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RESPONSE MATRICES IN THE DOUBLE P_N APPROXIMATION OF NEUTRON TRANSPORT THEORY

by

J.J. THOMPSON*

*Attached, from the University of New South Wales

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ABSTRACT

Relations are derived for reflection and transmission matrices of slabs in the double P_N approximation which permit their evaluation without the determination of particular eigenvalues and eigenvectors for each problem. Invariance principles also lead to results analogous to those of Chandrasekhar in his classical theory of radiative transfer.

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CONTENTS

1. INTRODUCTION

In a previous paper (Thompson 1963a), a matrix formulation and solution of the double P_N equations for monoenergetic neutron transport in slabs was presented.

The unknowns in this formulation are the vectors $B^+(x)$ and $B^-(x)$ whose elements are the coefficients of half range Legendre polynomial components of the angular flux directed in the (+)ve and (−)ve directions respectively.

In problem formulation and solution a key role is played by the slab response matrices which define the reflection and transmission properties of a slab by relations such as:

$$B^+(t) = \beta(t)\, B^-(t) + \gamma(t)\, B^+(0)$$

$$B^-(0) = \beta(t)\, B^+(0) + \gamma(t)\, B^-(t) \quad .$$

The reflection matrix $\beta(t)$ and the transmission matrix $\gamma(t)$ can be computed from a knowledge of the eigenvalues and eigenvectors of a basic matrix which is a function of the material properties specified by the paramater c, the number of secondary neutrons for collision, and, for anisotropic scattering, by the various moments of the scattering law.

An alternative solution by numerical integration of a set of coupled second order differential equations has also been shown to be quite efficient (Thompson 1963b). Certain difficulties are encountered, however, when the slabs are very thick, and when the response matrix approach involves the fundamental matrix β_∞, the infinite medium reflection matrix. Because the use of response matrices very greatly simplifies the problem formulation, to such an extent that in some cases of practical interest the solution can be written down directly in terms of such matrices, further work has been done to investigate the possibility of computing these matrices without the need to find sets of roots and vectors for all the different materials occurring in a problem.

The starting point was the theory of Chandrasekhar (1950), particularly his principles of invariance, and the later work on invariant imbedding by Bellman, e.g., Bellman, Kalaba and Wing (1960). It has been possible to obtain results in the double P_N approximation which are remarkably similar to those of Chandrasekhar and which greatly enlarge the usefulness of this method for practical nuclear engineering calculations.

2. REVIEW OF THE DOUBLE P_N FORMULATION

With isotropic source and distances in units of local mean free path, the basic equation is:

$$\mu \frac{\partial}{\partial x} \phi(x,\mu) + \phi(x,\mu) = \frac{1}{2} Q(x) + c \sum_{l=0}^{L} \alpha_l \frac{2l+1}{2} P_l(\mu) \int_{-1}^{+1} \phi(x,\mu') P_l(\mu') d\mu' \quad . \tag{1}$$

The expansion used is:

$$\begin{aligned} \phi(x,\mu) &= \sum_{n=0}^{N} (2n+1) B_n^+(x) P_n^+(\mu) \quad ; \quad 0 < \mu < 1 , \\ \phi(x,\mu) &= \sum_{n=0}^{N} (2n+1) B_n^-(x) P_n^-(\mu) \quad ; \quad -1 < \mu < 0 , \end{aligned} \tag{2}$$

with

$$\begin{aligned} P_n^+(\mu) &= P_n(2\mu - 1) , \\ P_n^-(\mu) &= P_n(-2\mu - 1) . \end{aligned} \tag{3}$$

In terms of the $B^+(x)$ and $B^-(x)$ vectors whose components are the $B_n^+(x)$ and $B_n^-(x)$ in order of increasing n, the flux and current are:

$$\begin{aligned} \phi(x) &= e_1' \left(B^+(x) + B^-(x)\right) , \\ J(x) &= \frac{1}{2}(e_1' + e_2') \left(B^+(x) - B^-(x)\right) . \end{aligned} \tag{4}$$

In the N'th order approximation, the 2(N + 1) equations derived from (1) by use of expansion (2) are written conveniently in matrix form as follows:

$$\begin{aligned} (MD + P) B^+ &= c \sum_{l=o}^{L} \alpha_l S_l \left(B^+ + (-1)^l B^-\right) + Q e_1 , \\ (-MD + P) B^- &= c \sum_{l=o}^{L} \alpha_l S_l \left(B^- + (-1)^l B^+\right) + Q e_1 . \end{aligned} \tag{5}$$

The matrices M, P, and S_l are defined in Appendix 1.

Note that each is symmetric, P is diagonal, and M is tridiagonal. D is the operator $\frac{d}{dx}$.

For numerical work it is convenient to define

$$X = B^+ + B^-, \quad Y = B^+ - B^- ,$$

representing generalized flux and current vectors, and to write (5) as:

$$\begin{aligned} MDX + P_o Y &= 0 \\ MDY + P_e X &= 2 Q e_1 . \end{aligned} \tag{6}$$

These equations lead to (N + 1) coupled second order differential equations:

$$M P_o^{-1} M D^2 X - P_e X + 2 Q e_1 = 0 . \tag{7}$$

Note that

$$\begin{aligned} P_o &= P - 2c \sum_{l=1}^{L/2} \alpha_{2l-1} S_{2l-1} , \\ P_e &= P - 2c \sum_{l=o}^{L/2} \alpha_{2l} S_{2l} . \end{aligned} \tag{8}$$

Solutions of (7) for a source-free medium involve the normalized eigenvectors and eigenvalues of the matrix:

$$G = M^{-1} P_o M^{-1} P_e . \tag{9}$$

Thus $$G \xi = \xi \Omega^2 , \tag{10}$$

where the elements of the matrix Ω are the absolute eigenvalues $|\omega_i|$ in increasing order of magnitude, and the columns of ξ are the corresponding eigenvectors, normalized by the condition:

$$\xi' P_e \xi = I . \tag{11}$$

Defining diagonal matrices

$$\Phi^{\pm}(x) = \sum_{i=1}^{N+1} e^{\pm|\omega_i| x} E_{ii} , \tag{12}$$

the source-free medium solution is

$$X = 2\xi\{\Phi^+(x)A^+ + \Phi^-(x)A^-\} ,$$

and $$Y = -2P_o^{-1}M\xi\Omega\{\Phi^+(x)A^+ - \Phi^-(x)A^-\} , \qquad (13)$$

where A^+ and A^- are arbitrary vectors.

The characteristic equation for the roots ω_i^2 is an (N + 1)th order polynomial in ω^2 with real roots, all of which are positive and non zero for $0 < c < 1$. Numerical values and formulae for P_1, P_2, and P_3 approximations are given by Thompson (1963a) for isotropic scattering.

The B vectors associated with solution (13) are:

$$\left.\begin{aligned} B^+(x) &= U\Phi^+(x)A^+ + V\Phi^-(x)A^- \\ B^-(x) &= V\Phi^+(x)A^+ + U\Phi^-(x)A^- \end{aligned}\right\} , \qquad (14)$$

with

$$\left.\begin{aligned} U &= \xi - P_o^{-1}M\xi\Omega = \xi - M^{-1}P_e\xi\Omega^{-1} \\ V &= \xi + P_o^{-1}M\xi\Omega = \xi + M^{-1}P_e\xi\Omega^{-1} \end{aligned}\right\} . \qquad (15)$$

3. REFLECTION AND TRANSMISSION MATRICES

If a neutron flux, defined by the vector $B^+(0)$ is incident on a medium occupying the region $(0 < x < \infty)$, the emergent or reflected flux distribution $B^-(0)$ defines an infinite medium reflection matrix β_∞.

Thus $$B^-(0) = \beta_\infty B^+(0) . \qquad (16)$$

From solution (14) it follows that

$$\beta_\infty = UV^{-1} , \qquad (17)$$

and hence may be computed from a knowledge of ξ and Ω.

For a slab occupying the region $(0 < x < t)$, the response to the incident $B^+(0)$ defines reflection and transmission matrices which are functions of t.

Thus

$$\left.\begin{aligned} B^+(t) &= \gamma(t)B^+(0) , \\ \text{and} \quad B^-(0) &= \beta(t)B^+(0) . \end{aligned}\right. \qquad (18)$$

Expressions for $\beta(t)$ and $\gamma(t)$ may also be obtained in terms of the basic eigenvalues and eigenvectors which depend on the properties of the slab. However this paper is concerned with alternative methods for calculating these response matrices, and formal solutions like (13) and (14) will be needed only to establish more fundamental results.

In many problems of practical interest, for example the flux depression in a repeating fuel – moderator reactor lattice, the response matrix of interest is $\beta(t) + \gamma(t)$. Consider a slab occupying the region $(-t < x < t)$ and let

$$R(2t) = \beta(2t) + \gamma(2t) . \qquad (19)$$

From solution (14),

$$R(2t) = \left(U\Phi^{+}(t) + V\,\Phi^{-}(t)\right)\left(V\,\Phi^{+}(t) + U\,\Phi^{-}(t) + U\,\Phi^{-}(t)\right)^{-1} \quad . \tag{20}$$

Let T be the diagonal matrix with elements $\tanh\left(|\omega_i|t\right)$ in increasing order. Using the explicit forms of U and V from (15),

$$R(2t) = I - 2\,(I + \xi\, T^{-1}\Omega\,\xi' M)^{-1} \ . \tag{21}$$

In the limit as $t \to \infty$,

$$\beta_\infty = I - 2\,(I + \xi\,\Omega\,\xi' M)^{-1} \ , \tag{22}$$

proving the important result,

$$M\beta_\infty = \beta'_\infty M \ . \tag{23}$$

Equation (22) is equivalent to the relation:

$$(I - \beta_\infty)^{-1}\,(I + \beta_\infty) = \xi\,\Omega\,\xi' M \ , \tag{24}$$

and therefore

$$(I - \beta_\infty)^{-1}(I + \beta_\infty)\,M^{-1}\,P_e\,\xi = \xi\,\Omega \ . \tag{25}$$

4. DETERMINATION OF $\beta(t)$ AND $\gamma(t)$

For material of constant c, the elements of the reflection and transmission matrices are continuous functions of the parameter t. Knowing the governing differential equations, numerical integration would yield the solution for any given thickness. To establish these differential equations it is convenient to rewrite the basic equations (5) for the B vectors in the simpler form:

$$\left.\begin{aligned} DB^{+} &= f\,B^{-} - g\,B^{+} \\ DB^{-} &= -f\,B^{+} + g\,B^{-} \end{aligned}\right\} \ , \tag{26}$$

with

$$\begin{aligned} f &= \frac{1}{2}M^{-1}(P_o - P_e) = c\,M^{-1}\sum_{l=o}^{L}(-1)^l\,\alpha_l\,S_l \ , \\ g &= \frac{1}{2}M^{-1}(P_o + Pe) = M^{-1}\{P - c\sum_{l=o}^{L}\alpha_l\,S_l\} \ . \end{aligned} \tag{27}$$

In this form Equation 26 may be regarded as defining the reflection and transmission matrices of a slab of infinitesimal thickness δt. An incident flux B produces a reflected flux $\delta t f B$ and a transmitted flux $(I - \delta t g)B$. If now a slab of thickness $(t + \delta t)$ with an incident flux B_o is split into two parts, one of thickness t, the other of thickness δt, continuity at the interface requires that

$$B = \left(I - \delta t\,\beta(t)\,f\right)^{-1}\gamma(t)\,B_o \ .$$

The overall response matrices are therefore

$$\gamma(t + \delta t) = (I - \delta t g)\left(I - \delta t\,\beta(t)\,f\right)^{-1}\gamma(t)$$

$$\beta(t + \delta t) = \beta(t) + \gamma(t)\,\delta t f\left(I - \delta t\,\beta(t)f\right)^{-1}\gamma(t) \ .$$

In the limit the following equations are obtained:

$$\left.\begin{aligned} D\gamma(t) &= \left(\beta(t)f - g\right)\gamma(t) \quad ; \quad \gamma(0) = I \\ D\beta(t) &= \gamma(t) f \gamma(t) \quad ; \quad \beta(0) = 0 \end{aligned}\right\} \tag{28}$$

Similar relations, but in energy space, and known as Stokes' relations, are given by Bellman, Kalaba, and Wing (1960).

These differential equations may be integrated for slabs of reasonable thickness. For very thick slabs the technique of successive doubling of thickness may be used. If suffixes 1 and 2 denote slabs of thickness t and 2t respectively, then it is easily shown that

$$\begin{aligned} \beta_2 &= \beta_1 + \gamma_1 (I - \beta_1^2)^{-1} \beta_1 \gamma_1 \\ \gamma_2 &= \gamma_1 (I - \beta_1^2)^{-1} \gamma_1 \end{aligned} \tag{29}$$

In many problems β_∞ enters as a basic response matrix. While integration of equations (28) and then successive doubling, till satisfactory convergence, could be used, it is desirable to have a more efficient technique available.

5. THE INFINITE MEDIUM REFLECTION MATRIX β_∞

In order to utilize the matrices f and g, we apply the principle of invariance in the form: Reflection from an infinite medium is invariant with respect to addition of a slab of thickness δt.

Consider a thin slab in the region $(0 < x < \delta t)$, and an infinite medium in the region $(\delta t < x < \infty)$. The infinite medium is defined by the relations

$$B_\infty^- (\delta t) = \beta_\infty B_\infty^+ (\delta t) \quad ,$$

while for the thin slab, alone,

$$B^-(0) = (I - \delta t\, g) B^-(\delta t) + \delta t f B^+(0) \quad ,$$

and

$$B^+(\delta t) = (I - \delta t\, g) B^+(0) + \delta t f B^-(\delta t) \quad .$$

For continuity at $x = \delta t$,

$$\beta_\infty B_\infty^+ (\delta t) = B^-(\delta t) \quad ,$$

and

$$B_\infty^+(\delta t) = (I - \delta t\, g) B^+(0) + \delta t f B^-(\delta t) \quad .$$

Therefore

$$B^-(\delta t) = (I - \beta_\infty \delta t f)^{-1} \beta_\infty (I - \delta t\, g) B^+(0) \quad .$$

The principle of invariance gives, for this value of $B^-(\delta t)$,

$$B^-(0) = \beta_\infty B^+(0) \quad ,$$

and, to first order in δt ,

$$\delta t (f + \beta_\infty f \beta_\infty - g \beta_\infty - \beta_\infty g) = 0 \quad .$$

The fundamental result is then

$$(I + \beta_\infty) f (I + \beta_\infty) = \beta_\infty (g + f) + (g + f) \beta_\infty \quad . \tag{30}$$

The method of solving this equation for β_∞ will be illustrated for the practically important case of isotropic scattering. Once the analogy with Chandrasekhar's work is established, the extension to anisotropic scattering should not prove difficult.

If scattering is isotropic,

$$f = c M^{-1} e_1 e_1' \quad , \quad g = M^{-1} P - c M^{-1} e_1 e_1' \quad , \tag{31}$$

and relation (30) becomes

$$c (I + \beta_\infty) M^{-1} e_1 e_1' (\beta_\infty + I) = \beta_\infty M^{-1} P + M^{-1} P \beta_\infty \ .$$

By introducing the vector H, defined as

$$H = P^{-1} (I + \beta_\infty') e_1 \ ,$$

and using the fact that $M \beta_\infty$ is symmetric, relation (30) reduces to

$$c H H' = \beta_\infty P^{-1} + P^{-1} \beta_\infty' \ . \tag{32}$$

It will be shown in Section 5 that H is the vector describing the emergent flux in Milne's problem with $c = 1$. It is therefore closely related to Chandrasekhar's $H(\mu)$ function, which is precisely the emergent flux as a continuous function of μ in that case. A result analogous to Chandrasekhar's, that is,

$$\int_0^1 H(\mu) d\mu = \frac{2}{c}(1 - \sqrt{1-c}) \ ,$$

can be established at this stage. As $(P^{-1})_{11} = 1/2$,

$$H_1 = \frac{1}{2} \left(I + (\beta_\infty)_{11} \right) \ .$$

Relation (32) yields

$$c H_1^2 = (\beta_\infty)_{11} = 2 H_1 - 1$$

with the solution, consistent with $(\beta_\infty)_{11} < 1$ for $c < 1$,

$$H_1 = \frac{1}{c}(1 - \sqrt{1-c}) \ . \tag{33}$$

To solve (32), let $\beta_\infty = KM$ with K necessarily symmetric, and let $\theta = P^{-1} M$.

Then

$$c H H' = \theta K + K \theta' \ ,$$

and

$$H = \frac{1}{2} e_1 + \theta K e_1 \ . \tag{34}$$

The matrix θ is independent of c. Let λ_i and ψ_i be the i'th eigenvalue and eigenvector of θ.

Then

$$\theta \psi_i = \lambda_i \psi_i \quad \text{or} \quad M \psi_i = \lambda_i P \psi_i \ , \tag{35}$$

with the normalization

$$\psi_i' M \psi_j = \delta_{ij} \ . \tag{36}$$

With the transformation

$$2\lambda = 1 - \zeta \quad ,$$

the characteristic equation (35) in the P_N approximation reduces to the condition

$$(N+1)!\, P_{N+1}(\zeta_i) = 0 \quad . \tag{37}$$

The eigenvalues ζ_i are then the roots of the (N + 1)th order Legendre polynomial and it follows that

$$0 < \lambda_i < 1 \quad , \quad (i = 1,\ldots,N+1) \quad .$$

The k'th component of the eigenvector ψ_i is then

$$\psi_{ik} \propto (-1)^{k-1} P_{k-1}(\zeta_i) \, , \quad (k = 1,\ldots,N+1) \quad . \tag{38}$$

The values λ_i and ψ_i may then be assumed known, and they are of course independent of c .

A solution for K and hence β_∞ may now be obtained in terms of these known vectors ψ_i and roots λ_i .

With

$$H = \sum_i h_i \psi_i \quad ,$$

it follows from equations (34) that

$$K = c \sum_i \sum_j \frac{h_i h_j}{\lambda_i + \lambda_j} \psi_i \psi_j' , \tag{39}$$

and

$$h_i = \frac{1}{2} \left\{ (\psi_{i1} + \psi_{i2}) + c \sum_j \frac{\lambda_i h_i h_j}{\lambda_i + \lambda_j} \psi_{j1} \right\} \quad . \tag{40}$$

Solution of the non-linear Equation 40 yields the h coefficients, and from Equation 39, the infinite medium reflection matrix:

$$\beta_\infty = c \left[\sum_i \sum_j \frac{\lambda_i}{\lambda_i + \lambda_j} h_i h_j \psi_i \psi_j' \right] P \quad . \tag{41}$$

Corresponding to Equation 40 is Chandrasekhar's integral equation

$$H(\mu) = 1 + \frac{1}{2} c \mu H(\mu) \int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu' \quad . \tag{42}$$

6. THE EMERGENT FLUX IN MILNE'S PROBLEM

A source at $(+\infty)$ in an isotropically scattering medium, with c = 1 , which occupies the region $(0 < x < \infty)$ produces an emergent flux defined by the vector $B^-(0)$ in the double P_N approximation. This vector is invariant with respect to the addition of a slab of any finite thickness. By the addition of a slab of infinitesimal thickness δt, a new incident flux $B_\infty^+(0)$ and reflected flux $\beta_\infty B_\infty^+(0)$ is introduced. The thin slab alone, in the region $(-\delta t < x < 0)$, is characterized by the surface fluxes

$$\left.\begin{array}{lll} B^-(0) \; , \; B^+(0) = \delta t\, f\, B^-(0) & \text{at } x = 0 & \\ \text{and} \quad B^-(\delta t) = (I - \delta t\, g)\, B^-(0) & \text{at } x = -\delta t \; . & \end{array}\right\}$$

For continuity at $x = 0$,

$$B_\infty^+(0) = \delta t\, f\, B^-(0) ,$$

$$B^-(0) = B_\infty^-(0) + \beta_\infty B_\infty^+(0) ,$$

and therefore

$$B^-(0) = (I + \beta_\infty \delta t\, f) B_\infty^-(0) .$$

The principle of invariance gives

$$B^-(\delta t) = B_\infty^-(0)$$

$$\therefore (I - \delta t\, g)(I + \beta_\infty \delta t\, f) B_\infty^-(0) = B_\infty^-(0) ,$$

which implies that

$$\beta_\infty f B_\infty^-(0) = g B_\infty^-(0) . \tag{43}$$

For isotropic scattering

$$(I + \beta_\infty) M^{-1} e_1 e_1' B_\infty^-(0) = M^{-1} P B_\infty^-(0)$$

and if $B_\infty^-(0)$, the emergent flux vector, is chosen to have leading element unity, then

$$B_\infty^-(0) = P^{-1}(I + \beta_\infty') e_1 = H. \tag{44}$$

Chandrasekhar's $H(\mu)$ function on the other hand, is normalized to $H(0) = 1$.

For $c \neq 1$, the principle of invariance is weakened, to the extent that the emergent flux obtained by addition of a thin slab is only similar to the original emergent flux. Thus

$$B^-(\delta t) = (1 - \alpha\, \delta t) B_\infty^-(0)$$

where α is some constant. This leads to the result

$$(g - \beta_\infty f) B_\infty^-(0) = \alpha B_\infty^-(0) ,$$

or (45)

$$(I - c H e_1') B_\infty^-(0) = \alpha P^{-1} M B_\infty^-(0) .$$

As ψ_i $(i = 1,\ldots,N+1)$ are the eigenvectors of $\theta = P^{-1} M$ and assumed known, the expansion

$$B_\infty^-(0) = \sum_i b_i \psi_i$$

in (45) leads to the relation

$$b_i = \frac{c\, h_i}{1 - \alpha \lambda_i} \sum_j b_j \psi_{j1} \tag{46}$$

giving the characteristic equation for α,

$$c \sum_i \frac{h_i \psi_{i1}}{1 - \alpha \lambda_i} = 1 , \tag{47}$$

and the solution

$$b_i = \frac{c\,h_i}{1-\alpha\,\lambda_i} \quad . \tag{48}$$

With the appropriate root α, the emergent flux vector with leading element unity is

$$B_\infty^-(0) = c \sum_i \frac{h_i\,\psi_i}{1-\alpha\,\lambda_i} \quad . \tag{49}$$

Because of relation (47), this can be written more concisely as

$$B_\infty^-(0) = c\,(I - \alpha\, P^{-1}\, M)\, H \quad . \tag{50}$$

For $c = 1$, the correct value of α is zero. Now it is known that for $c = 1$ the root ω_1^2 of the matrix G (Equation 10) is zero, which suggests a relation between α and $|\omega_1|$. Physically this must be so, as the asymptotic flux in a source-free medium is a single exponential varying as $\exp(\pm\,|\omega_1|\,x)$.

The proof of the identity of α and $|\omega_1|$ can be established as follows:

The analytical solution of Milne's problem, with distributions far from the boundary taken as

$$\left.\begin{aligned} B_\infty^+(x) &= U\,\Phi^+(x)\,e_1 \\ B_\infty^-(x) &= V\,\Phi^+(x)\,e_1 \end{aligned}\right\} ,$$

gives the emergent flux

$$\begin{aligned} B_\infty^-(0) &= (V - U\,V^{-1}\,U)\,e_1 \,, \\ &= 2\,(I - \beta_\infty)\,\xi_{\cdot 1} \quad . \end{aligned} \tag{51}$$

by the use of relations (15) and (17). By Equation 45 and the fundamental invariance principle expressed by relation (30) this leads to the result

$$(I - \beta_\infty)^{-1}(I + \beta_\infty)\,M^{-1}\,P_e\;\xi_{\cdot 1} = \alpha\,\xi_{\cdot 1} \quad . \tag{52}$$

Comparison with the previous result (25) shows that $\alpha = |\omega_1|$. The emergent flux for Milne's problem with $c \neq 1$ is therefore

$$B_\infty^-(0) = c \sum_i \frac{h_i\,\psi_i}{1 - |\omega_1|\,\lambda_i} \quad . \tag{53}$$

The corresponding result of Chandrasekhar gives an emergent flux distribution varying as

$$H(\mu) \big/ (1 - |\omega_1|\,\mu) \quad . \tag{54}$$

7. CONCLUSION

The outstanding virtue of the response matrix formulation of the double P_N approximation is the ease of problem formulation and solution with multiple arrays. In neutron transport theory, problems in slab geometry become similar to response problems in other branches of engineering. This paper has shown how the same approach using little more than algebra and a few results on eigenvalues and eigenvectors of matrices can lead to a theoretical development completely analogous to the classical theory of radiative transfer. In particular it has shown how infinite medium reflection matrices may be computed from a basic system of vectors and roots which are independent of the c parameter of the material, when the scattering is isotropic.

The derivation of the partial differential equations for the reflection and transmission matrices has made available an alternative method for practical problem solution. It is possible to proceed via analytical solutions involving the source-free medium solutions $\Phi^{\pm}(x)$, by a completely numerical approach solving the differential equations for the generalized flux vector X(x), or by numerical integration of the response matrix equations in any one region for values to be substituted in response theory solutions. This latter method is clearly advantageous when the problem involves multiple slabs of the same material, as β and γ are obtained as functions of thickness.

In developing the theory, the eigenvalues λ_i were seen to be the roots of the Legendre polynomial of order (N + 1). The values λ_i thus correspond to a set of discrete values of μ, or neutron directions, in each of the hemispheres in which $P_n^{\pm}(\mu)$ are defined. It is apparent therefore, that the results developed here again demonstrate the basic similarity of the methods of discrete ordinates and of spherical harmonics in treating neutron transport problems.

8. NOTATION

$A^{\pm}$	Arbitrary vectors in the source-free medium solution.
$B_n^{\pm}(x)$	Coefficients in the half-range Legendre polynomial expansions of the angular fluxes in the $\pm x$ directions, $(n = 0,\ldots,N)$.
$B^{\pm}(x)$	Vectors with components $(B^{\pm})_i = B_{i-1}^{\pm}$, $(i = 1,\ldots,N+1)$.
c	Ratio of scattering to total cross section (Σ_s/Σ_t).
D	Differential operator $\frac{d}{dx}$.
e_i	Unity vector, that is, $(e_i)_j = \delta_{ij}$.
E_{ij}	Unity matrix, that is, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.
f, g	Basic coefficient matrices for the derivatives of $B^{\pm}(x)$.
H	A particular $B^{\pm}$ vector, corresponding to the emergent flux distribution in Milne's problem with $c = 1$.
h_i	Coefficients of the ψ_i vectors in the expansion of H.
I	Unit matrix.
J(x)	Neutron current in the +x direction.
$P_l(\mu)$	l'th Legendre polynomial in μ.
$P_n^{\pm}(\mu)$	n'th half-range Legendre polynomials in the $\pm x$ directions.
Q	Isotropic neutron source strength for unit length in total mean free paths.
t	Thickness of slab.
x	Spatial coordinate related to true distance z by $dx = \Sigma_t\, dz$.
α_l	Ratio of l'th Legendre coefficient of the differential scattering cross section, to total scattering cross section.
$\beta(t)$	Reflection matrix for slab of thickness t.
β_∞	Infinite medium reflection matrix.

$\gamma(t)$	Transmission matrix for thickness t .
λ_i	i'th eigenvalue of matrix $\theta = P^{-1} M$.
μ	Cosine of angle between neutron direction vector and x axis.
ξ	Modal matrix for matrix G, with j'th column $\xi_{.j}$ the j'th eigenvector.
$\phi(x,\mu)$	2π(neutron angular flux for unit solid angle).
$\phi(x)$	Total scalar flux.
$\Phi^{\pm}(x)$	Diagonal matrix of source–free medium solutions exp $(\pm \lvert\omega_i\rvert x)$.
ψ_i	i'th eigenvector of matrix θ , with first and second components ψ_{i1} and ψ_{i2}
$\lvert\omega_i\rvert$	Absolute value of square root of i'th eigenvalue of G.
Ω^2	Spectral matrix of G with elements ω_i^2 .
A', A^{-1}	Transpose and inverse of matrix A .

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APPENDIX 1

MATRIX ELEMENTS

M matrix

$$P_3 \left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right. \begin{bmatrix} 1 & 1 & & & & \\ 1 & 3 & 2 & & & \\ & 2 & 5 & 3 & & \\ & & 3 & 7 & 4 & \\ & & & \ddots & \ddots & \ddots \\ & & & n-1 & 2n-1 & n \\ & & & & \ddots & \ddots \end{bmatrix}$$

In terms of the unit matrices E_{ij}

$$M = \sum_{i=1}^{N+1} \left[(i-1) E_{i-1,i} + (2i-1) E_{ii} + i \, E_{i,i+1} \right] \quad .$$

Similarly,

$$P = 2 \sum_{i=1}^{N+1} (2i-1) E_{ii}$$

$$S_0 = E_{11}$$

$$S_1 = \frac{3}{4} (E_{11} + E_{12} + E_{21} + E_{22})$$

$$S_2 \quad \frac{5}{16} (9E_{22} + 3E_{23} + 3E_{32}) \quad .$$