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ABSTRACT

A method is given for solving the inverse reaction problem to obtain complex potentials as in the optical model of the nucleus. The method will reproduce reaction data to the accuracy with which the reaction matrix can be least squares fitted to a sum of simple poles.

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CONTENTS

1. INTRODUCTION

In a previous paper (Cook 1970) it was shown that given the complex phase shifts in each channel of a reaction, a set of real channel wave functions and potentials could be evaluated which reproduced the phase shifts to the accuracy with which one could least squares fit the reaction matrix to a sum of poles. This formulation can be generalized to accommodate spin-orbit coupling, and to derive an equivalent alternative theory which uses complex potentials, as in the optical model (Preston 1962).

2. THE OPTICAL MODEL

The basic radial wave equation for the optical model wave function $U_{c\ell}$ for the incident channel c and orbital angular momentum ℓ is (Preston 1962)

$$\frac{d^2U_{c\ell}}{dr^2} + \left[k_c^2 - V_c + \frac{\ell(\ell+1)}{r^2}\right] U_{c\ell} = 0 \quad . \tag{1}$$

This wave function has the asymptotic form

$$\begin{aligned} U_{c\ell}\ (r \to \infty) \to \mathfrak{D}_{c\ell}\Big[&(1 + e^{-2\beta_{c\ell}})\sin\left(k_c r - \frac{\ell\pi}{2} + \alpha_{c\ell}\right) \\ &+ i\,(1 - e^{-2\beta_{c\ell}})\cos\left(k_c r - \frac{\ell\pi}{2} + \alpha_{c\ell}\right)\Big] \ , \end{aligned} \tag{2}$$

where

$\mathfrak{D}_{c\ell}$ = an arbitrary complex constant

$\alpha_{c\ell}$ = the real part of the phase shift

$\beta_{c\ell}$ = the imaginary part of the phase shift

$\delta_{c\ell} = \alpha_{c\ell} + i\,\beta_{c\ell}$.

Neglecting spin for the present, the cross sections are as follows:

(a) Elastic:

$$\sigma_{cc} = \frac{\pi}{k_c^2}\sum_\ell (2\ell+1)\,\left|1 - e^{2i\delta_{c\ell}}\right|^2 \ .$$

(b) Absorption:

$$\sigma_{ca} = \frac{\pi}{k_c^2}\sum_\ell (2\ell+1)\,\left(1 - e^{-4\beta_{c\ell}}\right)$$

(c) Total:

$$\sigma_{cT} = \frac{2\pi}{k_c^2}\sum_\ell (2\ell+1)\left[1 - \mathrm{Re}\ e^{2i\delta_{c\ell}}\right] \quad . \tag{3}$$

Let us suppose there are two channels in the reaction. We may associate with the cross sections a two-channel S-matrix

$$S_{cc'} \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

such that $S_{12} \equiv S_{21}$, as required by time reversal invariance,

and $S_{11} = e^{2i\delta_{1\ell}}$

$$|S_{12}|^2 = 1 - e^{-4\beta_{1\ell}} \quad . \tag{4}$$

Unitarity and time reversal invariance ensure only three independent elements. Thus a phase in S_{22} and the phase of S_{12} are arbitrary. In order to determine these elements, we introduce the equiphase assumption on the reduced transition matrix (Cook 1967), $\overset{\prime}{\underline{T}}$.

The $\underline{T}_\ell$ matrix is defined as

$$\underline{T}_\ell = \frac{1}{2i}(\underline{S}_\ell - \underline{I}) \ . \tag{5}$$

The reduced transition matrix is defined by the equation

$$\underline{T}_\ell = \frac{1}{2i}(\underline{\Omega}_\ell^2 - \underline{I}) + \underline{\Omega}_\ell \, \overset{\prime}{\underline{T}}_\ell \, \underline{\Omega}_\ell \quad , \tag{6}$$

where $\underline{\Omega}$ is the diagonal matrix (Lane and Thomas 1958)

$$\underline{\Omega}_\ell = \begin{pmatrix} e^{-i\theta_{1\ell}} & 0 \\ 0 & e^{-i\theta_{2\ell}} \end{pmatrix} \quad , \tag{7}$$

and $\theta_{1\ell}$, $\theta_{2\ell}$ are the hard sphere phase shifts for channels 1 and 2. This assumption ensures that the Bohr (1936) compound nucleus condition holds for the resonant part of the reaction. Since the $\underline{S}$ matrix is unitary, it can be shown readily that the $\overset{\prime}{\underline{T}}$ matrix must be of the form

$$\overset{\prime}{\underline{T}}_\ell = \frac{1}{2i}\begin{pmatrix} \eta e^{2i\overset{\prime}{\alpha}_{1\ell}} - 1 & \sqrt{1-\eta^2}\, e^{2i\phi_\ell} \\ \sqrt{1-\eta^2}\, e^{2i\phi_\ell} & -\eta e^{2i(2\phi_\ell - \overset{\prime}{\alpha}_{1\ell})} - 1 \end{pmatrix}, \tag{8}$$

where $\qquad \eta_\ell = e^{-2\beta_{1\ell}}$

ϕ_ℓ is an independent function

and $\qquad \overset{\prime}{\alpha}_{c\ell} = \alpha_{c\ell} + \theta_{c\ell}$ = the reduced phase shift.

The equiphase condition yields

$$\mathrm{Tan}\ 2\phi = \frac{-\eta \sin 2\overset{\prime}{\alpha}_{1\ell}}{1 - \eta\cos 2\overset{\prime}{\alpha}_{1\ell}} \quad , \quad \phi_\ell = \frac{\overset{\prime}{\alpha}_{1\ell} + \overset{\prime}{\alpha}_{2\ell}}{2} \quad , \tag{9}$$

from which S_{22} and the phase of S_{12} can be found. The second condition follows from unitarity and time reversal alone. The first allows the determination of ϕ. However, in solving the inverse reaction problem, one associates a real channel wave function $\psi_{c\ell}$ with each phase shift $\delta_{c\ell}$ such that each $\psi_{c\ell}$ obeys Schrodinger's equation.

$$\frac{d^2\psi_{1\ell}}{dr^2} + \left[k_1^2 - V_1 + \frac{\ell(\ell+1)}{r^2}\right]\psi_{1\ell} = 0 \tag{10a}$$

$$\frac{d^2\psi_{2\ell}}{dr^2} + \left[k_2^2 - V_2 + \frac{\ell(\ell+1)}{r^2}\right]\psi_{2\ell} = 0 \tag{10b}$$

$\psi_{1\ell}$ and $\psi_{2\ell}$ have asymptotic forms

$$\psi_{1\ell} \rightarrow C_{1\ell} \sin\left(k_1 r - \frac{\ell\pi}{2} + \alpha_{1\ell}\right)$$

$$\psi_{2\ell} \rightarrow C_{2\ell} \sin\left(k_2 r - \frac{\ell\pi}{2} + \alpha_{2\ell}\right) \quad .$$

$\psi_{1\ell}$ and $\psi_{2\ell}$ are not independent of each other but are coupled through their asymptotic normalisation constants, $C_{1\ell}$ and $C_{2\ell}$. This is how one determines η_ℓ.

The wave functions $\psi_{1\ell}$ and $\psi_{2\ell}$ may be re-normalised to yield functions $\phi_{1\ell}$ and $\phi_{2\ell}$ which satisfy (10a) and (b) and which have the asymptotic forms

$$\begin{aligned} \phi_{1\ell} &\rightarrow \sin\left(k_1 r - \frac{\ell\pi}{2} + \alpha_{1\ell}\right) \\ \phi_{2\ell} &\rightarrow \sin\left(k_2 r - \frac{\ell\pi}{2} + \alpha_{2\ell}\right) \quad . \end{aligned} \tag{11}$$

3. THE EQUIVALENT OPTICAL MODEL

Instead of using $(\alpha_{1\ell}, \eta_\ell, \phi_\ell)$ we will use the equivalent variables $(\alpha_{1\ell}, \alpha_{2\ell}, \eta_\ell)$. The problem is to find for given $\alpha_{1\ell}$, $\alpha_{2\ell}$ and η_ℓ the complex potential in Equation 1 which yields the same T_ℓ matrix as the wave functions $\phi_{1\ell}$ and $\phi_{2\ell}$. This can be done as follows: The inverse reaction calculation (Cook 1970) is used to determine effective potentials $\overline{V}_1$ and $\overline{V}_2$ such that wave functions $\overline{\phi}_{1\ell}$ and $\overline{\phi}_{2\ell}$ satisfy the transposed energy equations

$$\begin{aligned} \frac{d^2\overline{\phi}_{1\ell}}{dr^2} + \left[k_2^2 - \overline{V}_1 + \frac{\ell(\ell+1)}{r^2}\right] \overline{\phi}_{1\ell} &= 0 \\ \frac{d^2\overline{\phi}_{2\ell}}{dr^2} + \left[k_1^2 - \overline{V}_2 + \frac{\ell(\ell+1)}{r^2}\right] \overline{\phi}_{2\ell} &= 0 \quad , \end{aligned} \tag{12}$$

with asymptotic forms

$$\overline{\phi}_{1\ell} \xrightarrow[r \to \infty]{} \sin\left(k_2 r - \frac{\ell\pi}{2} + \alpha_{1\ell}\right) \tag{13a}$$

$$\overline{\phi}_{2\ell} \xrightarrow[r \to \infty]{} \sin\left(k_1 r - \frac{\ell\pi}{2} + \alpha_{2\ell}\right) \quad . \tag{13b}$$

We now introduce asymptotic normalisation constants $A_{1\ell}$, $A_{2\ell}$, $B_{1\ell}$ and $B_{2\ell}$ such that for large r

$$A_{1\ell}\,\phi_{1\ell} + B_{1\ell}\,\overline{\phi}_{2\ell} \rightarrow \sin\left(k_1 r - \frac{\ell\pi}{2} + \delta_{1\ell}\right)$$

$$A_{2\ell}\,\phi_{2\ell} + B_{2\ell}\,\overline{\phi}_{1\ell} \rightarrow \sin\left(k_2 r - \frac{\ell\pi}{2} + \delta_{2\ell}\right) \quad .$$

Hence we find

$$A_{1\ell} = \frac{\sin(\delta_{1\ell} - \alpha_{2\ell})}{\sin(\alpha_{1\ell} - \alpha_{2\ell})}$$

$$A_{2\ell} = \frac{\sin(\delta_{2\ell} - \alpha_{1\ell})}{\sin(\alpha_{2\ell} - \alpha_{1\ell})}$$

$$B_{1\ell} = -\frac{\sin(\delta_{1\ell}-\alpha_{1\ell})}{\sin(\alpha_{1\ell}-\alpha_{2\ell})} = -i\,\frac{\sin h\,\beta_{1\ell}}{\sin(\alpha_{1\ell}-\alpha_{2\ell})} \tag{14}$$

$$B_{2\ell} = -i\,\frac{\sin h\,\beta_{2\ell}}{\sin(\alpha_{2\ell}-\alpha_{1\ell})} \quad .$$

The functions $U_{1\ell}$ and $U_{2\ell}$ defined as

$$\begin{aligned} U_{1\ell} &= A_{1\ell}\,\phi_{1\ell}\,(k_1 r) + B_{2\ell}\,\bar{\phi}_{2\ell}\,(k_1 r) \\ U_{2\ell} &= B_{1\ell}\,\bar{\phi}_{1\ell}\,(k_2 r) + A_{2\ell}\,\phi_{2\ell}\,(k_2 r) \end{aligned} \tag{15}$$

will satisfy Equation 1 provided

$$\begin{aligned} \mathcal{V}_1 &= \frac{A_1 V_1 \phi_1 + B_2 \bar{V}_2 \bar{\phi}_2}{A_1 \phi_1 + B_2 \bar{\phi}_2} \\ \mathcal{V}_2 &= \frac{B_1 \bar{V}_1 \bar{\phi}_1 + A_2 V_2 \phi_2}{B_1 \bar{\phi}_1 + A_2 \phi_2} \end{aligned} \tag{16}$$

are the complex potentials in each channel.

More generally if we use source terms

$$V_c\,\phi_{\ell c} = \rho_c \quad , \quad \mathcal{V}_c\,U_{\ell c} = \xi_c \tag{17}$$

and the multichannel equations with the given asymptotic conditions (13) and (2), we will find

$$\begin{aligned} e^{i\delta_{c'\ell}} &= A_c\, e^{i\alpha_{c\ell}} + \sum_{c'\neq c} B_{cc'}\, e^{i\alpha_{c'\ell}} \\ e^{-i\delta_{c\ell}} &= A_c\, e^{-i\alpha_{c\ell}} + \sum_{c'\neq c} B_{cc'}\, e^{-i\alpha_{c'\ell}} \quad , \end{aligned} \tag{18}$$

from which the complex constants A_c and $B_{cc'}$ can be evaluated only in the case where $n = 2$. For $n > 2$, the $B_{cc'}$ are undetermined without additional constraints. We will then be able to find the general complex source in each channel from the equations

$$\xi_c = A_c \rho_c + \sum_{c'\neq c} B_{cc'}\,\bar{\rho}_{c'} \quad . \tag{19}$$

The spin-orbit contribution can be isolated by defining $\phi_{cJ\ell}$ for each J, ℓ, and establishing the contribution from the difference of $\rho_{cJJ+\frac{1}{2}}$ and $\rho_{cJJ-\frac{1}{2}}$.

4. CONCLUSION

A method has been given for calculating optical potentials in multichannel problems where only the complex scattering phase shifts are known. The method gives both the complex potential required to reproduce the phase shift and an optical model wave function.

Although such functions reproduce all of the data, they are not, of course, unique. This difficult question of uniqueness is worthy of further investigation. We point out, however, that the calculation of a set of consistent ρ_c's has been performed successfully for a wide range of phase shifts (Clayton and Cook,AAEC report in preparation), and given this information, the above method cannot fail to reproduce the data.

The constants A_c and $B_{cc'}$ are not uniquely determined because the optical model groups all reaction channels together and cannot give information on these outgoing channels other than the sum of the cross sections.

5. REFERENCES

Bohr, N. (1936). – Nature 137: 344.

Cook, J.L. (1967). – Aust. J. Phys. 20: 355.

Cook, J.L. (1970). – AAEC/TM560.

Lane, A.M. and Thomas, R.G. (1958). – Rev. Mod. Phys. 30: 257.

Preston, M.A. (1962). – Physics of the Nucleus. Addison-Wesley, Massachusetts.