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HIGH ENERGY NEUTRON SPECTRA IN INFINITE HOMOGENEOUS REACTOR SYSTEMS MODERATED BY BERYLLIUM OR BERYLLIA

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ABSTRACT

A programme is described for determining the neutron enhancement due to the (n,2n) reaction in reactor systems moderated by beryllium. For moderation by pure beryllium the enhancement has been found to be 9.7 per cent.

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CONTENTS

1. INTRODUCTION

A theoretical prediction of the neutron enhancement due to the (n,2n) reaction in beryllium depends to a great extent on the values of the cross sections used in the calculation and on the mechanism assumed for the reaction. Estimates of the enhancement in pure beryllium and beryllium oxide have been obtained by Hafele (1959), Hafele and Tsagaris (1959), Aline et al. (1960) and Hines and Pollard (1962). However experimental determinations of the enhancement in various reactor assemblies (Krasin et al. 1958; and Benoist et al. 1958) indicate deviations from the theoretical predictions for pure moderators which could be explained by the presence of fertile and fissile materials in the assemblies.

The work reported here was done to assess the effect of various concentrations of fertile and fissile materials on the (n,2n) enhancement in infinite homogeneous systems moderated by Be or BeO. To compare neutron enhancements in various mixtures the neutron spectrum at high energies is required in detail, with full allowance made for the presence of neutron sources derived from fission and other neutron-emitting reactions. The problem is further complicated by the presence of a large number of competing non-elastic reactions and because the angular distribution of scattered neutrons is anisotropic in the centre of mass system.

The spectrum calculation is basically an extension of a calculation by Pollard (1960), in which he determined the spectrum for beryllium, treating the (n, α) and (n,2n) reactions as pure absorption and in which the effect of the anisotropy of the differential elastic scattering cross section in the centre of mass system was neglected. This computation gave an estimate of the minimum possible enhancement of the slowing down density in pure beryllium. The validity of an improved version of the Greuling-Goertzel approximation, which included the derivative of the source and allowed for absorption, was checked against a computer solution of the integral equation and almost complete agreement was established.

The method of expanding the Laplace transform as a power series in the transform variable in order to obtain the flux in a single nuclear species as an explicit function of the source was given by Keane (1961), while the extension of the method to include anisotropic scattering has been treated more recently by Keane and Pollard (1962). A resume of this theory is given in Section 2 and it is then extended to infinite homogeneous absorbing mixtures.

Using the theory developed in the first part of the paper a programme was written to calculate the flux for any mixture of Be9, O16, Th232, U233, U235, U238, and Pu239 at intervals of 0.1 MeV in the range 0.1 MeV to 10 MeV. The neutron enhancements due to (n,f) and (n,2n) reactions as well as losses due to (n, α) and (n, γ) reactions were also calculated. All cross section data and the energies of neutrons resulting from (n,n') and (n,2n) reactions were taken from Buckingham, Parker, and Pendlebury (1961). The specific mixtures considered are given in Section 6 and the results summarised in Table IV.

2. THE NEUTRON SPECTRUM IN A SINGLE NUCLEAR SPECIES

2.1 Anisotropic Scattering

The angular distribution function for the differential scattering cross section of neutrons on collision with scattering nuclei is assumed to be given by $g(\mu)$, where μ is the cosine of θ, the scattering angle in the centre of mass system. In terms of the initial and final lethargies u' and u,

$\mu = \frac{2}{1-\alpha} e^{-(u-u')} - \frac{1+\alpha}{1-\alpha}$, where $1-\alpha$ is the maximum fractional energy loss on a collision. This assumes that the differential scattering cross section depends only on the difference between the initial and final lethargies of the neutron and is otherwise independent of the lethargy of the incident neutron.

The neutron balance equation in the lethargy interval du at u, in the absence of absorption, is

$$\Sigma_s(u)\,\phi(u) = \int_{u-u_m}^{u} \Sigma_s(u')\,\phi(u')\,e^{-(u-u')}\,2\,g(\mu)\,\frac{du'}{1-\alpha} + S(u) \qquad 2.1$$

where $\phi(u)$ is the flux, $\Sigma_s(u)$ the scattering cross section, and S(u) the source; all at lethargy u, and $u_m = \ln 1/\alpha$.

Taking the Laplace transform of equation 2.1 and rearranging the terms:

$$L\{\Sigma_s(u)\,\phi(u)\} = \frac{1}{1-K(p)}\,L\{S(u)\}\,, \tag{2.2}$$

where $$K(p) = \int_0^{u_m} e^{-(p+1)u}\,2g\left(\frac{1}{\delta}e^{-u} - \frac{1}{\delta} + 1\right)\frac{du}{1-\alpha}$$

$$= \int_{-1}^{1} \{1-\delta(1-\mu)\}^p\, g(\mu)\,d\mu\,, \tag{2.3}$$

and $$\delta = \frac{1}{2}(1-\alpha)\,. \tag{2.4}$$

The normalising condition $\int_{-1}^{1} g(\mu)\,d\mu = 1$ is equivalent to $K(0) = 1$. Thus, expanding $\{1-K(p)\}^{-1}$ in powers of p:

$$\frac{1}{1-K(p)} = \frac{1}{\xi p} + \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \ldots\,, \tag{2.5}$$

where the coefficients are given by

$$\frac{1}{\xi} = \lim_{p\to 0}\frac{p}{1-K(p)} = -\frac{1}{K'(0)}\,, \tag{2.6}$$

and $$\gamma_n = \lim_{p\to 0}\frac{1}{p^n}\left[\frac{1}{1-K(p)} - \frac{1}{\xi p} - \gamma_0 - \gamma_1 p - \ldots - \gamma_{n-1}\,p^{n-1}\right]\,,$$

or $$\gamma_n\{K'(0)\}^2 = \frac{K^{(n+2)}(0)}{(n+2)!} - K'(0)\left[\frac{\gamma_0 K^{(n+1)}(0)}{(n+1)!} + \frac{\gamma_1 K^{(n)}(0)}{n!} + \ldots + \gamma_{n-1}\frac{K''(0)}{2!}\right]\,, \tag{2.7}$$

where from equation 2.3:

$$K^{(r)}(0) = \int_{-1}^{1}\{\ln[1-\delta(1-\mu)]\}^r\, g(\mu)\,d\mu\,. \tag{2.8}$$

Inserting the expansion 2.5 into equation 2.2 and inverting the Laplace transform:

$$\Sigma_s\,\phi(u) = \frac{1}{\xi}\int_0^u S(u)\,du + \gamma_0 S(u) + \gamma_1 S'(u) + \ldots \tag{2.9}$$

As is usual we have used the one-sided Laplace transform which imposes the condition that $S(u) = 0$ for $u < o$. Since the zero of lethargy must be taken at a finite energy we have automatically truncated the source. To overcome the difficulty we should use the two-sided Laplace transform which would give the range of the integral in equation 2.9 as $(-\infty, u)$.

The expansion of the Laplace transform in powers of p, leading to equation 2.5, only takes the pole at $p = o$ into account. The neglected poles are complex and give oscillatory terms damped exponentially. Thus we are neglecting the oscillations in the Placzek function and assuming that the flux immediately attains its asymptotic form. Since the source is continuous there would be considerable interference between the oscillatory contributions, and further since only a small source of neutrons occurs in each infinitesimal range the oscillations would give only an insignificant perturbation to the flux.

For heavy scattering nuclei $\delta \to 0$ and so

$$K^{(r)}(0) = \int_{-1}^{1} (-\delta)^r (1-\mu)^r g(\mu)\, d\mu$$

$$= (-\delta)^r \overline{(1-\mu)^r} \quad ,$$

where the bar denotes the average value. Thus

$$\left.\begin{aligned} \xi &= \delta\, \overline{(1-\mu)} \\ \gamma_0 &= \frac{\overline{(1-\mu)^2}}{2\,\overline{(1-\mu)}^2} \\ \gamma_1 &= \delta\left[\gamma_0 \frac{\overline{(1-\mu)^2}}{2\,\overline{(1-\mu)}} - \frac{\overline{(1-\mu)^3}}{6\,\overline{(1-\mu)^2}}\right] \end{aligned}\right\} \quad 2.10$$

showing that when δ is small, ξ and γ_1 are proportional to δ while γ_0 is constant.

2.2 Isotropic Scattering

When the angular distribution of scattered neutrons is isotropic in the centre of mass system we have $g(\mu) = \frac{1}{2}$, and hence from equation 2.3:

$$K(p) = \int_0^{u_m} e^{-(p+1)u} \frac{du}{1-\alpha} \quad .$$

Thus

$$K^{(r)}(0) = \int_0^{u_m} (-u)^r e^{-u} \frac{du}{1-\alpha}$$

$$= \frac{-\alpha}{1-\alpha} (\ln \alpha)^r - r\, K^{(r-1)}(0)$$

$$= (1-\xi) \ln \alpha)^{r-1} - r\, K^{(r-1)}(0) \quad , \qquad 2.11$$

since

$$\xi = \int_0^{u_m} u\, e^{-u} \frac{du}{1-\alpha} = 1 + \frac{\alpha}{1-\alpha} \ln \alpha \quad . \qquad 2.12$$

From equation 2.7 we have

$$\gamma_0 = \frac{K''(0)}{2\{K'(0)\}^2} \quad ,$$

whence on using 2.11

$$\gamma_0 = \frac{1}{\xi}\left[1 + \frac{1-\xi}{\xi} \frac{\ln \alpha}{2}\right]. \qquad 2.13$$

Substituting from 2.11 into 2.7 we find that for $n \geq 1$, γ_n is given by

$$\gamma_n = \frac{1-\xi}{\xi^2}\left[\frac{\ln^{n+1}\alpha}{(n+2)!} + \gamma_0 \frac{\xi \ln^n \alpha}{(n+1)!} + \gamma_1 \frac{\xi \ln^{n-1}\alpha}{n!} + \ldots + \gamma_{n-1} \frac{\xi \ln \alpha}{2!}\right] \qquad 2.14$$

For a heavy nucleus $\alpha \to 1$ and the expression in square brackets in equation 2.14 must tend to zero if γ_n is to be finite. Hence

$$\gamma_{n-1}\, \xi^2 = \gamma_{n-2} \frac{\xi^3 2^2}{3!} - \gamma_{n-3} \frac{\xi^4 2^3}{4!} + \ldots (-1)^{n+1} \frac{\xi^{n+1} 2^{n+1}}{(n+2)!} \quad ,$$

where we have used the approximation $\ln \alpha \to -2\,\xi$ as $\alpha \to 1$.

Thus, correct to the lowest power of ξ,

$$\gamma_0 = \frac{2}{3}\,, \quad \gamma_1 = \frac{1}{9}\xi\,, \quad \gamma_2 = -\frac{2}{135}\xi^2$$

$$\gamma_3 = -\frac{1}{405}\,\xi^3\,, \quad \gamma_4 = \frac{2}{1701}\,\xi^4 \quad .$$

3. THE NEUTRON SPECTRUM IN NON–ABSORBING MIXTURES

In a mixture of nuclides the neutron balance equation in the lethargy interval du at u is

$$\sum_i \Sigma_{si}\,\phi(u) = \sum_i \int_{u-u_i}^{u} \Sigma_{si}(u')\,\phi(u')\,e^{-(u-u')}\,2\,g_i(\mu)\,\frac{du'}{1-\alpha_i} + S(u) \quad , \qquad 3.1$$

where the subscript i denotes parameters associated with the i–th nuclear species.

Taking the Laplace transform of 3.1 gives

$$\sum_i L\{\Sigma_{si}\,\phi(u)\} = \sum_i L\{\Sigma_{si}\,\phi(u)\}\,K_i(p) + L\{S(u)\}$$

$$\sum_i L\{\Sigma_{si}\,\phi(u)\}\,[1-K_i(p)] = L\{S(u)\}$$

$$\sum_i L\{\Sigma_{si}\,\phi(u)\}\prod_{j\neq i}\left[\frac{1}{1-K_j(p)}\right] = \prod_i\left[\frac{1}{1-K_i(p)}\right] L\{S(u)\} \quad .$$

Now expand $\{1-K_i(p)\}^{-1}$ as a power series in p and multiply both sides by $p^{n-1}\prod_i \xi_i$ to obtain

$$\sum_i \xi_i\, L\{\Sigma_{si}\phi(u)\}\,[1+p\sum_{j\neq i}\xi_j\,\gamma_{0j} + p^2\{\sum_{j\neq i}\xi_j\,\gamma_{1j} + \sum_{j\neq k\neq i}\xi_j\,\xi_i\,\gamma_{0j}\,\gamma_{0k}\} + \ldots]$$

$$= L\{S(u)\}\,[\frac{1}{p} + \sum_i \xi_i\,\gamma_{0i} + p\,\{\sum_i \xi_i\,\gamma_{1i} + \sum_{i\neq k}\xi_i\,\xi_k\,\gamma_{0i}\,\gamma_{0k}\} + \ldots] \quad .$$

To simplify this expression it is necessary to assume that the scattering cross sections Σ_{si} are independent of lethargy, whence

$$\sum_i \xi_i\,\Sigma_{si}\,L\{\phi(u)\} = L\{S(u)\}\,[\frac{1}{p} + \sum_i \xi_i\,\gamma_{0i} + p\,\{\sum_i \xi_i\,\gamma_{1i} + \sum_{i\neq k}\xi_i\,\xi_k\,\gamma_{0i}\,\gamma_{0k}\} + \ldots]$$

$$\times\left[1 + p\,\frac{\sum_i \xi_i\Sigma_{si}\sum_{j\neq i}\xi_j\,\gamma_{0j}}{\sum_i \xi_i\,\Sigma_{si}} + p^2\,\frac{\sum_i \xi_i\,\Sigma_{si}\sum_{j\neq i}\xi_j\,\gamma_{1j} + \sum_{j\neq k\neq i}\xi_j\,\xi_k\,\gamma_{0j}\,\gamma_{0k}}{\sum_i \xi_i\,\Sigma_{si}} + \ldots\right]^{-1} ,$$

yielding

$$\Sigma_s\,\{L\,\phi(u)\} = \frac{1}{\bar{\xi}p} + \bar{\gamma}_0 + \bar{\gamma}_1\,p + \ldots \quad , \qquad 3.2$$

where

$$\left.\begin{aligned} \Sigma_s &= \sum_i \Sigma_{si} \quad , \quad \bar{\xi} = \frac{\sum_i \xi_i\Sigma_{si}}{\Sigma_s} \\ \bar{\gamma}_0 &= \frac{1}{\bar{\xi}^2\Sigma_s}\sum_i \xi_i^2\,\gamma_{0i}\,\Sigma_{si} \\ \bar{\gamma}_1 &= \frac{1}{\bar{\xi}^2\Sigma_s}\left(\sum_i \xi_i^2\,\gamma_{1i}\,\Sigma_{si} - \sum_i \xi_i^3\,\gamma_{0i}\,\Sigma_{si}\right) + \bar{\xi}\,\bar{\gamma}_0^2 \quad . \end{aligned}\right\} \qquad 3.3$$

Inversion of equation 3.2 leads to

$$\Sigma_s \phi(u) = \frac{1}{\bar{\xi}} \int_o^u S(u)\,du + \bar{\gamma}_o S(u) + \bar{\gamma}_1 S'(u) + \ldots \quad , \qquad 3.4$$

which is the same as equation 2.9 for a single species except that we must use the average slowing down parameters as given by equations 3.3.

The assumption that the scattering cross sections are independent of lethargy is not necessary. It is sufficient that all the scattering cross sections have the same lethargy dependence, an eventuality which is extremely unlikely. However close inspection reveals that $\bar{\xi}$ as in equation 3.3 is independent of any assumption, so that the dominant term on the right hand side of equation 3.4 is not subject to any restriction. The other parameters $\bar{\gamma}_o$ and $\bar{\gamma}_1$ are only slightly in error if all but one of the nuclear species are present in small proportions or if the scattering cross sections of the major constituents are only slowly varying functions of lethargy.

4. SLOWING DOWN WITH ABSORPTION

If the system has a macroscopic absorption cross section Σ_a, then we may treat the absorption as a negative source so that the source term in equation 3.4 is $\delta(u) = S(u) - \Sigma_a \phi(u)$, where $S(u)$ is the positive neutron source term. Thus

$$\Sigma_s \phi(u) = \frac{1}{\bar{\xi}} \int_o^u \delta(u)\,du + \bar{\gamma}_o \delta(u) + \bar{\gamma}_1 \delta'(u) + \ldots$$

$$= \frac{1}{\bar{\xi}} \int_o^u S(u)\,du + \bar{\gamma}_o S(u) + \bar{\gamma}_1 S'(u) + \ldots$$

$$- \frac{1}{\bar{\xi}} \int_o^u \Sigma_a \phi(u)\,du - \bar{\gamma}_o \Sigma_a \phi(u) - \bar{\gamma}_1 \{\Sigma_a \phi(u)\}' - \ldots \qquad 4.1$$

On taking Laplace transforms and collecting terms

$$\left[\frac{1}{\bar{\xi} p} + \left(\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_o \right) + \bar{\gamma}_1 p + \bar{\gamma}_2 p^2 + \ldots \right] L\{\Sigma_a \phi(u)\}$$

$$= \left[\frac{1}{\bar{\xi} p} + \bar{\gamma}_o + \bar{\gamma}_1 p + \bar{\gamma}_2 p^2 + \ldots \right] L\{S(u)\} \quad , \qquad 4.2$$

in which step we have assumed that Σ_a and Σ_s have the same energy dependence.

We now multiply both sides of equation 4.2 by the power series $\sum_{n=o}^{\infty} A_n p^n$ to obtain

$$\left[\frac{A_o}{\bar{\xi} p} + \left\{ A_o \left(\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_o \right) + \frac{A_1}{\bar{\xi}} \right\} + \left\{ A_o \bar{\gamma}_1 + A_1 \left(\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_o \right) + \frac{A_2}{\bar{\xi}} \right\} p + \ldots \right] L\{\Sigma_a \phi(u)\}$$

$$= \left[\frac{A_o}{\bar{\xi} p} + \left\{ A_o \bar{\gamma}_o + \frac{A_1}{\bar{\xi}} \right\} + \left\{ A_o \bar{\gamma}_1 + A_1 \bar{\gamma}_o + \frac{A_2}{\bar{\xi}} \right\} p + \ldots \right] L\{S(u)\} \quad ,$$

and then choose the constants A_n so that

$$A_o \bar{\gamma}_{n-1} + A_1 \bar{\gamma}_{n-2} + \ldots + A_{n-1} \left(\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_o \right) + \frac{A_n}{\bar{\xi}} = 0$$

$$\text{for } n \geq 2 \qquad 4.3$$

in order to terminate the first series. Thus

$$\left[\frac{A_0}{\bar{\xi} p} + \{ A_0 \frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0 + \frac{A_1}{\bar{\xi}} \} \right] L \{ \Sigma_a \phi(u) \}$$

$$= \left[\frac{A_0}{\bar{\xi} p} + \left(A_0 \bar{\gamma}_0 + \frac{A_1}{\bar{\xi}} \right) - \frac{\Sigma_s}{\Sigma_a} A_1 p - \frac{\Sigma_s}{\Sigma_a} A_2 p^2 - \ldots \right] L \{ S(u) \} \quad . \qquad 4.4$$

Since A_0 is a common factor we may take $A_0 = 1$ and then as a final restriction take $A_2 = 0$ in order to eliminate the p^2 term in the series of equation 4.4.

The condition $A_2 = 0$ when combined with equation 4.3 for $n = 2$ gives

$$A_1 = - \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \quad ,$$

so that equation 4.4 reduces to

$$\left[\frac{1}{\bar{\xi} p} + \left\{ \frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \right\} \right] L \{ \Sigma_a \phi(u) \}$$

$$= \left[\frac{1}{\bar{\xi} p} + \left\{ \bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \right\} + \frac{\Sigma_s}{\Sigma_a} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} p + 0(p^3) \right] L \{ S(u) \} \quad . \qquad 4.5$$

On inverting the Laplace transform we obtain

$$\frac{1}{\bar{\xi}} \int_0^u \Sigma_a \phi(u) \, du + \Sigma_s \phi(u) + \left(\bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \right) \Sigma_a \phi(u)$$

$$= \frac{1}{\bar{\xi} p} \int_0^u S(u) du + \left(\bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \right) S(u) + \frac{\Sigma_s}{\Sigma_a} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} S'(u) + \ldots \quad ,$$

or

$$\Sigma_s \phi(u) \left[1 + \frac{\Sigma_a}{\Sigma_s} \left(\bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \right) \right]$$

$$= \frac{1}{\bar{\xi}} \int_0^u S(u) du - \frac{1}{\bar{\xi}} \int_0^u \Sigma_a \phi(u) \, du + \left(\bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\gamma_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \right) S(u)$$

$$+ \frac{\bar{\gamma}_1}{1 + \bar{\gamma}_0 \frac{\Sigma_a}{\Sigma_s}} S'(u) + \ldots \qquad 4.6$$

To examine the effect of assuming that Σ_a and Σ_s have the same energy dependence note firstly that equations 4.1 and 4.6 both reduce to

$$(\Sigma_s + \bar{\gamma}_0 \Sigma_a) \phi(u) = \frac{1}{\bar{\xi}} \int_0^u \{ S(u) - \Sigma_a \phi(u) \} du + \bar{\gamma}_0 S(u) \quad , \qquad 4.7$$

if terms in γ_1 and smaller are neglected. Thus equation 4.6 is at least correct to the order of the Greuling Goertzel approximation. Since γ_1 is $0(\bar{\xi})$ the terms neglected in equation 4.7 will be

insignificant for heavy nuclei and for high energies where the angular dependence of the scattering cross section is markedly anisotropic.

The term in equation 4.1 which is multiplied by $\bar{\gamma}_1$ is $\frac{d}{du}[S(u) - \Sigma_a \phi(u)]$ from which $d\phi/du$ may be eliminated by using the approximate value of $\phi(u)$ given by 4.7 . Thus

$$\frac{d}{du}[S(u) - \Sigma_a \phi(u)] = \left[\frac{1}{\bar{\xi}} \frac{\Sigma_a}{\Sigma_s + \bar{\gamma}_0 \Sigma_a}(\Sigma_a \phi(u) - S(u)) + \frac{\Sigma_s}{\Sigma_s + \bar{\gamma}_0 \Sigma_a} S'(u)\right]$$

$$+ (\Sigma_s + \bar{\gamma}_0 \Sigma_a)\, \phi(u) \frac{d}{du}\left(\frac{\Sigma_a}{\Sigma_s + \bar{\gamma}_0 \Sigma_a}\right) .$$

If we neglect the term containing the derivative of $\Sigma_a/(\Sigma_s + \bar{\gamma}_0 \Sigma_a)$ the remaining terms are the same as those that appear multiplied by $\bar{\gamma}_1$ in equation 4.6 . Hence the assumption that Σ_a and Σ_s have the same energy dependence is equivalent to assuming that

$$\frac{d}{du}\left(\frac{\Sigma_a}{\Sigma_s + \bar{\gamma}_0 \Sigma_a}\right) << \frac{1}{\bar{\xi}}\left(\frac{\Sigma_a}{\Sigma_s + \bar{\gamma}_0 \Sigma_a}\right)^2 ,$$

or

$$\Sigma_s \frac{d\Sigma_a}{du} - \Sigma_a \frac{d\Sigma_s}{du} << \frac{1}{\bar{\xi}}(\Sigma_a)^2 .$$

Such an inequality is satisfied for slowly varying cross sections and introduces the greatest error at a few isolated points such as near the threshold of various high energy reactions. Even so the error is in a term contributing only a small percentage of the flux.

The equation 4.6 was used for the calculation of the high energy flux in an absorbing mixture. All the parameters are functions of the mixture and of energy, while the source term S(u) contains, in addition to the fission source, those neutrons appearing at lethargy u as a result of non-elastic collisions at lower lethargies.

5. METHOD OF CALCULATION

Taking $E_0 = 10$ MeV as the zero of lethargy and converting to energy $E = E_0 e^{-u}$, equation 4.6 becomes

$$\Sigma_s \phi(E)\left[1 + \frac{\Sigma_a}{\Sigma_s}\left(\bar{\gamma}_0 + \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0}\right)\right] = \frac{1}{\bar{\xi} E}\int_E^{E_0} [S(E') - \Sigma_a \phi(E')]\, dE'$$

$$+ \left(\bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} - \frac{\bar{\gamma}_1}{1 - \bar{\gamma}_0 \frac{\Sigma_a}{\Sigma_s}}\right) S(E) - \left(\frac{\bar{\gamma}_1}{1 + \bar{\gamma}_0 \frac{\Sigma_a}{\Sigma_s}}\right) E\, S'(E) \qquad 5.1$$

To evaluate the flux from equation 5.1 we march in equal intervals $\Delta E = 0.1$ MeV commencing at $E_0 = 10$ MeV and continuing the process down to 0.1 MeV. Only 0.1 per cent. of the fission spectrum is above 10 MeV so that it was considered reasonable to account for these neutrons in $S(E_0)$. Thus we took $S(E_0) = \int_{E_0}^{\infty} S(E)dE$ while $S'(E_0)$ was given the value obtained from the analytic expression for the fission spectrum.

Approximating the integral $\int_{E_n}^{E_0} [S(E') - \Sigma_a \phi(E')]\, dE'$ by the trapezoidal rule and taking the slope between adjacent points as the derivative at the lower point, then the flux at E_n is given by

$$\phi(E_n) = \left\{ \frac{1}{\bar{\xi} E_n} \frac{\Delta E}{2} \left[S(E_n) + 2S(E_{n-1}) + \dots + 2S(E_1) + S(E_0) - 2\Sigma_a \phi(E_{n-1}) - \right. \right.$$

$$\left. \dots - 2\Sigma_a \phi(E_1) - \Sigma_a \phi(E_0) \right] + \left(\bar{\gamma}_0 - \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} - \frac{\bar{\gamma}_1}{1 + \bar{\gamma}_0 \frac{\Sigma_a}{\Sigma_s}} \right) S(E_n)$$

$$\left. - \left(\frac{\bar{\gamma}_1}{1 + \bar{\gamma}_0 \frac{\Sigma_a}{\Sigma_s}} \right) E_n \left[\frac{S(E_{n-1}) - S(E_n)}{\Delta E} \right] \right\} \left[\Sigma_s + \Sigma_a \left(\bar{\gamma}_0 + \frac{1}{\bar{\xi}} \frac{\bar{\gamma}_1}{\frac{\Sigma_s}{\Sigma_a} + \bar{\gamma}_0} \right) + \frac{1}{2} \Sigma_a \frac{\Delta E}{\bar{\xi} E_n} \right]^{-1} \qquad 5.2$$

The data required for the calculation are the slowing down parameters $\bar{\xi}$, $\bar{\gamma}_0$, $\bar{\gamma}_1$, the cross sections Σ_a, Σ_s, and the source contributions at the hundred points at which the flux is computed. For consistency all data used have been taken from the graphs by Buckingham, Parker, and Pendlebury (1961).

5.1 Slowing Down Parameters

The data given for the differential elastic cross sections, $g(\mu)$, are in the form of average values over irregular energy intervals. The average values of $K'(0)$, $K''(0)$, $K'''(0)$ over the energies at which the data are given were calculated from equation 2.8 using Simpson's rule at intervals of 0.2 in μ. Then the slowing down parameters were found from 2.6 and 2.7. Since the same set of curves is given for thorium, uranium, and plutonium, the slowing down parameters ξ, γ_0, γ, were calculated for Th232 and corrected for the other fuels by using equations 2.10, which shows that ξ and γ_1 are proportional to $\frac{1}{2}(1-\alpha)$ while γ_0 is constant.

The average slowing down parameters for Be9, 016, and Th232 as calculated from the data of Buckingham, Parker, and Pendlebury (1961) are shown in Tables I, II, and III.

The values of the slowing down parameters at the points required for the calculation were obtained by linear interpolation after assuming that the values obtained in the given ranges are the values at the mid-points of the ranges.

5.2 High Energy Reactions

The non-elastic cross sections allowed for in the calculation were:

Be9	(n,2n), (n,α)
016	(n,n'), (n,α)
Th_{232}	(n,f), (n,n'), (n,2n), (n,γ)
U_{233}	(n,f), (n,n'), (n,γ)
U_{235}	(n,f), (n,n')
U_{238}	(n,f), (n,n'), (n,γ)
Pu_{239}	(n,f), (n,n')

The cross sections were read at such energies that linear interpolation is reasonable at intermediate points. Since beryllium and oxygen would be most abundant in any mixture a large number of points were taken for these nuclei to achieve greater accuracy. Small discrepancies in the data for the fissile nuclei would not seriously affect the overall accuracy of the calculation so that linear interpolation is valid over much larger energy ranges.

All the non-elastic reactions were treated in the computation as absorption and, where necessary, allowance was made for the emerging neutrons in the form of secondary sources added on to the primary fission source. The original fission source was taken from the Argonne National Laboratory's (1958) compilation of reactor physics constants.

The secondary sources arising from non-elastic reactions were treated as follows:

(i) (n,f) Reactions

The number of neutrons emerging from fast fission reactions with the i-th nuclear species is $\tau_i = \int_0^u \nu_i(u') N_i \sigma_{fi}(u') \phi(u') du'$, where u is above the threshold for the reaction, and very little error is introduced if we assume that the energy spectrum of these neutrons is the same as the initial fission spectrum. Since the initial fission source is normalised to unity and gives rise to τ_i fast fission neutrons, these τ_i neutrons will lead in turn to a further τ_i^2 neutrons. Consequently the total number of neutrons appearing with the energy distribution of the fission spectrum equals $1 + \tau_i + \tau_i^2 + \ldots = 1/1 - \tau_i$.

It is implicit in the above definition of τ_i that the flux $\phi(u)$ is that calculated on the assumption that neutrons entering fast fission reactions are lost to the system. Thus the flux in the mixture when allowance is made for the neutrons resulting from the fast fission reaction is $\frac{1}{1-\tau}\phi(u)$ where $\tau = \Sigma \tau_i$. This simple relation is due to the fact that the flux, as calculated from equation 5.1 on the assumption that neutrons entering the (n,f) reaction are lost to the system, is directly proportional to the normalisation of the fission spectrum. For this reason the flux was initially calculated with the fast fission cross section treated as pure absorption and no secondary sources were introduced. The flux obtained on this basis was consequently increased at all energies by the factor $1/1-\tau$ at the conclusion of the calculation.

(ii) (n,n') Reactions

For most neutrons undergoing inelastic scattering the final energy is a simple function of the initial energy and can be calculated directly. Such neutrons can then be added on to the fission source at the energy at which they emerge from the reaction.

However for certain ranges of initial neutron energy E, the neutrons resulting for (n,n') reactions appear in the range E' to E' + dE' with probability given by

$$Q(x)\,dx = 16x\,e^{-4x}\,dx \quad ,$$

where $x = E'/\sqrt{E}$. Such a distribution is derived theoretically and has the obvious fault that it allows for a small percentage of neutrons to emerge above the initial energy. Apart from the fact that an increase of energy is impossible it would lead to almost insurmountable difficulties in the computation.

Because the neutrons initiating the (n,n') reaction are distributed continuously and since only a small fraction of the total neutrons enter this reaction it seems reasonable to approximate the distribution Q(x) in the following way:

Assume that the neutrons emerge at three discrete energies given by x_0, x_1, x_2 where x_0 gives the mean energy of emergence for the most degraded 25 per cent. of neutrons having an (n,n') reaction at energy E, x_1 gives the mean for the intermediate 50 per cent. and x_2 the mean for the 25 per cent. emerging with the highest energies.

Hence

$$x_0 = 4\int_0^a x\,Q(x)\,dx \text{ with } \int_0^a Q(x)\,dx = \tfrac{1}{4} \text{ or } x_0 = 0.1466$$

$$x_1 = 2\int_a^b x\,Q(x)\,dx \text{ with } \int_a^b Q(x)\,dx = \tfrac{1}{2} \text{ or } x_1 = 0.4313$$

$$x_2 = 4\int_b^\infty x\,Q(x)\,dx \text{ with } \int_b^\infty Q(x)\,dx = \tfrac{1}{4} \text{ or } x_2 = 0.9909.$$

Thus we take one quarter of the neutrons entering the reaction at E to emerge at $E' = x_0\sqrt{E}$, half to emerge at $E' = x_1\sqrt{E}$ and the remaining quarter at $E' = x_2\sqrt{E}$.

(iii) (n,2n) Reactions

The final energies of neutrons entering (n,2n) reactions can be calculated directly in terms of their initial energies by following Buckingham, Parker, and Pendlebury (1961) in taking the distribution of secondary neutrons as:

Threshold< E< 2.7 MeV — Each neutron is taken to have equal probabilities of all energies from 0–1.7 MeV. A suitable representation is to assume that each neutron has equal probabilities (0.1 per neutron) of having energies α(E–1.7)MeV, where α = 0.05(0.1) 0.95.

2.7<E<13.0 MeV — The first neutron is taken to have energies:

0.1E in 0.08 of reactions

0.3E in 0.08 of reactions

0.5E in 0.08 of reactions

0.7E in 0.08 of reactions

0.86(E–2.55)MeV in 0.68 of reactions.

The second neutron is taken to have energies:

0.1E in 0.2 of reactions

0.2E in 0.2 of reactions

0.4E in 0.1 of reactions

0.8 MeV in 0.5 of reactions.

5.3 Programme Notes

The programme for the computation was written in FORTRAN for the IBM 1620 computer. The input required is the number of Be9, 016, Th232, U233, U235, U238, Pu239 nuclei in units of 10^{24}cm, in the mixture under consideration. The output obtained is the flux as a function of energy between 0.1 and 10MeV, together with the integrals $\int_{0.1}^{10} \Sigma_{ij}\, \phi(u')\, du'$ where j denotes one of the reactions (n,2n), (n, α), (n,f), or (n,γ) and i specifies a particular nuclear species. The total number of neutrons from fission or (n,2n) reactions entering the system at energies below 0.1MeV is also obtained as output.

6. NUMERICAL RESULTS

The programme was run for both pure beryllium with 0.12354×10^{24} nuclei/cm^3 and for beryllium oxide with 0.06741×10^{24} molecules/cm^3. Results are given in tabular form in Table IV and the fluxes compared graphically in Figure 1.

The following systems were also considered:

(a) U233, Th232, Be9 in the ratio 1 : 30 : 3000

(b) Pu239, Th232, Be9 in the ratio 1 : 30 : 2000

(c) Pu239, U238, Be9 in the ratio 1 : 10 : 2000

(d) U235, Th232, BeO in the ratio 1 : 30 : 1500

(e) Th232, Be9, in the ratio 1 : 20.

Since the exact number of nuclei in a complex system depends not only on the temperature but also on the particular compounds formed, it was decided to consider that 1.0×10^{24} beryllium nuclei were present in each cubic centimetre and to take the number of other nuclei in the given ratio. An incorrect input in the number of nuclei/cm^3 will change the flux by a constant factor but since the same factor will occur in the macroscopic cross sections we still obtain the correct reaction rates. For this reason the entries in Table IV, where the results for the various mixtures are recorded, depend only on the ratios of the numbers of nuclei in the system. No graphs have been given for the fluxes in these mixtures.

7. CONCLUDING REMARKS

For pure beryllium the enhancement from the (n,2n) reaction is 14.0 per cent. while the loss to the (n,α) reaction is 4.3 per cent. leaving an overall enhancement of 9.7 per cent. which is 1 per cent. greater than the value found by Hines and Pollard (1962). They calculated a gain of 13.6 per cent. from the (n,2n) reaction and a loss of 4.9 per cent. to the (n, α) reaction. The difference is due almost entirely to the choice of cross section data, since in the present work the (n,2n) cross section is greater and the (n,α) cross section less than the values chosen by Hines and Pollard.

The total enhancement for BeO is 6.5 per cent. which is composed of 10.6 per cent. from the (n,2n) reaction in beryllium less 3.2 per cent. for the (n,α) reaction in beryllium and 0.9 per cent. for the (n,α) reaction in oxygen. This is to be compared with a 5.1 per cent. enhancement found by Hines and Pollard (unpublished) made up of 9.4 per cent. from the Be(n,2n) reaction less 3.8 per cent. and 0.5 per cent. from the (n,α) reactions in beryllium and oxygen.

The main reaction resulting from the addition of fuel and fertile material to beryllium is the thorium (n,γ) reaction, which reduces the enhancement by 0.9 per cent. for each thorium nucleus per hundred beryllium nuclei. While apparently reducing the enhancement such a reaction is an advantage in a breeder reactor since it does not represent a loss, but rather a gain in conversion. This effect would be considerably greater for a heterogeneous system where the high energy neutrons in the rods are in a more concentrated mixture of thorium and beryllium. Spinrad (1956) has pointed out that measured values of the conversion ratio in natural uranium heterogeneous systems are on the high side compared to the values predicted by reactivity calculations, and has assigned the discrepancy to high energy (n,γ) reactions in U238.

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TABLE I

SLOWING DOWN PARAMETERS OF Be9

Energy Range	ξ	γ_0	γ_1
0.25eV – 0.2MeV	0.2066	0.6925	0.0234
0.2 – 0.4MeV	0.1996	0.7036	0.0231
0.4 –0.6MeV	0.2051	0.6911	0.0232
0.6 – 1.0MeV	0.2136	0.7064	0.0262
1.0 – 1.5MeV	0.1829	0.7328	0.0103
1.5 – 2.0MeV	0.1781	0.7595	0.0220
2.0 – 2.6MeV	0.2096	0.7814	0.0324
2.6 – 2.8MeV	0.1230	1.0788	0.0228
2.8 – 3.25MeV	0.1489	0.9771	0.0300
3.25 – 3.75MeV	0.1282	1.0543	0.0243
3.75 – 4.5MeV	0.1215	1.0813	0.0208
4.5 – 5.5MeV	0.1051	1.1453	0.0144
5.5 – 6.5MeV	0.0947	1.2506	0.0145
6.5 – 9.0MeV	0.0823	1.3500	0.0070
9.0 – 12.0MeV	0.0611	1.2828	0.0143

TABLE II

SLOWING DOWN PARAMETERS OF O16

Energy Range	ξ	γ_0	γ_1
0.025eV – 0.25MeV	0.1199	0.6811	0.0065
0.25 – 1.5MeV	0.1059	0.7930	0.0056
1.5 – 3.0MeV	0.0666	0.9989	0.0095
3.0 – 7.0MeV	0.0590	1.0460	0.0059
7.0 – 15.0MeV	0.0516	1.2122	0.0047

TABLE III

SLOWING DOWN PARAMETERS OF Th_{232}

Energy Range	ξ	γ_0	γ_1
0.05 – 0.15 MeV	0.0079	0.7090	0.0010
0.15 – 0.35MeV	0.0070	0.7508	0.0009
0.35 – 0.65MeV	0.0059	0.8730	0.0008
0.65 – 0.85MeV	0.0055	0.8648	0.0008
0.85 – 1.25MeV	0.0054	0.9324	0.0009
1.25 – 1.75MeV	0.0051	1.0017	0.0010
1.75 – 2.25MeV	0.0037	1.3368	0.0012
2.25 – 2.75MeV	0.0030	1.5515	0.0009
2.75 – 3.50MeV	0.0022	1.9208	0.0006
3.5 – 4.5MeV	0.0016	2.5545	0.0007
4.5 – 5.5MeV	0.0014	3.0306	0.0012
5.5 – 6.5MeV	0.0016	3.0114	0.0019
6.5 – 7.5MeV	0.0013	3.9388	0.0039
7.5 – 8.5MeV	0.0012	4.2394	0.0044
8.5 – 9.5MeV	0.0013	3.9279	0.0031
9.5 – 12.0MeV	0.0014	3.4114	0.0017

TABLE IV

COMPARISON OF RESULTS FOR VARIOUS MIXTURES

System	Slowing Down Density at 0.1 MeV	Source Below 0.1 MeV	Enhancement	Fast Fission Factor
Be9	1.077	0.020	9.7%	1.000
BeO	1.045	0.020	6.5%	1.000
U233:Th232:Be9 :: 1:30:3000	1.059	0.029	8.8%	1.009
Pu239:Th232:Be9:: 1:30:2000	1.050	0.033	8.3%	1.013
Pu239:U238:Be9:: 1:10:2000	1.072	0.025	9.7%	1.016
U235:Th232:BeO:: 1:30:1500	1.021	0.032	5.3%	1.009
Th232:Be9:: 1:20	0.991	0.060	5.1%	1.011

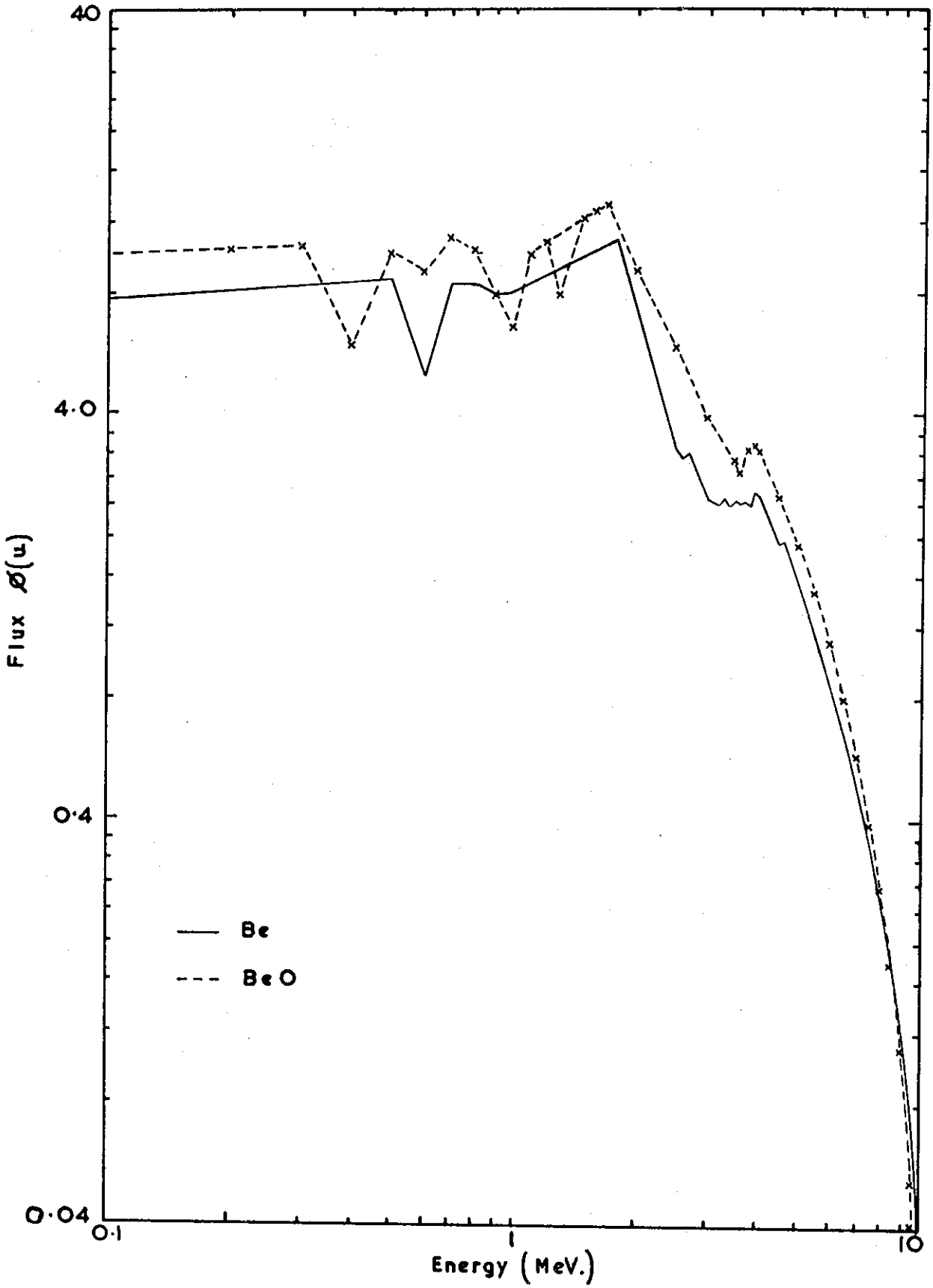

FIGURE 1. FLUX ø(u) IN Be & BeO FROM A UNIT FISSION SOURCE.