mechanical energy is ignored.

Having found G° and x°, the mass flow rates of the individual phases follow from

$$G^\circ_L = (1 - x^\circ)\,G^\circ \tag{8.7}$$

and

$$G^\circ_V = x^\circ\,G^\circ\ . \tag{8.8}$$

Also,

$$J^\circ = G^\circ_L/\rho_L + G^\circ_V/\rho_V \tag{8.9}$$

so that U°_V and α° can be calculated from

$$U^\circ_V = c\,J^\circ + v \tag{8.10}$$

and

$$\alpha^\circ = G^\circ_V\,(\rho_V U^\circ_V)^{-1}. \tag{8.11}$$

Note that at an area change, because of the presence of the constant drift velocity v,

$$(\alpha^\circ_{in})_n \neq (\alpha^\circ_{out})_{n-1}$$

unless v is zero.

Relationships for perturbations in the various quantities follow from

$$(A\,\bar{G}_{in})_n = (A\,\bar{G}_{out})_{n-1} \tag{8.12}$$

$$(\bar{x}_{in})_n = (\bar{x}_{out})_{n-1}\ . \tag{8.13}$$

Thus

$$\bar{G}_L = (1 - x^\circ)\,\bar{G} - G^\circ\,\bar{x}$$

$$\bar{G}_v = x^\circ\,\bar{G} + G^\circ\,\bar{x}$$

$$\bar{J} = \bar{G}_L/\rho_L + \bar{G}_v/\rho_v$$

$$\bar{U}_v = c\,\bar{J}$$

$$\frac{\bar{\alpha}}{\alpha^\circ} = \frac{\bar{G}_v}{G_v^\circ} - \frac{\bar{U}_v}{U_v^\circ} \; .$$

It follows that all velocity perturbations are scaled by the area ratio, for example,

$$(A\,\bar{J}_{in})_n = (A\,\bar{J}_{out})_{n-1} \tag{8.14}$$

and that void fraction perturbation is proportional to steady-state voidage, that is,

$$\left(\frac{\bar{\alpha}_{in}}{\alpha^\circ_{in}}\right)_n = \left(\frac{\bar{\alpha}_{out}}{\alpha^\circ_{out}}\right)_{n-1} \; . \tag{8.15}$$

9. SINGLE-PHASE, TWO-PHASE BOUNDARY

Consider a vapour/two-phase boundary and a liquid/two-phase boundary; in each case the two-phase section is downstream. Substitute L or v for the subscript s depending on whether the upstream section is liquid or vapour.

Locate the steady-state boundary, determine perturbations in the position of the boundary arising from enthalpy perturbations at the steady-state boundary, and then derive velocity and void-fraction perturbations to be used as boundary conditions for the two-phase section but applied at the position of the steady-state boundary. This procedure can lead to negative void fractions but void-fraction will be positive down-stream from the perturbed boundary.

Steady-State Boundary

From equation 4.5, the steady-state boundary is located at

$$Z_b^\circ = \frac{\rho_s\,U_{in}^\circ}{Q}\,(h_s - h_{in}^\circ) \tag{9.1}$$

Perturbed Boundary

Figure 1a shows that at Z_b° ,

$$\bar{h}_{out} = -\,\bar{Z}_b\left(\frac{dh^\circ}{dZ}\right)_{Z=Z_b^\circ} \; .$$

Using equation 4.5,

$$\bar{Z}_b = -\,\frac{\rho_s\,U_{in}^\circ}{Q}\,\bar{h}_{out} \tag{9.2}$$

Steady-State Velocities

From mass conservation

$$J_{in}^\circ = U_{out}^\circ \; . \tag{9.3}$$

The liquid and vapour velocities are obtained from equations 5.15 and 5.14 respectively.

Perturbed Velocity

Figure 1b shows that at Z_b°,

$$\bar{J}_{in} = \bar{U}_{out} - \bar{Z}_b \left(\frac{dJ^\circ}{dZ}\right)_{Z=Z_b^\circ} .$$

Using equation 5.8,

$$\bar{J}_{in} = \bar{U}_{out} - q \bar{Z}_b \quad . \tag{9.4}$$

Steady-State Void Fraction

Physically,

$$x_{in}^\circ = 0 \text{ or } 1 \tag{9.5}$$

and $\quad \alpha_{in}^\circ = 0 \text{ or } 1$

depending on whether the upstream section is liquid or vapour. However, with the slip correlation used, a void fraction of unity will not necessarily correspond to a quality of unity. For consistency in the mathematical model, the void fraction must be allowed to depart from unity even though quality is unity. From equation 7.2, when $x = 1$,

$$\alpha_{in}^\circ = \left(\frac{G}{\rho_v U_v}\right)_{in}^\circ$$

$$= \frac{J_{in}^\circ}{c J_{in}^\circ + v} \tag{9.6}$$

Perturbed Void Fraction

Figure 1c shows that at Z_b° ,

$$\bar{\alpha}_{in} = - \bar{Z}_b \left(\frac{d\alpha^\circ}{dZ}\right)_{Z=Z_b}$$

$$= - \frac{\bar{Z}_b}{U_{v_{in}}^\circ} \left(\frac{d\alpha^\circ}{d\tau_2}\right)_{in} .$$

Using equation 5.29

$$\bar{\alpha}_{in} = - \frac{\bar{Z}_b}{U_{v_{in}}^\circ} \left(\frac{q \rho_L}{\Delta \rho} - c q \alpha_{in}^\circ\right) . \tag{9.7}$$

10. TWO-PHASE, SINGLE-PHASE BOUNDARY

Consider a two-phase/vapour boundary and a two-phase/liquid boundary; in each case the two-phase section is upstream. Substitute L or v for the subscript s depending on whether the downstream section is liquid or vapour.

Locate the steady-state boundary, determine perturbations in the position of the boundary arising from void-fraction perturbations at the steady-state boundary, and then derive velocity and enthalpy perturbations to be used as boundary conditions for the single-phase section but applied at the position of the steady-state boundary.

Steady-State Boundary

When the energy equation 5.2 is integrated, the steady-state boundary is found to be located at

$$Z_b^\circ = \frac{(h_V - h_L)\,G}{Q}\,(x_{out}^\circ - x_{in}^\circ)\ . \tag{10.1}$$

Use $x_{out}^\circ = 0$ or 1 depending on whether the downstream section is liquid or vapour. Having determined Z_b° find $U_{v_{out}}^\circ$ from equation 5.14, τ_2 from equation 5.22 and α_{out}° from equation 5.27.

Perturbed Boundary

The perturbation $\bar{Z}_b$ in the position of the boundary is derived from $\bar{\alpha}_{out}$, the void-fraction perturbation at the position of the steady-state boundary. Proceeding in similar fashion to the derivation of equation 9.7;

$$\bar{Z}_b = -\bar{\alpha}_{out}\,\frac{\Delta\rho\,U_{v_{out}}^\circ}{q(\rho_L - c\Delta\rho\,\alpha_{out}^\circ)} \tag{10.2}$$

Steady-State Velocity

From mass conservation

$$U_{in}^\circ = J_{out}^\circ \tag{10.3}$$

Perturbed Velocity

Proceeding in similar fashion to the derivation of equation 9.4;

$$\bar{U}_{in} = \bar{J}_{out} + q\,\bar{Z}_b \tag{10.4}$$

Steady-State Enthalpy

$$h_{in}^\circ = h_s \tag{10.5}$$

Perturbed Enthalpy

Proceeding in similar fashion to the derivation of equation 9.2;

$$\bar{h}_{in} = -\frac{Q}{\rho_s\,U_{in}^\circ}\,\bar{Z}_b\ . \tag{10.6}$$

11. PRESSURE DROP IN SINGLE-PHASE FLOW

The momentum equation in a single-phase region is

$$\frac{\partial}{\partial Z}(\rho U^2) + \frac{\partial}{\partial t}(\rho U) = -\frac{\partial P}{\partial Z} - \rho g - K\rho U^2 - K_r\,\rho U^2\,\delta_+(Z - Z_r) \tag{11.1}$$

g is the acceleration due to gravity resolved along the negative Z direction and single-phase friction is represented by $K\rho U^2$. K will be permitted to vary with mass flow G in accordance with the relationship of Colebrook (1938, 1939):

$$K = \frac{2f}{D_e};\quad f^{-1/2} = C;\quad C = -1.73716\,\mathrm{Ln}\left(\frac{e/D_e}{3.7} + \frac{1.26C}{R_e}\right)$$

$$R_e = \frac{G\,D_e}{\mu_L} \tag{11.2}$$

e/D_e is the surface roughness relative to the equivalent diameter D_e. For localised restrictions K_r will be assumed independent of mass flow. K_r includes both frictional and momentum components of a localised pressure drop. At changes in flow area K_r is related to the downstream velocity; hence the "+" in $\delta_+(Z - Z_r)$.

The use of mass conservation, equation 4.1, permits the momentum equation to be written as

$$\rho U \frac{\partial U}{\partial Z} + \rho \frac{\partial U}{\partial t} = -\frac{\partial P}{\partial Z} - \rho g - K\rho U^2 - K_r \rho U^2 \delta_+(Z - Z_r) \qquad (11.3)$$

Gravity Pressure Drop

Steady-State

$$-\frac{dP_g^\circ}{dZ} = \rho g$$

$$-\Delta P_g^\circ = \rho g Z$$

Perturbed

$$-\Delta P_g^* = \int_{Z_{b_{in}}^*}^{Z+Z_{b_{out}}^*} \left(-\frac{\partial P_g}{\partial Z}\right) dZ - \int_0^Z \left(-\frac{\partial P_g^\circ}{\partial Z}\right) dZ$$

$$= \rho g \, (Z_{b_{out}}^* - Z_{b_{in}}^*)$$

$$-\Delta \bar{P}_g = \rho g \, (\bar{Z}_{b_{out}} - \bar{Z}_{b_{in}}) \qquad (11.4)$$

Momentum Pressure Drop

Steady-State

$$-\frac{dP_m^\circ}{dZ} = \rho U^\circ \frac{dU^\circ}{dZ} \approx 0$$

$$-\Delta P_m^\circ = 0 \qquad (11.5)$$

Perturbed

$$-\Delta P_m^* = \int_{Z_{b_{in}}^*}^{Z+Z_{b_{out}}^*} \left(-\frac{\partial P_m}{\partial Z}\right) dZ - \int_0^Z \left(-\frac{\partial P_m^\circ}{\partial Z}\right) dZ$$

$$= \int_0^Z \frac{\partial U^*}{\partial t} \, dZ$$

$$-\Delta \bar{P}_m = s \tau_1 \rho \bar{U} U^\circ , \qquad (11.6)$$

where the variable τ_1 defined in equation 4.7 is used.

Frictional Pressure Drop

Steady-State

$$-\frac{dP_f^\circ}{dZ} = K^\circ \ \rho (U^\circ)^2$$

$$-\Delta P_f^\circ = K^\circ \ \rho (U^\circ)^2 Z \tag{11.7}$$

Perturbed

$$-\Delta P_f^* = \int_{Z_{b_{in}}^*}^{Z+Z_{b_{out}}^*} \left(-\frac{\partial P_f}{\partial Z}\right) dZ - \int_0^Z \left(-\frac{\partial P_f^\circ}{\partial Z}\right) dZ$$

$$= K^\circ \ \rho (U^\circ)^2 (Z_{b_{out}}^* - Z_{b_{in}}^*) + \int_0^Z [2K^\circ \rho \ U^\circ U^* + K^* \rho (U^\circ)^2] \, dZ \tag{11.8}$$

From equations 11.2

$$\frac{K^*}{K^\circ} = \frac{f^*}{f^\circ} = -2 \ \frac{C^*}{C^\circ} \quad ,$$

$$\frac{C^*}{C^\circ} = \frac{\beta}{1+\beta} \ \frac{R_e^*}{R_e^\circ}$$

where

$$\beta = 1.26 \times 1.73716 \left[R_e^\circ \left(\frac{e/D_e}{3.7} + \frac{1.26 \ C^\circ}{R_e^\circ}\right)\right]^{-1} \quad , \tag{11.9}$$

and

$$\frac{R_e^*}{R_e^\circ} = \frac{G^*}{G^\circ} = \frac{U^*}{U^\circ} \quad .$$

Expressing K^*/K° in terms of U^*/U° and substituting into equation 11.8;

$$-\Delta P_f^* = K^\circ \ \rho (U^\circ)^2 \ (Z_{b_{out}}^* - Z_{b_{in}}^*) + \frac{1}{1+\beta} \int_0^Z 2K^\circ \ \rho U^\circ U^* \, dZ \ .$$

Take Laplace transforms and use equation 4.7 for τ_1;

$$-\Delta \bar{P}_f = K^\circ \ \rho (U^\circ)^2 \left(\bar{Z}_{b_{out}} - \bar{Z}_{b_{in}} + \frac{2\tau_1 \bar{U}}{1+\beta}\right) \tag{11.10}$$

Restriction Pressure Drop

Steady-State

$$-\frac{d P_r^\circ}{dZ} = K_r \ \rho (U^\circ)^2 \ \delta_+(Z - Z_r)$$

$$-\Delta P_r^\circ = K_r \ \rho (U^\circ)^2 \tag{11.11}$$

Perturbation

$$-\Delta \bar{P}_r = 2 K_r \rho \ U^\circ \ \bar{U} \ . \tag{11.12}$$

12. PRESSURE DROP IN TWO-PHASE FLOW

The momentum equation in a two-phase region is

$$\frac{\partial}{\partial Z}[\rho_L(1-\alpha)U_L^2 + \rho_V\,\alpha\,U_V^2] + \frac{\partial}{\partial t}[\rho_L(1-\alpha)U_L + \rho_V\,\alpha\,U_V] =$$

$$-\frac{\partial P}{\partial Z} - g[\rho_L(1-\alpha) + \rho_V\,\alpha] - \chi(x)\left[\frac{KG^2}{\rho_L} + \frac{K_r G^2}{\rho_L}\,\delta_+(Z - Z_r)\right] \qquad (12.1)$$

K_r is constant and K varies with mass flow G in accordance with equation 11.2. The two-phase-friction multiplier χ is taken as a quadratic in quality;

$$\chi(x) = 1 + ax + bx^2 \qquad (12.2)$$

where a and b are constant.

Gravity Pressure Drop

$$-\frac{\partial P_g}{\partial Z} = g\,[\rho_L(1-\alpha) + \rho_V\,\alpha] = g\,(\rho_L - \Delta\rho\,\alpha) \qquad (12.3)$$

Steady-State

$$-\Delta P_g^\circ = g\int_0^Z (\rho_L - \Delta\rho\,\alpha^\circ)\,dZ \quad .$$

Using the variable τ_2 defined in equation 5.21;

$$-\Delta P_g^\circ = g\int_0^{\tau_2} (\rho_L - \Delta\rho\,\alpha^\circ)\,U_V^\circ\,d\tau_2$$

therefore $$-\Delta P_g^\circ = g\,U_{V_{in}}^\circ \int_0^{\tau_2} (\rho_L - \Delta\rho\,\alpha^\circ)\,e^{cq\tau_2}\,d\tau_2 \quad ,$$

where equation 5.23 is used.

Substituting for α° from equation 5.27 ,

$$-\Delta P_g^\circ = g\,U_{V_{in}}^\circ \int_0^{\tau_2}\left[\rho_L - \frac{1}{c}(\rho_L - De^{-cq\tau_2})\right] e^{cq\tau_2}\,d\tau_2$$

$$= g\,U_{V_{in}}^\circ\left[\rho_L\left(1 - \frac{1}{c}\right)\frac{e^{cq\tau_2} - 1}{cq} + \frac{D\tau_2}{c}\right] \qquad (12.4)$$

Perturbed

$$-\Delta P_g^* = \int_{Z_{b_{in}}^*}^{Z + Z_{b_{out}}^*} g\,[\rho_L - \Delta\rho\,(\alpha^\circ + \alpha^*)]\,dZ - \int_0^Z g\,[\rho_L - \Delta\rho\,\alpha^\circ]\,dZ$$

Therefore,

$$-\Delta\overline{P}_g = g\,(\rho_L - \alpha^\circ_{out}\,\Delta\rho)\,\overline{Z}_{b_{out}} - g\,(\rho_L - \alpha^\circ_{in}\,\Delta\rho)\,\overline{Z}_{b_{in}} - g\,\Delta\rho\int_0^{\tau_2}\overline{\alpha}\,U_V^\circ\,d\tau_2 .$$

Using equation 5.30

$$- \Delta\overline{P}_g = g(\rho_L - \alpha^\circ_{out}\ \Delta\rho)\overline{Z}_{b_{out}} - g(\rho_L - \alpha^\circ_{in}\ \Delta\rho)\overline{Z}_{b_{in}} -$$

$$- g\,\Delta\rho\ U^\circ_{v_{in}}\overline{\alpha}_{in}\ \frac{1-e^{-s\tau_2}}{s} + \frac{g\,cq\,D\,\overline{J}_{in}}{cq-s}\left(\frac{1-e^{-s\tau_2}}{s} - \frac{1-e^{-cq\tau_2}}{cq}\right) \tag{12.5}$$

Momentum Pressure Drop

$$- \frac{\partial P_m}{\partial Z} = \frac{\partial}{\partial Z}\left[\rho_L(1-\alpha)U_L^2 + \rho_v\alpha U_v^2\right] + \frac{\partial}{\partial t}\left[\rho_L(1-\alpha)U_L + \rho_v\alpha U_v\right]$$

$$= \frac{\partial}{\partial Z}(G_L U_L + G_v U_v) + \frac{\partial}{\partial t}(G_L + G_v)$$

Steady-State

$$- \Delta P^\circ_m = (G^\circ_L U^\circ_L + G^\circ_v U^\circ_v)_{out} - (G^\circ_L U^\circ_L + G^\circ_v U^\circ_v)_{in} \tag{12.6}$$

Perturbed

$$- \Delta P^*_m = \int_{Z^*_{b_{in}}}^{Z+Z^*_{b_{out}}}\left(-\frac{\partial P_m}{\partial Z}\right)dZ - \int_0^Z\left(-\frac{\partial P^\circ_m}{\partial Z}\right)dZ$$

$$= Z^*_{b_{out}}\left[\frac{d}{dZ}(G^\circ_L U^\circ_L + G^\circ_v U^\circ_v)\right]_{\alpha=\alpha_{out}} - Z^*_{b_{in}}\left[\frac{d}{dZ}(G^\circ_L U^\circ_L + G^\circ_v U^\circ_v)\right]_{\alpha=\alpha_{in}} +$$

$$+ \int_0^Z \frac{\partial}{\partial Z}(G^\circ_L U^*_L + G^*_L U^\circ_L + G^\circ_v U^*_v + G^*_v U^\circ_v)\,dZ + \int_0^Z \frac{\partial G^*}{\partial t}\,dZ$$

$$- \Delta\overline{P}_m = \overline{Z}_{b_{out}}\left[\frac{d}{dZ}(G^\circ_L U^\circ_L + G^\circ_v U^\circ_v)\right]_{\alpha=\alpha_{out}} - \overline{Z}_{b_{in}}\left[\frac{d}{dZ}(G^\circ_L U^\circ_L + G^\circ_v U^\circ_v)\right]_{\alpha=\alpha_{in}} +$$

$$+ (G^\circ_L \overline{U}_L + \overline{G}_L U^\circ_L + G^\circ_v \overline{U}_v + \overline{G}_v U^\circ_v)_{out} -$$

$$- (G^\circ_L \overline{U}_L + \overline{G}_L U^\circ_L + G^\circ_v \overline{U}_v + \overline{G}_v U^\circ_v)_{in} + s\int_0^Z \overline{G}\,dZ \tag{12.7}$$

As $\overline{G}_L$, $\overline{G}_v$, $\overline{U}_L$ and $\overline{U}_v$ have been previously determined,

$$\left.\begin{array}{l}\frac{d}{dZ}(G^\circ_L U^\circ_L)\\[1ex] \frac{d}{dZ}(G^\circ_v U^\circ_v)\end{array}\right\}\quad \text{at } \alpha^\circ = 0 \text{ and } 1$$

and $\int_0^Z \overline{G}\,dz$

remain to be evaluated.

$$\frac{d}{dZ}(G_L^\circ U_L^\circ) = \frac{d}{dZ}[\rho_L(1-\alpha^\circ) U_L^\circ U_L^\circ]$$

$$= \rho_L\left\{(1-\alpha^\circ) U_L^\circ \frac{d U_L^\circ}{dZ} + U_L^\circ \frac{d}{dZ}[(1-\alpha^\circ) U_L^\circ]\right\} \quad . \tag{12.8}$$

From equation 5.6

$$\frac{d}{dZ}[(1-\alpha^\circ) U_L^\circ] = -\frac{q\rho_V}{\Delta\rho} \quad . \tag{12.9}$$

This gives the second term on the right of equation 12.8. Continue with equation 12.9 to get the first term of equation 12.8.

$$(1-\alpha^\circ)\frac{d U_L^\circ}{dZ} = U_L^\circ \frac{d\alpha^\circ}{dZ} - \frac{q\rho_V}{\Delta\rho} \quad . \tag{12.10}$$

From equation 5.3

$$\frac{d}{dZ}(\alpha^\circ U_V^\circ) = \frac{q\rho_L}{\Delta\rho} \quad .$$

Therefore,

$$\alpha^\circ \frac{d U_V^\circ}{dZ} + U_V^\circ \frac{d\alpha^\circ}{dZ} = \frac{q\rho_L}{\Delta\rho} \quad .$$

Equation 5.14 gives

$$\frac{d U_V^\circ}{dZ} = cq \quad .$$

Therefore,

$$\frac{d\alpha^\circ}{dZ} = \frac{1}{U_V^\circ}\left(\frac{q\rho_L}{\Delta\rho} - \alpha^\circ cq\right) \quad .$$

Substitute in equation 12.10

$$(1-\alpha^\circ)\frac{d U_L^\circ}{dZ} = \frac{U_L^\circ}{U_V^\circ}\left(\frac{q\rho_L}{\Delta\rho} - \alpha^\circ cq\right) - \frac{q\rho_V}{\Delta\rho} \quad . \tag{12.11}$$

Substitute from equations 12.9 and 12.11 into equation 12.8;

$$\frac{d}{dZ}(G_L^\circ U_L^\circ) = \frac{q\rho_L U_L^\circ}{\Delta\rho U_V^\circ}\left\{U_L^\circ(\rho_L - c\alpha^\circ\Delta\rho) - 2U_V^\circ \rho_V\right\} \tag{12.12}$$

Now consider $\frac{d}{dZ}(G_V^\circ U_V^\circ)$. From equation 6.6

$$\frac{d}{dZ}(G_V^\circ U_V^\circ) = \frac{d}{dZ}(\rho_V \alpha^\circ U_V^\circ U_V^\circ)$$

$$= \rho_V\left[\alpha^\circ U_V^\circ \frac{d U_V^\circ}{dZ} + U_V^\circ \frac{d}{dZ}(\alpha^\circ U_V^\circ)\right] \quad .$$

From equation 5.14

$$\frac{d U_V^\circ}{dZ} = cq \quad .$$

From equation 5.3

$$\frac{d}{dZ}(\alpha^\circ U_V^\circ) = \frac{q\rho_L}{\Delta\rho} \quad .$$

Therefore $\frac{d}{dZ} \; (G_v^\circ U_v^\circ) = \frac{q \rho_v U_v^\circ}{\Delta\rho} (\rho_L + c\,\alpha^\circ \,\Delta\rho)$. (12.13)

Finally consider $\int_0^Z \bar{G}\, dZ$. With τ_2 defined by equation 5.21

$$\int_0^Z \bar{G}\, dZ = \int_0^{\tau_2} \bar{G}\, U_v^\circ \, d\tau_2$$

$$= U_{v_{in}}^\circ \int_0^{\tau_2} \bar{G}\, e^{cq\tau_2}\, d\tau_2 \quad .$$

Substitute for $\bar{G}$ from equation 6.10 and integrate;

$$\int_0^Z \bar{G}\, dZ = U_{v_{in}}^\circ \left\{ D\,\bar{J}_{in}\; \tau_2 - \Delta\rho\, U_{v_{in}}^\circ \,\bar{\alpha}_{in} \; \frac{e^{(cq-s)\tau_2} - 1}{cq-s} + \frac{cq\, D\, \bar{J}_{in}'}{cq-s} \left[\frac{e^{(cq-s)\tau_2} - 1}{cq-s} - \tau_2 \right] \right\} \qquad (12.14)$$

Frictional Pressure Drop

$$\frac{\partial P_f}{\partial Z} = \chi(x)\; \frac{K\, G^2}{\rho_L}$$

where $\chi(x) = 1 + ax + bx^2$.

Steady-State

$$-\;\Delta P_f^\circ = \frac{K^\circ (G^\circ)^2}{\rho_L} \int_0^Z \chi^\circ \; dZ = \frac{K^\circ (G^\circ)^2\, U_{v_{in}}^\circ}{\rho_L} \int_0^{\tau_2} \chi^\circ\, e^{cq\tau_2}\, d\tau_2$$

where $\chi^\circ = 1 + ax^\circ + b(x^\circ)^2$.

Substitute for x° from equation 7.3;

$$\chi^\circ = 1 + a \, \frac{\rho_v U_{v_{in}}^\circ}{G^\circ c \Delta\rho} \left(\rho_L \, e^{cq\tau_2} - D \right) + b \left(\frac{\rho_v \, U_{v_{in}}^\circ}{G^\circ c \Delta\rho} \right)^2 \left(\rho_L \, e^{cq\tau_2} - D \right)^2$$

$$= \sum_{n=1}^{3} \chi_n^\circ \; e^{(n-1)cq\tau_2} \qquad (12.15)$$

where

$$\chi_1^\circ = 1 - a \, \frac{\rho_v \; U_{v_{in}}^\circ \; D}{G^\circ \; c\Delta\rho} + b \left(\frac{\rho_v \; U_{v_{in}}^\circ \; D}{G^\circ \; c\Delta\rho} \right)^2 \qquad (12.16)$$

$$\chi_2^\circ = a \, \frac{\rho_v \; U_{v_{in}}^\circ \; \rho_L}{G^\circ c \Delta\rho} - 2b \left(\frac{\rho_v \; U_{v_{in}}^\circ}{G^\circ c \Delta\rho} \right)^2 \rho_L \; D \qquad (12.17)$$

$$\chi_3^\circ = b \left(\frac{\rho_v \; U_{v_{in}}^\circ \; \rho_L}{G^\circ c \Delta\rho} \right)^2 \qquad (12.18)$$

$$-\Delta P_f^\circ = \frac{K^\circ(G^\circ)^2 U^\circ_{v_{in}}}{\rho_L} \int_0^{\tau_2} \sum_{n=1}^{3} \chi_n^\circ e^{ncq\tau_2} d\tau_2$$

$$= \frac{K^\circ(G^\circ)^2 U^\circ_{v_{in}}}{\rho_L} \sum_{n=1}^{3} \chi_n^\circ \frac{e^{ncq\tau_2}-1}{ncq} \qquad (12.19)$$

Perturbed

$$-\Delta P_f^* = \int_{Z^*_{b_{in}}}^{Z+Z^*_{b_{out}}} \left(-\frac{\partial P_f}{\partial Z}\right) dZ - \int_0^Z \left(-\frac{\partial P_f^\circ}{\partial Z}\right) dZ$$

$$= \frac{K^\circ(G^\circ)^2}{\rho_L} (\chi^\circ_{out} Z^*_{b_{out}} - \chi^\circ_{in} Z^*_{b_{in}}) +$$

$$+ \frac{1}{\rho_L} \int_0^Z [\chi^* K^\circ(G^\circ)^2 + \chi^\circ K^*(G^\circ)^2 + 2\chi^\circ K^\circ G^\circ G^*] \, dZ . \qquad (12.20)$$

Use $\chi^\circ_{out} = 1$ or $(1 + a + b)$ depending on whether the downstream section is liquid or vapour. Similarly χ°_{in} depends on the upstream section. Express K^* in terms of G^* using equation 11.2 and substitute into 12.20:

$$-\Delta\bar{P}_f = \frac{K^\circ(G^\circ)^2}{\rho_L} (\chi^\circ_{out} \bar{Z}_{b_{out}} - \chi^\circ_{in} \bar{Z}_{b_{in}}) +$$

$$+ \frac{K^\circ(G^\circ)^2}{\rho_L} \int_0^Z \bar{\chi} \, dZ + \frac{2K^\circ G^\circ}{\rho_L(1+\beta)} \int_0^Z \chi^\circ \bar{G} \, dZ \qquad (12.21)$$

where β is defined by equation 11.9.

The two integrals of 12.21 remain to be evaluated;

$$\int_0^Z \chi^\circ \bar{G} \, dZ = U^\circ_{v_{in}} \int_0^{\tau_2} \chi^\circ \bar{G} e^{cq\tau_2} d\tau_2 .$$

Substitute for χ° from equation 12.15, for $\bar{G}$ from equation 6.10 and integrate,

$$\int_0^Z \chi^\circ \bar{G} \, dZ = U^\circ_{v_{in}} \left\{ \left[\frac{q_i D C}{cq-s} \bar{J}_{in} - \Delta\rho U^\circ_{v_{in}} \bar{\alpha}_{in}\right] \sum_{n=1}^{3} \left(\chi_n^\circ \frac{e^{(ncq-s)\tau_2}-1}{ncq-s}\right) - \right.$$

$$\left. - \frac{D \bar{J}_{in} s}{cq-s} \left[\chi_1^\circ \tau_2' + \sum_{n=2}^{3} \chi_n^\circ \frac{e^{(n-1)cq\tau_2}-1}{(n-1)cq}\right]\right\} . \qquad (12.22)$$

Determine $\int_0^Z (G^\circ)^2 \bar{\chi} \, dZ$.

$$\bar{\chi} = (a + 2bx^\circ) \bar{x}$$

$$= (f_1 + f_2 e^{cq\tau_2}) \bar{x}$$

where, from equation 7.3

$$f_1 = a - 2b \frac{\rho_v U^\circ_{v_{in}} D}{G^\circ c \Delta\rho} \tag{12.23}$$

and

$$f_2 = 2b \frac{\rho_v U^\circ_{v_{in}} \rho_L}{G^\circ c \Delta\rho} . \tag{12.24}$$

$$\int_0^Z (G^\circ)^2 \overline{X}\, dZ = U^\circ_{v_{in}} \int_0^{\tau_2} (G^\circ)^2 \overline{X}\, e^{cq\tau_2}\, d\tau_2$$

$$= U^\circ_{v_{in}} \int_0^{\tau_2} (G^\circ)^2 \overline{x} \sum_{n=1}^{2} f_n e^{ncq\tau_2}\, d\tau_2 .$$

Substitute for $\overline{x}$ from equation 7.5 and integrate;

$$\int_0^Z (G^\circ)^2 \overline{X}\, dZ = U^\circ_{v_{in}} \rho_v \sum_{n=1}^{2} f_n \Bigg\{ \frac{G^\circ \overline{J}_{in}}{\Delta\rho} \left[\rho_L \frac{e^{ncq\tau_2} - 1}{ncq} - D \frac{e^{(n-1)cq\tau_2} - 1}{(n-1)\, cq} \right] +$$

$$+ G^\circ U^\circ_{v_{in}} \overline{\alpha}_{in} \frac{e^{(ncq-s)\tau_2} - 1}{ncq-s} - \frac{G^\circ\, cq\, D\, \overline{J}_{in}}{\Delta\rho\,(cq-s)} \left[\frac{e^{(ncq-s)\tau_2} - 1}{ncq-s} - \frac{e^{(n-1)cq\tau_2} - 1}{(n-1)\, cq} \right] -$$

$$- \frac{U^\circ_{v_{in}}}{c\Delta\rho} \left(\frac{q D c \overline{J}_{in}}{cq-s} - \Delta\rho\, U^\circ_{v_{in}} \overline{\alpha}_{in} \right) \left[\rho_L \frac{e^{[(n+1)cq-s]\tau_2} - 1}{(n+1)cq-s} - D \frac{e^{(ncq-s)\tau_2} - 1}{ncq-s} \right] +$$

$$+ \frac{U^\circ_{v_{in}}}{c\Delta\rho} \frac{D \overline{J}_{in} s}{cq-s} \left[\rho_L \frac{e^{ncq\tau_2} - 1}{ncq} - D \frac{e^{(n-1)cq\tau_2} - 1}{(n-1)\, cq} \right] \Bigg\} \tag{12.25}$$

where it is understood that

$$\left(\frac{e^{ncq\tau_2} - 1}{ncq} \right)_{n=0} \text{ means } \lim_{n \to 0} \frac{e^{ncq\tau_2} - 1}{ncq} = \tau_2$$

<u>Restriction Pressure Drop</u>

$$-\Delta P_r = \chi(x_r) \frac{K_r G^2}{\rho_L}$$

<u>Steady-State</u>

$$-\Delta P_r^\circ = \chi^\circ (x_r^\circ) \frac{K_r (G^\circ)^2}{\rho_L} \tag{12.26}$$

<u>Perturbed</u>

$$-\Delta \overline{P}_r = \frac{K_r (G^\circ)^2}{\rho_L} \overline{X} + \chi^\circ (x_r^\circ) \frac{2 K_r G^\circ \overline{G}(Z_r)}{\rho_L} \tag{12.27}$$

13. CONCLUSIONS

Although the pressure-drop perturbation equations become quite cumbersome in this analytical approach, the time required to compute the transfer function is small in comparison with numerical techniques. Another advantage is that mesh fineness or numerical stability worries do not arise.

The main disadvantage of this approach lies in the difficulty of using more complex void-fraction and two-phase-friction correlations. Since even the complex correlations do not have experimental backing in the general dynamic situation, it is questionable whether much additional accuracy would be gained by their use. Of course this situation will change if complex dynamic correlations become available.

The author is convinced that the treatment cannot be extended to allow non-uniform power, variable fluid properties and frictional heating. Variation of fluid properties is probably the only worry here, but then only when pressure drop is comparable to system pressure. In this case an adjustment can be made as discussed in Section 2.2.

Useful further work could be done in extending the analysis to include improved slip and two-phase-friction correlations, a subcooled boiling model, and/or a flow dependent heat source. The slip correlation in use is deficient at unit void fraction and the two-phase-friction multiplier could show some flow dependence.

14. NOTATION

A = flow area

c = coefficient in slip correlation

C = see equation 11.2

D = see equation 5.25

D_e = equivalent pipe diameter

e = surface roughness

f = Fanning friction factor. Also see equations 12.23 and 12.24

F = transfer function relating change in pressure drop to an inlet velocity perturbation

g = acceleration due to gravity resolved in direction of motion

G = mass velocity

h = enthalpy

J = volume average velocity, see equation 5.7

K = generally used to express pressure drop in terms of velocity head

P = pressure

q = see equation 5.4

Q = rate of heat input per unit volume of fluid

R_e = Reynolds number

s = Laplace transform variable

t = time

U = velocity

v = drift velocity in slip correlation

x = quality

Z = space co-ordinate

α = void fraction

β = see equation 11.9

μ = viscosity

ρ = density

τ = propagation time (for vapour if flow is two-phase)

χ = two-phase-friction multiplier

ω = frequency

Subscripts, superscripts, etc.

in = at inlet to pipe section

out = at outlet from pipe section

1 = single-phase flow

2 = two-phase flow

L = liquid

v = vapour

s = L or v

b = at single-phase/two-phase boundary

r = at localised restriction

g = from gravitational effect

m = from inertial effect

f = from frictional effect

o = steady-state

* = perturbation

— = Laplace transformed perturbation

Δ = finite change

15. REFERENCES

Bankoff, S.G. (1960). – A Variable Density Single-Fluid Model for Two Phase Flow. Trans. ASME, 82, 265.

Colebrook, C.F. (1938/1939). – J. Inst. Civil Engrs. (London), 11, 133.

Davies, A.L. and Potter, R. (1966). – Hydraulic Stability, An Analysis of the Causes of Unstable Flow in Parallel Channels. AEEW-R446.

Jones, A.B. (1961). – Hydrodynamic Stability of a Boiling Channel. KAPL 2170.

Jones, A.B. and Dight, D.G. (1962). – Hydrodynamic Stability of a Boiling Channel, Part 2, KAPL 2208.

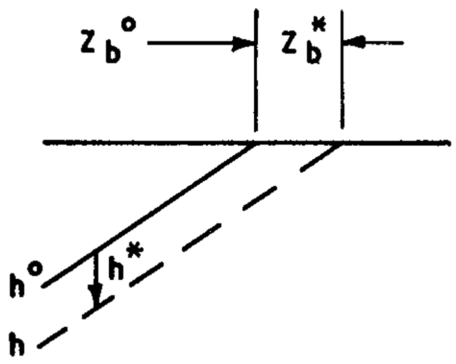
Martinelli, R.C. and Nelson, D.B. (1948). – Prediction of Pressure Drop During Forced Circulation Boiling of Water. Trans. ASME, 70, 695.

Neal, L.G. and Zivi, S.M. (1965). – Hydrodynamic Stability of Natural Circulation Boiling Systems; A Comparative Study of Analytical Models and Experimental Data. STL 372–14 (1).

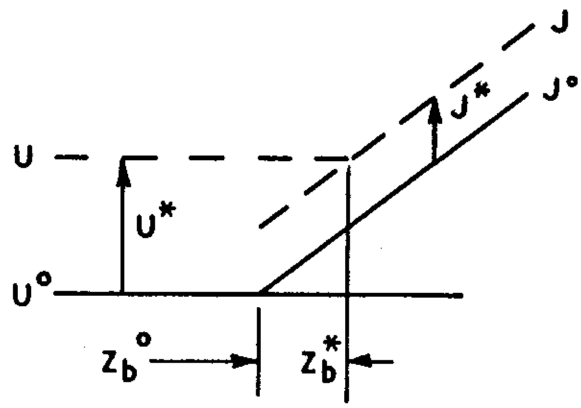
Shotkin, L.M. (1967). – Stability Considerations in Two-Phase Flow. Nucl. Sci. Eng. 28, 317.

Spinks, N. (1968). – The Dependence of Power on Inlet Subcooling for the Threshold of Flow Instability in a Natural-Circulation Loop. Trans. Am. Nucl. Soc. 11, 344.

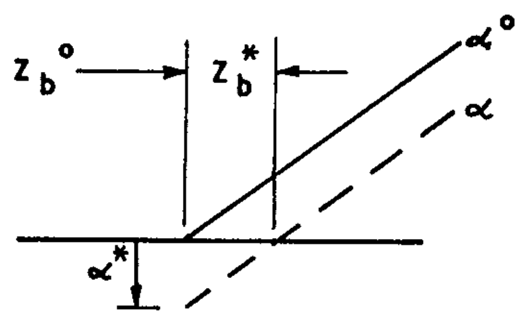
Zuber, N. and Findlay, J.A. (1965). – Average Volumetric Concentration in Two-Phase Flow Systems. Journal of Heat Transfer, 87, 453.



a. Boundary Perturbation induced by Enthalpy Perturbation



b. Effect of Boundary Perturbation on Perturbed Velocities



c. Void Fraction Perturbation induced by Boundary Perturbation

FIGURE 1. PERTURBATIONS AT SINGLE-PHASE/TWO-PHASE BOUNDARIES