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AXIAL TEMPERATURE DISTRIBUTIONS IN CONCENTRIC
COOLING CHANNELS SURROUNDING A
HEAT GENERATING SOURCE

by

F. H. CARR

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ABSTRACT

A set of simultaneous differential equations is established to describe the temperature distribution for coolant flow in three concentric channels separated by walls of finite thermal conductivity and surrounding a cylindrical heat source. The solution of this set of differential equations is dependent on the heat source function $Q(z)$ which must be known or specified. An assumed function $Q(z) = \text{constant}$ is taken as being a representative case and the resultant solutions are applied to several geometric arrangements.

The system of three channel flow reduces to two channel flow when there is zero heat flow across the outer intermediate wall. This condition may arise if the wall is a perfect insulator, or if the flow in the third channel is zero. For the former case the temperature in the third channel is constant over its length, and in the second the stationary coolant assumes the temperature of the coolant in the middle channel. From the set of differential equations established for two channel flow, treatment similar to that used for three channel flow is adopted and the resultant solutions are applied to several geometric arrangements. Brief mention is made of single channel flow. By assuming that the heat source is zero in the case of two channel flow, expressions are derived for the temperature distribution in parallel flow heat exchangers.

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APPENDIX I - Value of α (Section 3.2) When $C_1 = C_2$

Figure 1 Three concentric coolant channels surrounding a heat generating source

Figure 2 Three channel (re-entrant) flow

Figure 3 Counter and unidirectional flow heat exchange

1. INTRODUCTION

The purpose of the following analysis is to derive a method for determining the axial temperature distribution in a system of three concentric coolant channels separated by walls of heat conducting media and surrounding a cylindrical heat generating source. The following assumptions are made:

- (i) The physical properties of the coolant fluids may be represented by the values corresponding to the mean temperature of the coolant over the channel length.
- (ii) The enclosing boundaries of the system between the arbitrarily fixed flow boundaries are thermally insulated.
- (iii) The channel flow area is constant with respect to channel length.
- (iv) All heat flow within the channel walls is normal to the coolant stream.

2. THEORY

2.1 Derivation of Equations

Figure 1 shows the general arrangement of a cylindrical heat generating source surrounded by three concentric coolant channels a, b, and d, and the outer wall of channel d forms the boundary of the system. Let δT be the coolant temperature rise corresponding to δZ ,

$\delta S = P \delta Z$, be the surface area corresponding to δZ , and
 $Q(Z)$ be the heat generated per unit length.

Taking longitudinal and transverse heat balances the following equations can be written down:

Channel a

$$Q(z) \delta z = W_a C_a \delta T_a + U_i P_i \delta z (T_a - T_b) ,$$

$$\text{or } Q(z) = W_a C_a \frac{dT_a}{dz} + U_i P_i (T_a - T_b) ,$$

$$\text{and } \frac{dT_a}{dz} = A Q(z) - C_1 (T_a - T_b) , \quad (2.1)$$

$$\text{where } C_1 = \frac{U_i P_i}{W_a C_a} , \quad A = \frac{1}{W_a C_a} .$$

Channel b

$$U_i P_i \delta z (T_a - T_b) = -W_b C_b \delta T_b + U_o P_o \delta z (T_b - T_d) ,$$

$$\text{or } U_i P_i (T_a - T_b) = -W_b C_b \frac{dT_b}{dz} + U_o P_o (T_b - T_d) ,$$

$$\text{and } \frac{dT_b}{dz} = -C_2 (T_a - T_b) + C_3 (T_b - T_d) , \quad (2.2)$$

$$\text{where } C_2 = \frac{U_i P_i}{W_b C_b} , \quad C_3 = \frac{U_o P_o}{W_b C_b}$$

Channel d

$$U_o P_o \delta z (T_b - T_d) = W_d C_d \delta T_d \quad ,$$

$$\text{or } U_o P_o (T_b - T_d) = W_d C_d \frac{dT_d}{dz} \quad ,$$

$$\text{and } \frac{dT_d}{dz} = C_4 (T_b - T_d) \quad , \tag{2.3}$$

$$\text{where } C_4 = \frac{U_o P_o}{W_d C_d} \quad .$$

2.2 Analysis

2.2.1 Three channel flow

The system of simultaneous differential equations:

$$(2.1) \quad (D + C_1) T_a - C_1 T_b = AQ(z)$$

$$(2.2) \quad C_2 T_a + (D - C_2 - C_3) T_b + C_3 T_d = 0$$

$$(2.3) \quad -C_4 T_b + (D + C_4) T_d = 0 \quad ,$$

reduces to:

$$\left. \begin{aligned} D(D + \lambda_1)(D - \lambda_2) T_a &= \{D(D + C_4 - C_3 - C_2 - C_2 C_4)\} AQ(z) \\ D(D + \lambda_1)(D - \lambda_2) T_b &= -\{C_2(D + C_4)\} AQ(z) \\ D(D + \lambda_1)(D - \lambda_2) T_d &= -C_2 C_4 AQ(z) \quad , \end{aligned} \right\} \tag{2.4}$$

$$\text{where } b = C_1 - C_2 - C_3 + C_4 \quad ,$$

$$C = C_1 C_4 - C_2 C_4 - C_1 C_3 \quad ,$$

$$\lambda_1 = \frac{b + \sqrt{b^2 - 4C}}{2} \quad ,$$

$$\lambda_2 = \frac{-b + \sqrt{b^2 - 4C}}{2} \quad .$$

The solution of (2.4) depends on the form of Q(z), and the value and combination of the roots 0, $-\lambda_1$, and λ_2 .

The variation of Q(z) must be specified and for this analysis will be assumed to be constant. However, whatever value of Q(z) is given, provided it is amenable to analytical treatment, the method of solution of (2.4) will be similar to the following:

For Q(z) = constant, (2.4) reduces to

$$\left. \begin{aligned} D(D + \lambda_1)(D - \lambda_2) T_a &= -C_2 C_4 AQ \\ D(D + \lambda_1)(D - \lambda_2) T_b &= -C_2 C_4 AQ \\ D(D + \lambda_1)(D - \lambda_2) T_d &= -C_2 C_4 AQ \end{aligned} \right\} \tag{2.5}$$

The roots $-\lambda_1$ and λ_2 are real and unequal provided that $b^2 - 4C > 0$.

Now,

$$\begin{aligned} b^2 - 4C &= (C_1 - C_2)^2 + (C_3 - C_4)^2 + 2(C_1C_3 - C_2C_3 + C_2C_4 - C_1C_4) + 4C_2C_3 \\ &= \{(C_1 - C_2) + (C_3 - C_4)\}^2 + 4C_2C_3 \end{aligned} \quad (2.6)$$

which is always positive where $C_1, C_2, C_3,$ and $C_4 > 0$. The roots of Equation 2.5 therefore are 0, $-\lambda_1$, and λ_2 , where $-\lambda_1$ and λ_2 are real and unequal.

A further requirement for the solution of (2.5) is to examine the case where either $-\lambda_1$ or λ_2 is equal to zero. This can only occur if the term C in the expression $b^2 - 4C$ is equal to zero.

$$\text{Then } C = C_1C_4 - C_2C_4 - C_1C_3$$

$$\begin{aligned} &= \frac{U_i P_i \cdot U_o P_o}{W_a C_a \cdot W_d C_d} - \frac{U_i P_i \cdot U_o P_o}{W_b C_b \cdot W_d C_d} - \frac{U_i P_i \cdot U_o P_o}{W_a C_a \cdot W_b C_b} \\ &= \frac{U_i P_i \cdot U_o P_o}{W_a C_a \cdot W_b C_b \cdot W_d C_d} (W_b C_b - W_a C_a - W_d C_d) \end{aligned} \quad (2.7)$$

and for three channel flow (2.7) is equal to zero only if

$$W_b C_b = W_a C_a + W_d C_d .$$

The roots of Equation 2.5 will then be 0, $-\lambda_1$, and 0, since $\lambda_2 = 0$.

Solution of Equation 2.5 for $W_b C_b = W_a C_a + W_d C_d$ (roots: 0, $-\lambda_1, \lambda_2$)

i.e. λ_1 and $\lambda_2 \neq 0$

The solution of Equation 2.5 for this case will be of the form:

$$T_b = K_1 + K_2 e^{-\lambda_1 z} + K_3 e^{\lambda_2 z} - \frac{C_2 C_4 A Q z}{C} ,$$

$$T_a = K_4 + K_5 e^{-\lambda_1 z} + K_6 e^{\lambda_2 z} - \frac{C_2 C_4 A Q z}{C} ,$$

$$T_d = K_7 + K_8 e^{-\lambda_1 z} + K_9 e^{\lambda_2 z} - \frac{C_2 C_4 A Q z}{C} ,$$

where K_1, \dots, K_9 , are arbitrary constants, and since Equation 2.5 is of degree 3 in D, there are three arbitrary constants in the solution. To eliminate the other 6 arbitrary constants the above equations are substituted into Equations 2.1 and 2.3 to obtain

$$\left. \begin{aligned} T_a &= K_1 + \frac{C_1}{C_1 - \lambda_1} K_2 e^{-\lambda_1 z} + \frac{C_1}{C_1 + \lambda_2} K_3 e^{\lambda_2 z} + \frac{A Q}{C} [C_4 - C_3 - C_2 C_4 z] \\ T_b &= K_1 + K_2 e^{-\lambda_1 z} + K_3 e^{\lambda_2 z} - \frac{C_2 C_4 A Q z}{C} \\ T_d &= K_1 + \frac{C_4}{C_4 - \lambda_1} K_2 e^{-\lambda_1 z} + \frac{C_4}{C_4 + \lambda_2} K_3 e^{\lambda_2 z} + \frac{A Q}{C} [C_2 - C_2 C_4 z] \end{aligned} \right\} \quad (2.8)$$

where:

$$b = C_1 - C_2 - C_3 + C_4$$

$$C = C_1 C_4 - C_2 C_4 - C_1 C_3$$

$$\lambda_1 = \frac{b + \sqrt{b^2 - 4C}}{2} \neq 0$$

$$\lambda_2 = \frac{-b + \sqrt{b^2 - 4C}}{2} \neq 0$$

Solution of Equation 2.5 for $W_b C_b = W_a C_a + W_d C_d$ (roots, 0, $-\lambda_1$, 0)

The solution of Equation 2.5 will be of the form:

$$T_b = B_1 + B_2 z + B_3 e^{-\lambda_1 z} - \frac{C_2 C_4 A Q z^2}{2 \lambda_1},$$

$$T_a = B_4 + B_5 z + B_6 e^{-\lambda_1 z} - \frac{C_2 C_4 A Q z^2}{2 \lambda_1},$$

$$T_d = B_7 + B_8 z + B_9 e^{-\lambda_1 z} - \frac{C_2 C_4 A Q z^2}{2 \lambda_1},$$

where B_1, \dots, B_9 are arbitrary constants.

Substitution of the above equations into Equations 2.1 and 2.3 to eliminate the constants B_4, \dots, B_9 , gives:

$$\left. \begin{aligned} T_a &= B_1 + B_2 \left(z - \frac{1}{C_1}\right) + \frac{C_1}{C_1 - \lambda_1} B_3 e^{-\lambda_1 z} - \frac{C_2 A Q}{\lambda_1} \left[\frac{C_4 z^2}{2} - \frac{C_4 z}{C_1} + \frac{C_4}{C_1^2} - \frac{\lambda_1}{C_1 C_2} \right] \\ T_b &= B_1 + B_2 z + B_3 e^{-\lambda_1 z} - \frac{C_2 C_4 A Q z^2}{2 \lambda_1} \\ T_d &= B_1 + B_2 \left(z - \frac{1}{C_4}\right) + \frac{C_4}{C_4 - \lambda_1} B_3 e^{-\lambda_1 z} - \frac{C_2 A Q}{\lambda_1} \left[\frac{1}{C_4} z + \frac{C_4 z^2}{2} \right] \end{aligned} \right\} (2.9)$$

where $\lambda_1 = b = C_1 - C_2 - C_3 + C_4$.

2.2.2 Two channel flow

It is of interest to examine the condition where the heat flow across the wall b-d is zero. From Section 2.1, channel b we have:

$$U_i P_i (T_a - T_b) = -W_b C_b \frac{dT_b}{dz} + U_o P_o (T_b - T_d)$$

where the last term of this equation is the heat source term for channel d. This term can be zero if:

- (i) $U_o = 0$, in which case C_3 and $C_4 = 0$ and from (2.3), $T_d = \text{constant}$, and
- (ii) $T_b - T_d = 0$, or $T_b = T_d$, in which case

$$U_o P_o (T_b - T_d) = W_d C_d \frac{dT_d}{dz} = 0$$

Therefore $W_d = 0$.

Hence, the two conditions for zero heat flow across the wall b - d are:

- (a) When $U_o = 0$, then $C_3 = C_4 = 0$, and $T_d = \text{constant}$.
- (b) When $W_d = 0$, then $T_d = T_b$.

Equations (2.1) and (2.2) for both cases reduce to:

$$\left. \begin{aligned} (D + C_1) T_a - C_1 T_b &= A Q(z) \\ C_2 T_a + (D - C_2) T_b &= 0 \end{aligned} \right\} \quad (2.10)$$

Solution of this set of simultaneous differential equations gives:

$$\left. \begin{aligned} D(D + b) T_a &= (D - C_2) A Q(z) \\ D(D + b) T_b &= -C_2 A Q(z) \end{aligned} \right\} \quad (2.11)$$

where $b = C_1 - C_2$.

The solution of (2.11) depends on the variation of the heat source Q with respect to z, and also on the conditions $C_1 \neq C_2$ or $C_1 = C_2$.

As for the case of three channel flow, Q(z) will be assumed constant.

Where $C_1 \neq C_2$:

The solution of Equation 2.11 will be of the form:

$$\left. \begin{aligned} T_b &= K_1 + K_2 e^{-bz} - \frac{C_2 A Q z}{b} \\ T_a &= K_3 + K_4 e^{-bz} - \frac{C_2 A Q z}{b} \end{aligned} \right\} ,$$

which upon substitution into Equations 2.10 to eliminate the constants K_3 and K_4 gives

$$\left. \begin{aligned} T_a &= K_1 + \frac{C_1}{C_2} K_2 e^{-bz} + \frac{A Q}{b} (1 - C_2 z) \\ T_b &= K_1 + K_2 e^{-bz} - \frac{C_2 A Q z}{b} \end{aligned} \right\} \quad (2.12)$$

where $b = C_1 - C_2$ and K_1 and K_2 are arbitrary constants.

Where $C_1 = C_2$:

The solution of Equation 2.11 will be of the form:

$$\left. \begin{aligned} T_b &= B_1 + B_2 z - \frac{C_1 A Q z^2}{2} \\ T_a &= B_3 + B_4 z - \frac{C_1 A Q z^2}{2} \end{aligned} \right\} ,$$

which upon substitution into Equation 2.10 to eliminate the constants B_3 and B_4 gives

$$\left. \begin{aligned} T_a &= B_1 + B_2 \left(z - \frac{1}{C_1} \right) + AQ \left(z - \frac{C_1 z^2}{2} \right) \\ T_b &= B_1 + B_2 z - \frac{C_1 AQ z^2}{2} \end{aligned} \right\}, \quad (2.13)$$

where B_1 and B_2 are arbitrary constants.

2.2.3 Single channel flow

For the case of one channel flow it is obvious from Equation 2.1 that there will be zero heat flow across the wall a-b if $U_i = 0$, or $W_b = 0$, giving the expression for channel a:

$$\frac{dT_a}{dz} = AQ(z), \quad (2.14)$$

where $Q(z)$ must be defined.

2.3 Heat Source Surface Temperature

The surface temperature distribution of the heat source may be determined by the use of the relationship

$$T_s(z) - T_a(z) = \frac{Q(z)}{hP},$$

where P is the perimeter of the heat source, and h is the heat transfer coefficient. If $Q(z) = \text{constant}$, the value of $T_a(z)$ may be determined from one of the previous results appropriate to the system under consideration.

3. APPLICATION TO GIVEN GEOMETRIC ASSEMBLIES

In this section the expressions derived in Section 2 are applied to several assemblies of interest, to determine the respective arbitrary constants.

3.1 Three Channel (Independent) Flow (Figure 1) Application of Equation 2.8, when $W_b C_b \neq W_a C_a + W_d C_d$

Equation 2.8 may be stated for convenience as:

$$\left. \begin{aligned} T_a &= K_1 + d K_2 e^{-\lambda_1 z} + f K_3 e^{\lambda_2 z} + P_2 - P_1 \frac{z}{L} \\ T_b &= K_1 + K_2 e^{-\lambda_1 z} + K_3 e^{\lambda_2 z} - P_1 \frac{z}{L} \\ T_d &= K_1 + g K_2 e^{-\lambda_1 z} + h K_3 e^{\lambda_2 z} + P_3 - P_1 \frac{z}{L} \end{aligned} \right\}, \quad (2.8a)$$

$$\text{where: } d = \frac{C_1}{C_1 - \lambda_1}, \quad f = \frac{C_1}{C_1 + \lambda_2}, \quad g = \frac{C_4}{C_4 - \lambda_1}, \quad h = \frac{C_4}{C_4 + \lambda_2}$$

$$P_1 = \frac{AQ C_2 C_4 L}{C}, \quad P_2 = \frac{AQ}{C} (C_4 - C_3), \quad P_3 = \frac{AQ C_2}{C}.$$

Then applying the boundary conditions:

- (i) $T_a = T_{a_i}$ at $z = 0$
- (ii) $T_b = T_{b_i}$ at $z = L$
- (iii) $T_d = T_{d_i}$ at $z = 0$

to Equation 2.8a gives:

$$\begin{aligned} K_1 + dK_2 + fK_3 &= T_{a_i} - P_2 \\ K_1 + K_2 e^{-\lambda_1 L} + K_3 e^{\lambda_2 L} &= T_{b_i} + P_1 \\ K_1 + gK_2 + hK_3 &= T_{d_i} - P_3 \end{aligned}$$

and the solution of this set of simultaneous equations is:

$$\left. \begin{aligned} K_1 &= \frac{(T_{a_i} - P_2)\alpha + (T_{b_i} + P_1)\beta + (T_{d_i} - P_3)\gamma}{\alpha + \beta + \gamma} \\ K_2 &= \frac{(T_{a_i} - P_2)(e^{\lambda_2 L} - h) + (T_{b_i} + P_1)(h - f) + (T_{d_i} - P_3)(f - e^{-\lambda_2 L})}{\alpha + \beta + \gamma} \\ K_3 &= \frac{(T_{a_i} - P_2)(g - e^{-\lambda_1 L}) + (T_{b_i} + P_1)(d - g) + (T_{d_i} - P_3)(e^{-\lambda_1 L} - d)}{\alpha + \beta + \gamma} \end{aligned} \right\} \quad (3.1)$$

where: $\alpha = (he^{-\lambda_1 L} - ge^{\lambda_2 L})$, $\beta = (gf - dh)$, and $\gamma = (de^{\lambda_2 L} - fe^{-\lambda_1 L})$.

The combination of Equations 2.8a and 3.1 describes the temperature distribution in the respective channels for this arrangement.

3.2 Three Channel (Re-entrant) Flow (Figure 2) Application of Equation 2.8, where $\frac{W_b C_b}{W_a C_a} \neq \frac{W_d C_d}{W_a C_a}$

Equation 2.8 is again stated as in Equation 2.8a (Section 3.1), and application of the boundary equations:

- (i) $T_a = T_b$ at $z = 0$
- (ii) $T_b = T_{b_i}$ at $z = L$
- (iii) $T_d = T_{d_i}$ at $z = 0$

to Equation 2.8a gives:

$$\begin{aligned} (1-d)K_2 + (1-f)K_3 &= P_2 \\ K_1 + e^{-\lambda_1 L}K_2 + e^{\lambda_2 L}K_3 &= T_{b_i} + P_1 \\ K_1 + gK_2 + hK_3 &= T_{b_i} - P_3 \end{aligned}$$

and the solution of this set of simultaneous equations is:

$$\left. \begin{aligned} K_1 &= \frac{P_2(he^{-\lambda_1 L} - ge^{\lambda_2 L}) - (T_{b_i} + P_1)\alpha + (T_{d_i} - P_3)\beta}{\beta - \alpha} \\ K_2 &= \frac{P_2(e^{\lambda_2 L} - h) - (T_{b_i} + P_1)(1-f) + (T_{d_i} - P_3)(1-f)}{\beta - \alpha} \\ K_3 &= \frac{P_2(g - e^{-\lambda_1 L}) + (T_{b_i} + P_1)(1-d) - (T_{d_i} - P_3)(1-d)}{\beta - \alpha} \end{aligned} \right\} \quad (3.2)$$

where: $\alpha = [h(1-d) - g(1-f)]$, and $\beta = [e^{\lambda_2 L}(1-d) - e^{-\lambda_1 L}(1-f)]$.

It should be noted for this arrangement that the difference between the constants C_1 and C_2 for channels a and b respectively is the change of the specific heat of the coolant due to the variation of temperature. If a mean value of the specific heat of the coolant in channels a and b is assumed then $C_1 = C_2$, and as shown in Appendix 1 the expression for α is zero.

$$\left. \begin{aligned} T_a &= B_1 + B_2 \left(z - \frac{1}{C_1}\right) + S_1 \frac{z}{L} - S_2 \frac{z^2}{L^2} \\ T_b &= B_1 + B_2 z - S_2 \frac{z^2}{L^2} \end{aligned} \right\} \quad (2.13a)$$

$$\text{where: } S_1 = AQL, \text{ and } S_2 = \frac{AQC_1L^2}{2}$$

Then applying the boundary conditions

$$(i) \quad T_a = T_{a_i} \text{ at } z = 0$$

$$(ii) \quad T_b = T_{b_i} \text{ at } z = L$$

to equation 2.13a gives

$$\left. \begin{aligned} B_1 - \frac{1}{C_1} B_2 &= T_{a_i} \\ B_1 + LB_2 &= T_{b_i} + S_2 \end{aligned} \right\}$$

and the solution for this set of simultaneous equations is:

$$\left. \begin{aligned} B_1 &= \frac{T_{a_i} C_1 L + (T_{b_i} + S_2)}{1 + C_1 L} \\ B_2 &= \frac{-T_{a_i} C_1 + (T_{b_i} + S_2) C_1}{1 + C_1 L} \end{aligned} \right\}$$

which on substitution in Equation 2.13a gives

$$\left. \begin{aligned} T_a &= \frac{T_{a_i} (1 + C_1 L - C_1 z) + C_1 z (T_{b_i} + S_2)}{1 + C_1 L} + S_1 \frac{z}{L} - S_2 \frac{z^2}{L^2} \\ T_b &= \frac{T_{a_i} C_1 (L - z) + (1 + C_1 z) (T_{b_i} + S_2)}{1 + C_1 L} - S_2 \frac{z^2}{L^2} \end{aligned} \right\} \quad (3.3.2)$$

3.4 Two Channel (Re-entrant) Flow (Figure 2)

3.4.1 Application of Equation 2.12, where $C_1 \neq C_2$

It should be noted that this application will be considered only when the variation of specific heat due to temperature in channels a and b is taken into account.

Equation 2.12 is again re-written as in Equation 2.12a, Section 3.3.1, as follows:

$$\left. \begin{aligned} T_a &= K_1 + \frac{C_1}{C_2} K_2 e^{-bz} + R_1 - R_2 \frac{z}{L} \\ T_b &= K_1 + K_2 e^{-bz} - R_2 \frac{z}{L} \end{aligned} \right\} \quad (2.12a)$$

$$\text{where } R_1 = \frac{AQ}{b}, \quad R_2 = \frac{AQC_2L}{b}, \text{ and } b = C_1 - C_2.$$

Then applying the boundary conditions:

(i) $T_a = T_b$ at $z = 0$

(ii) $T_b = T_{bi}$ at $z = L$

to Equation 2.12a gives

$$\left. \begin{aligned} K_2 \left(\frac{C_1}{C_2} - 1 \right) &= -R_1 \\ K_1 + K_2 e^{-bL} &= T_{bi} + R_2 \end{aligned} \right\},$$

and the solution of this set of simultaneous equations is

$$\left. \begin{aligned} K_1 &= \frac{R_1 C_2 e^{-bL}}{C_1 - C_2} + T_{bi} + R_2 \\ K_2 &= -\frac{R_1 C_2}{C_1 - C_2} \end{aligned} \right\},$$

which on substitution in Equation 2.12a gives

$$\left. \begin{aligned} T_a &= T_{bi} + R_2 \left(1 - \frac{z}{L} \right) + \frac{R_1}{C_1 - C_2} [C_2 (e^{-bL} - 1) + C_1 (1 - e^{-bz})] \\ T_b &= T_{bi} + R_2 \left(1 - \frac{z}{L} \right) + \frac{R_1 C_2}{C_1 - C_2} [e^{-bL} - e^{-bz}] \end{aligned} \right\} \quad (3.4.1)$$

3.4.2 Application of Equation 2.13, where $C_1 = C_2$

Equation 2.13 is again re-written as in Equation 2.13a, Section 3.3.2 as follows:

$$\left. \begin{aligned} T_a &= B_1 + B_2 \left(z - \frac{1}{C_1} \right) + S_1 \frac{z}{L} - S_2 \frac{z^2}{L^2} \\ T_b &= B_1 + B_2 z - S_2 \frac{z^2}{L^2} \end{aligned} \right\} \quad (2.13a)$$

where $S_1 = AQL$, and $S_2 = \frac{AQC_1 L^2}{2}$.

Then applying the boundary conditions:

(i) $T_a = T_b$ at $z = 0$

(ii) $T_b = T_{bi}$ at $z = 0$

to Equation 2.13a gives:

$$\left. \begin{aligned} B_1 &= T_{bi} + S_2 \\ B_2 &= 0 \end{aligned} \right\},$$

which on substitution in Equation 2.13a gives

$$\left. \begin{aligned} T_a &= T_{bi} + S_1 \frac{z}{L} + S_2 \left(1 - \frac{z^2}{L^2} \right) \\ T_b &= T_{bi} + S_2 \left(1 - \frac{z^2}{L^2} \right) \end{aligned} \right\} \quad (3.4.2)$$

3.5 Parallel Flow Heat Exchangers

If in the expression derived in Section 3.3 for two channel (independent) flow the heat source term Q is made equal to zero, then Equations 3.3.1 and 3.3.2 may be used to express the temperature distribution in counterflow and unidirectional flow heat exchangers.

Schematic diagrams are given in Figure 3.

3.5.1 Counterflow

Considering first the case where $C_1 \neq C_2$, we have from Equation 3.3.1 with $Q = 0$,

$$T_a = \frac{T_{a_i}(e^{-bz} - \frac{C_2}{C_1} e^{-bL}) + T_{b_i}(1 - e^{-bz})}{1 - \frac{C_2}{C_1} e^{-bL}}$$

$$T_b = \frac{T_{a_i} \frac{C_2}{C_1} (e^{-bz} - e^{-bL}) + T_{b_i}(1 - \frac{C_2}{C_1} e^{-bz})}{1 - \frac{C_2}{C_1} e^{-bL}} \quad (3.5.1)$$

$$\text{where: } b = C_1 - C_2$$

and on combining the above equations,

$$T_a - T_b = \frac{(T_{a_i} - T_{b_i}) [e^{-bz} (1 - \frac{C_2}{C_1})]}{1 - \frac{C_2}{C_1} e^{-bL}} \quad (3.5.2)$$

or

$$T_a - T_b = (T_{a_i} - T_{b_i}) e^{-bz} \quad ,$$

and if the efficiency of the heat exchanger is defined as:

$$\eta = \frac{T_{a_i} - T_{a_e}}{T_{a_i} - T_{b_i}} \quad ,$$

then T_{a_e} may be determined from Equation 3.5.1 at $z = L$,

and

$$\eta = \frac{1 - e^{-bL}}{1 - \frac{C_2}{C_1} e^{-bL}} \quad \text{for } C_1 \neq C_2 \quad (3.5.3)$$

Secondly, for the case where $C_1 = C_2$, from Equation 3.3.2 with $Q = 0$,

$$\left. \begin{aligned} T_a &= \frac{T_{a_i}(1 + C_1 L - C_1 z) + T_{b_i} C_1 z}{1 + C_1 L} \\ T_b &= \frac{T_{a_i} C_1 (L - z) + T_{b_i} (1 + C_1 z)}{1 + C_1 L} \end{aligned} \right\} \quad (3.5.4)$$

and on combining the above equations,

$$T_a - T_b = \frac{T_{a_i} - T_{b_i}}{1 + C_1 L} \quad (3.5.5)$$

Using Equation 3.5.4. to determine T_{ae} at $z = L$,

$$\eta = \frac{T_{ai} - T_{ae}}{T_{ai} - T_{bi}} = \frac{1}{1 + \frac{1}{C_1 L}} \quad \text{for } C_1 = C_2 \quad (3.5.6)$$

3.5.2 Unidirectional Flow

For this case the flow in channel b is reversed and the temperature increase is in accordance with the flow direction. This may be adjusted by replacing C_2 by $-C_2$, which is apparent from Equation 2.2.

Then $b = C_1 + C_2$ and Equation 2.12 is applicable for all cases, namely $C_1 = C_2$ and $C_1 \neq C_2$.

Equation 2.12 with $Q = 0$ reduces to

$$\left. \begin{aligned} T_a &= K_1 - \frac{C_1}{C_2} K_2 e^{-bz} \\ T_b &= K_1 + K_2 e^{-bz} \end{aligned} \right\} ,$$

and applying the boundary conditions

(i) $T_a = T_{ai}$ at $z = 0$

(ii) $T_b = T_{bi}$ at $z = 0$

the above equation becomes:

$$\left. \begin{aligned} T_a &= \frac{T_{ai} (1 + \frac{C_1}{C_2} e^{-bz}) + T_{bi} \frac{C_1}{C_2} (1 - e^{-bz})}{1 + \frac{C_1}{C_2}} \\ T_b &= \frac{T_{ai} (1 - e^{-bz}) + T_{bi} (\frac{C_1}{C_2} + e^{-bz})}{1 + \frac{C_1}{C_2}} \end{aligned} \right\} , \quad (3.5.7)$$

where $b = C_1 + C_2$,

and on combining the above equations,

$$T_a - T_b = (T_{ai} - T_{bi}) e^{-bz} \quad (3.5.8)$$

Using equation 3.5.7 to determine T_{ae} at $z = L$, gives

$$\eta = \frac{T_{ai} - T_{ae}}{T_{ai} - T_{bi}} = \frac{1 - e^{-bL}}{1 + \frac{C_2}{C_1}} \quad (3.5.9)$$

4. SUMMARY OF RESULTS

4.1 Analysis -Section 2.0

(i) Three Channel Flow

Differential Equations:

Equations

2.4

Soln: for $Q(z) = \text{constant}$.

2.8 and 2.9

<u>(ii) Two Channel Flow</u>	<u>Equations</u>
Differential Equations:	2.11
Solutions: for $Q(z) = \text{constant}$	2.12 and 2.13
<u>(iii) Single Channel Flow</u>	
Differential Equation:	2.14
 4.2 <u>Application of Analysis</u> – Section 3.0	
(i) Three Channel (Independent) Flow	– 3.1 in conjunction with 2.8
(ii) " " (Re-entrant) "	– 3.2 " " " 2.8
(iii) Two Channel (Independent) Flow	– 3.3.1 " " " 2.12
(iv) " " " "	– 3.3.2 " " " 2.13
(v) Two Channel (Re-entrant) Flow	– 3.4.1 " " " 2.12
(vi) " " " "	– 3.4.2 " " " 2.13
(vii) Parallel Heat Exchange $Q = 0$	
Counter Flow	– 3.5.1 " " " 2.12
"	– 3.5.2 " " " 2.13
Unidirectional Flow	– 3.5.7 " " " 2.12

5. NOTATION

a	– inner coolant channel
b	– middle coolant channel
d	– outer coolant channel
C_p	– specific heat at constant pressure of coolant. C_a, C_b, C_d refer to mean values of specific heat of coolant in channels a, b, and d respectively
L	– length of coolant channel
P	– channel perimeter. P_i and P_o refer to perimeter of inner and outer channel walls respectively
Q	– quantity of heat generated per unit length, per unit time
r	– radius
T	– coolant temperature. $T_{a_i}, T_{b_i},$ and T_{d_i} refer to the inlet temperature of coolants in channels a, b, and d respectively
U	– overall heat transfer coefficient, based on either inside or outside wall radii consistent with perimeter P. U_i and U_o refer to inner and outer channel walls respectively
W	– coolant mass flow rate
z	– distance along coolant channel

APPENDIX I

VALUE OF α (SECTION 3.2) WHEN $C_1 = C_2$

From Section 3.2:

$$\begin{aligned}\alpha &= h(1-d) - g(1-f) \\ &= \frac{C_4}{C_4 + \lambda_2} \left(1 - \frac{C_1}{C_1 - \lambda_1} \right) - \frac{C_4}{C_4 - \lambda_1} \left(1 - \frac{C_1}{C_1 + \lambda_2} \right) \\ &= \frac{-C_4 \lambda_1}{(C_4 + \lambda_2)(C_1 - \lambda_1)} - \frac{C_4 \lambda_2}{(C_4 - \lambda_1)(C_1 + \lambda_2)}\end{aligned}$$

But for this case

$$\lambda_1 \lambda_2 = C_1 C_3, \text{ and } \lambda_1 - \lambda_2 = C_4 - C_3,$$

from which

$$C_4 + \lambda_2 = \frac{\lambda_1}{C_1} (C_1 + \lambda_2),$$

and

$$C_4 - \lambda_1 = -\frac{\lambda_2}{C_1} (C_1 - \lambda_1).$$

Substituting in the above

$$\begin{aligned}\alpha &= -\frac{C_4 C_1}{\lambda_1 (C_1 + \lambda_2)} \cdot \frac{\lambda_1}{(C_1 - \lambda_1)} + \frac{C_4 C_1}{\lambda_2 (C_1 - \lambda_1)} \cdot \frac{\lambda_2}{(C_1 + \lambda_2)} \\ &= 0\end{aligned}$$

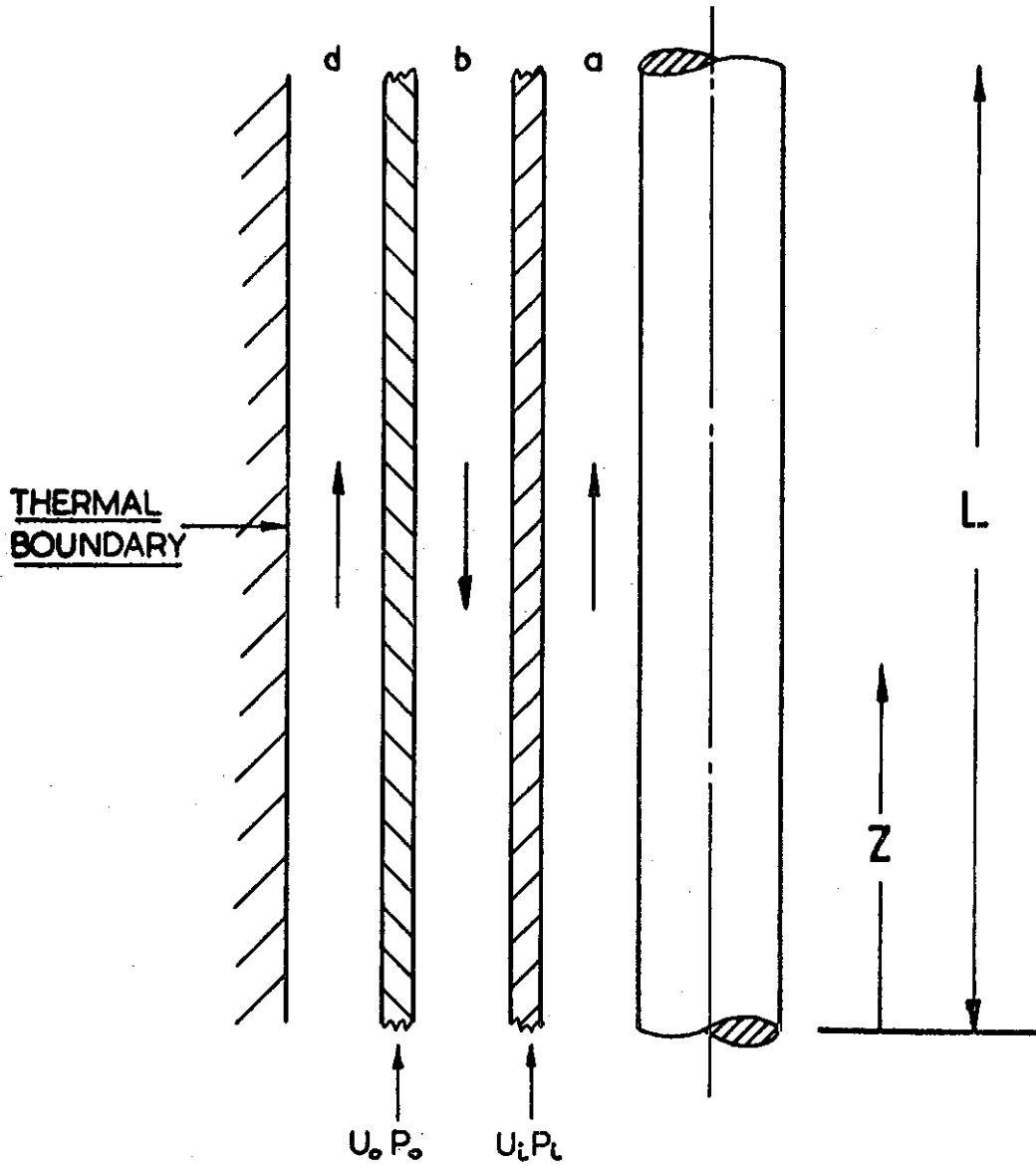


FIG. 1.
THREE CONCENTRIC COOLANT CHANNELS
SURROUNDING A HEAT GENERATING SOURCE. E.R.P. 17.

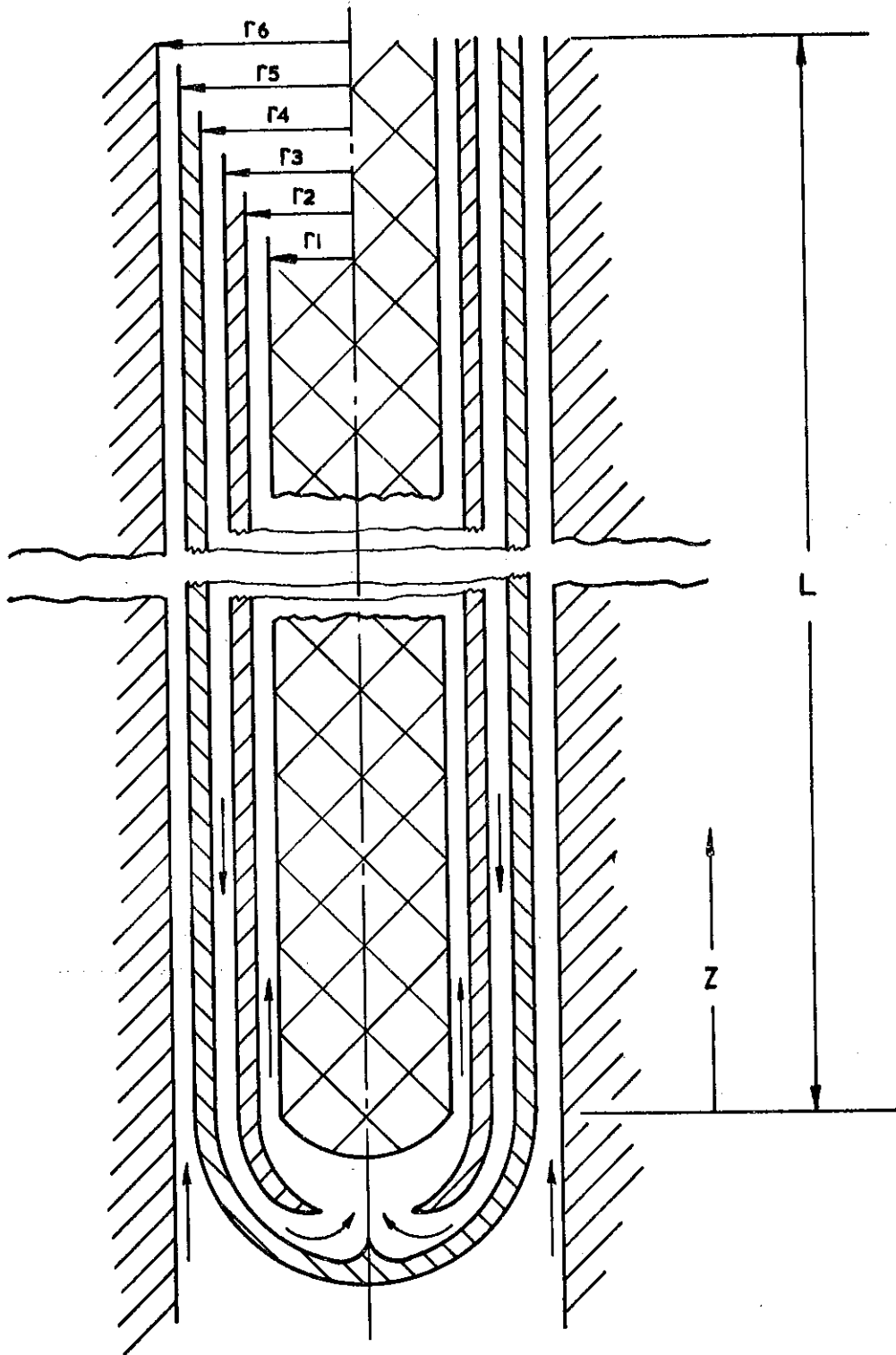
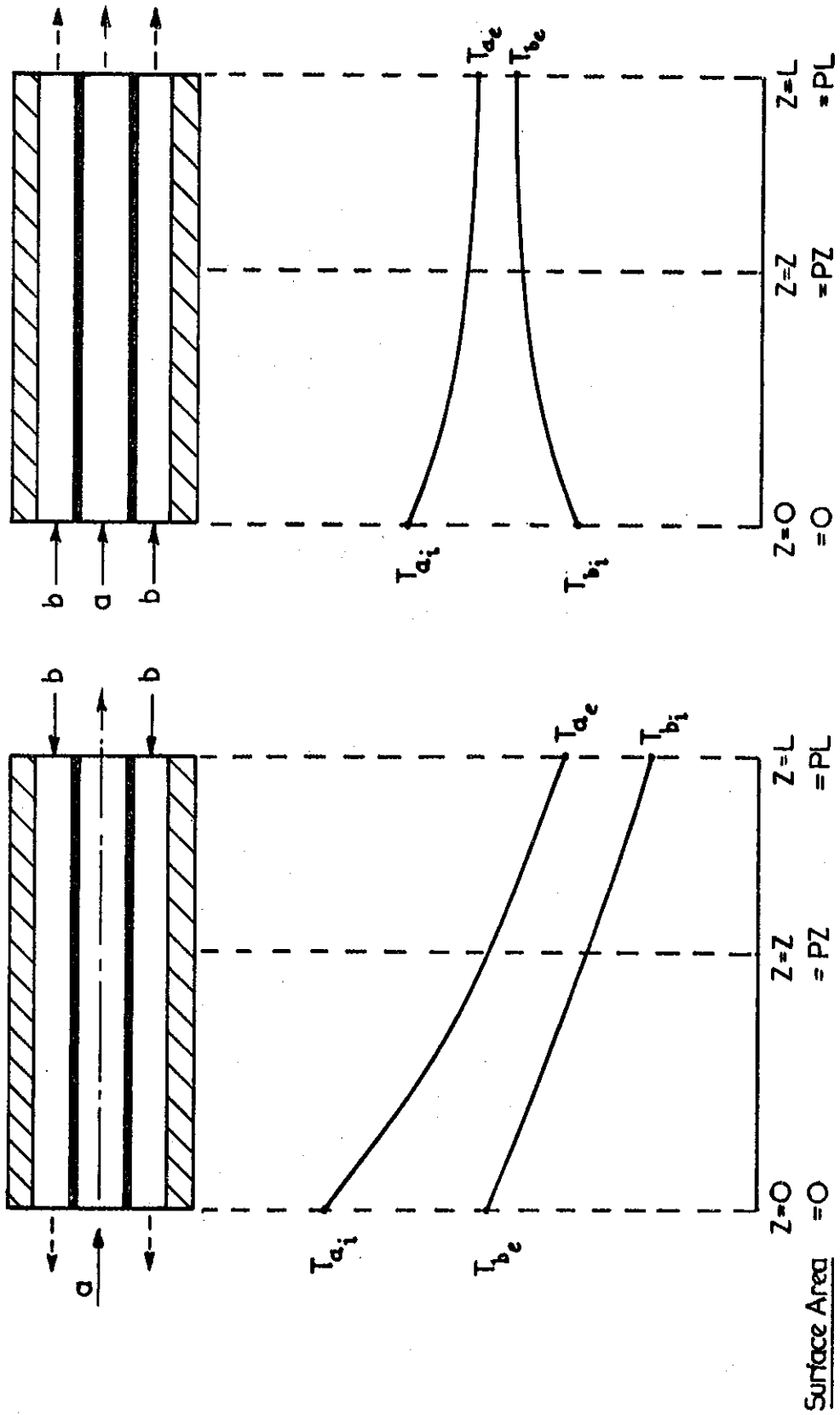


FIG.2.
THREE CHANNEL (RE-ENTRANT) FLOW



COUNTER FLOW

UNIDIRECTIONAL FLOW

FIG.3. COUNTER & UNIDIRECTIONAL FLOW HEAT EXCHANGE.

