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# A DERIVATION OF THE EFFECTIVE RESONANCE INTEGRAL IN HETEROGENEOUS SYSTEMS

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# A DERIVATION OF THE EFFECTIVE RESONANCE INTEGRAL IN HETEROGENEOUS SYSTEMS

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Abstract

This paper is concerned with the evaluation of the effective resonance integral for a close–packed lattice of fuel elements in beryllium oxide moderator. For this evaluation the usual theory of resonance absorption is extended to allow for flux depression in the moderator. Further, the fuel elements under consideration consist of clusters of seven rods, so that it is necessary to determine the effective surface area for such a cluster.

When the fuel elements are composed of U235, Th232 and Be atoms in the ratio of approximately 1:35:1200 and the BeO moderator occupies almost three and a half times the volume of the fuel rods, the effective resonance integral is found to be about 60% of the value for a homogeneous system of the same composition.

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## CONTENTS

## 1. INTRODUCTION

The problem solved in this report is the evaluation of the effective resonance integral for a close–packed lattice of fuel elements in beryllium oxide moderator. Each fuel element is taken to consist of a cluster of seven rods which are composed of U235, Th232 and Be atoms in the ratio of approximately 1:35:1200.

The addition of Be to the fuel simplifies the problem in so far as it will reduce the error involved in the use of the narrow resonance approximation for all the resonances of the Th232. However, the compactness of the lattice and the geometry of the fuel elements will introduce complications.

In Section 2 it is the intention to give an outline of the usual treatment of resonance absorption in heterogeneous lattices and to point out those of the usual approximations which are no longer valid for our purposes. Sections 3 to 5 are concerned with improvements to the usual theory; Section 6 is devoted to the evaluation of the effective resonance integral of a homogeneous mixture containing Th232, while Section 7 gives the numerical solution for the reactor system specified above.

A paper by Petrov (1) treats close packed heterogeneous lattices, but since this work is based on the theory of Gurevich and Pomeranchouk (2) which assumes that scattering by the fuel may be neglected, the results are not relevant to the present problem.

The Dancoff–Ginsburg theory (3) assumes that the fuel rods are black to the resonance flux. The theory developed in Sections (3) and (4) of the present report allows for mutual screening of the fuel rods without this assumption of blackness.

## 2. THE RESONANCE ESCAPE PROBABILITY FOR A HETEROGENEOUS LATTICE

An approximate formula for the resonance escape probability for a homogeneous mixture is

$$p = \exp\left( - \frac{1}{\xi} \int \frac{\sum_i N_i \sigma_{ai}}{\sum_i N_i (\sigma_{ai} + \sigma_{si} + \sigma_{pi})} \frac{dE}{E} \right) \tag{2.1}$$

where $N_i$ is the number of nuclei of the ith species per cc of the mixture, $\sigma_{ai}$ and $\sigma_{si}$ are the resonance absorption and scattering cross–sections of the ith nuclear species at energy E and $\sigma_{pi}$ is the potential scattering cross–section of the ith nuclear species. $\xi$ denotes the average logarithmic energy loss for a neutron on collision with an atom of the mixture.

For a mixture containing only one species of fertile atom, the equation (2.1) becomes

$$p = \exp\left( - \frac{1}{\xi} \int \frac{\sigma_a}{\sigma_a + \sigma_s + \sigma_m} \frac{dE}{E} \right) \tag{2.2}$$

where $\sigma_a$ and $\sigma_s$ are the resonance absorption and scattering cross–sections of a fertile nucleus and $\sigma_m$ is the total potential scattering cross–section of the mixture per fertile atom.

Assuming that the potential scattering cross–section is energy independent, we may rewrite (2.2) as

$$p = \exp\left( - \frac{1}{\xi \sigma_m} I \right) \tag{2.3}$$

where I, known as the effective resonance integral, is given by

$$I = \int \frac{\sigma_a}{1 + \frac{\sigma_a + \sigma_s}{\sigma_m}} \frac{dE}{E} \tag{2.4}$$

When treating a heterogeneous lattice, it is the usual practice to retain equation (2.1) for the resonance escape probability but to weight the cross–sections of the ith nuclear species with the average flux in the region containing nuclei of the ith species. Assuming that the space averaged flux in the region containing nuclei of the ith species is $\overline{\phi}_i$ we now have, instead of (2.1),

$$p = \exp\left\{ -\frac{1}{\overline{\xi}} \int \frac{\sum_i N_i \sigma_{ai} \overline{\phi}_i}{\sum_i N_i (\sigma_{ai} + \sigma_{si} + \sigma_{pi}) \overline{\phi}_i} \frac{dE}{E} \right\} \tag{2.5}$$

A heterogeneous lattice normally consists of two regions. The first of these is occupied by the fuel elements composed of fertile and fissile atoms together with some moderating material. We shall in future refer to the material in this region as the fuel. The second region is that between the fuel elements which, since it consists entirely of moderating material, will henceforth be referred to as the moderator.

For two regions with only one species of fertile atom in the fuel, the equation (2.5) becomes

$$p = \exp\left\{ -\frac{1}{\overline{\xi}} \int \frac{\sigma_a \overline{\phi}_f}{(\sigma_a + \sigma_s + \sigma_p)\overline{\phi}_f + \frac{V_m}{V_f} \frac{\Sigma_m}{N} \overline{\phi}_m} \frac{dE}{E} \right\} \tag{2.6}$$

where $\sigma_p$ is the potential scattering cross–section of the fuel per fertile atom, $\Sigma_m$ is the macroscopic cross–section of the moderator and N is the number of fertile atoms per cc of the fuel. $V_m/V_f$ is the volume ratio of moderator to fuel and

$$\overline{\phi}_f = \frac{1}{V_f} \int_{\text{fuel elements}} \phi(r) dV$$

with a similar definition for $\overline{\phi}_m$.

If the heterogeneous lattice is loosely packed it is usually assumed that the flux in the moderator is not perturbed by the absorption in the fuel elements. Thus the flux in the moderator may be considered as uniform, isotropic and constant in lethargy so that we may replace $\overline{\phi}_m$ by $\phi_o$. For a loosely packed lattice equation (2.6) reduces to

$$p = \exp\left\{ -\frac{1}{\overline{\xi}} \int \frac{\sigma_a \overline{\phi}_f / \phi_o}{(\sigma_a + \sigma_s + \sigma_p)\overline{\phi}_f / \phi_o + \frac{V_m}{V_f} \frac{\Sigma_m}{N}} \frac{dE}{E} \right\} \tag{2.7}$$

Another assumption, valid for a loosely packed lattice, is that the volume of moderator greatly exceeds the volume of fuel. Then (2.7) becomes

$$p = \exp\left( - \frac{1}{\bar{\xi}} \frac{V_f}{V_m} \frac{N}{\Sigma_m} \int \sigma_a \frac{\bar{\phi}_f}{\phi_o} \frac{dE}{E} \right) \tag{2.8}$$

and the integral

$$\int (\sigma_a)_{eff} \frac{dE}{E} = \int \sigma_a \frac{\bar{\phi}_f}{\phi_o} \frac{dE}{E} \tag{2.9}$$

is known as the effective resonance integral of the heterogeneous lattice.

To evaluate $\bar{\phi}_f / \phi_o$ Spinrad, Chernick and Corngold (4) have considered an isolated lump of fuel in a uniform isotropic neutron bath. They equate the total collisions in the fuel lump per unit time in the lethargy element du about u to collisions due to neutrons entering from the neutron bath plus the collisions due to neutrons achieving this lethargy within the fuel. Thus if S is the surface area of the fuel lump, $\alpha_i$ is the maximum fractional energy loss for a neutron on collision with a nucleus of the $i^{th}$ species in the fuel and we denote $\sum_i N_i (\sigma_{ai} + \sigma_{si} + \sigma_{pi})$ by $\Sigma$, then

$$\bar{\phi}_f \Sigma V_f = \frac{S}{4} \phi_o G_o + P_o^o V_f \sum_i \int_{u + \ln \alpha_i}^{u} \bar{\phi}_f(u') (N_i \sigma_{si}(u') + N_i \sigma_{pi}) \cdot e^{-(u-u')} \frac{du'}{1-\alpha_i} \tag{2.10}$$

where $V_f$ is the volume of the fuel lump, $G_o$ is the probability that a neutron impinging on the fuel lump at lethargy u undergoes a reaction in the lump and $P_o^o$ denotes the probability that a neutron achieving lethargy u within the lump has its next collision in the lump.

According to Case, de Hoffman and Placzek (5),

$$G_o = \int f(l)\,(1 - e^{-\Sigma l})\, dl \tag{2.11}$$

where f (l) is the distribution function of non–collision neutron paths in the lump, and the integral is over all paths.

If the resonances of the fertile material are narrow compared with the average energy lost by a neutron on collision with a nucleus in the fuel lump, then neutrons of lethargy u are produced uniformly throughout the fuel lump. Thus $1 - P_o^o$ is the escape probability investigated by Case, de Hoffman and Placzek (5), so that

$$P_o^o = 1 - \frac{S}{4 V_f \Sigma} G_o \tag{2.12}$$

Another consequence of narrow resonances is that the neutrons "born" at lethargy u in the fuel lump come mainly from non–resonance regions where the flux is $\phi_o$, the same as in the moderator, and where the resonance cross–section is effectively zero. Hence equation (2.10) becomes

$$\bar{\phi}_f \, \Sigma V_f = \frac{S}{4} \phi_o G_o + P_o^o V_f \sum_i \phi_o N_i \sigma_{pi} \int_{u+\ln \alpha_i}^{u} e^{-(u-u')} \frac{du'}{1-\alpha_i}$$

$$= \frac{S}{4} \phi_o G_o + P_o^o V_f \sum_i \phi_o N_i \sigma_{pi}$$

$$= \frac{S}{4} \phi_o G_o + P_o^o V_f \phi_o \Sigma_p \qquad (2.13)$$

where $\Sigma_p = \sum_i N_i \sigma_{pi}$ .

Substituting from (2.12) into (2.13) we have

$$\frac{\bar{\phi}_f}{\phi_o} = \frac{\Sigma_p}{\Sigma} + \frac{S}{4V_f} \frac{1}{\Sigma} (1 - \frac{\Sigma_p}{\Sigma}) G_o \qquad (2.14)$$

so that on substituting for $G_o$ from equation (2.11) and assuming the presence of only one species of fertile nuclei, we obtain the effective resonance integral of equation (2.9) in the form

$$\int (\sigma_a)_{eff} \frac{dE}{E}$$

$$= \int \frac{\sigma_a \sigma_p}{\sigma_a+\sigma_s+\sigma_p} \frac{dE}{E} + \frac{1}{N\bar{l}} \int \frac{\sigma_a (\sigma_a+\sigma_s)}{(\sigma_a+\sigma_s+\sigma_p)^2}$$

$$\left( \int f(l) (1 - e^{-\Sigma l}) \, dl \right) \frac{dE}{E} \qquad (2.15)$$

where $\bar{l}$ is the mean chord length of a fuel lump and has been shown by Case, de Hoffman and Placzek to equal $4V_f/S$. Equation (2.15) is identical with that obtained by Wigner et al (6) from an entirely different argument.

To obtain a simplified expression for (2.14) it is usual to introduce the approximations

$$G_o \doteq 1 - e^{-\Sigma \bar{l}} \doteq \frac{\Sigma \bar{l}}{1 + \Sigma \bar{l}} \qquad (2.16)$$

Substituting from (2.16) into (2.14), we obtain, after a little manipulation,

$$\frac{\bar{\phi}}{\phi_o} = \frac{1 + \Sigma_p \bar{l}}{1 + \Sigma \bar{l}} \qquad (2.17)$$

With this simplification the effective resonance integral becomes

$$\int (\sigma_a)_{eff} \frac{dE}{E} = \int \frac{\sigma_a}{1 + \dfrac{N\bar{l}(\sigma_a + \sigma_s)}{1 + N\bar{l}\,\sigma_p}} \frac{dE}{E} \tag{2.18}$$

which is the same as equation (2.4) if we make the substitution

$$\sigma_m = \frac{1 + N\bar{l}\,\sigma_p}{N\bar{l}} \tag{2.19}$$

Thus the effective resonance integral and therefore the resonance escape probability of a heterogeneous lattice may be determined from a consideration of an equivalent homogeneous system.

The above identification of heterogeneous and homogeneous systems depends mainly on the validity of the approximations in equation (2.16). It is therefore necessary to examine in some detail the error introduced by these approximations. Section 3 of the present report will be devoted to such an investigation.

The reactor system in which we are interested consists of a close-packed lattice of fuel elements, the elements themselves consisting of clusters of rods. In order to adapt the above theory to our needs we must investigate the following problems:-

(i) The assumption that $V_m >> V_f$, used to obtain equation (2.8) from (2.7), is no longer valid, nor is the assumption that the moderator flux is uniform and isotropic, used in deriving equation (2.7). It is therefore necessary to carry through the analysis without these simplifications.

Equation (2.10) must be modified to allow for neutrons "born" in one fuel element being absorbed in another fuel element, i.e. we can no longer consider an isolated fuel lump immersed in moderator. All these close packing corrections are considered in Section 4.

(ii) Since the fuel elements consist of clusters of rods we must determine the mean neutron path through a cluster and also the effective surface area of a cluster of fuel rods. This investigation is given in Section 5.

(iii) A certain amount of moderating material is to be incorporated into the fuel rods, and this will reduce the error involved in assuming that all the resonances of the fertile nuclei may be treated as narrow. Since the fertile material is Th232 we must evaluate the effective resonance integral (2.4) for a homogeneous mixture containing Th232 and may use the narrow resonance approximation. The evaluation of the effective resonance integral at zero degrees absolute as a function of the scattering cross-section per fertile atom is given in Section 6.

(iv) The doppler increase in the effective resonance integral as a function of temperature is required. This problem will be the subject of a later report.

## 3. A CORRECTION TO THE EFFECTIVE RESONANCE INTEGRAL

We consider now the error involved in the approximations (2.16), and derive a correction factor to be applied to the effective resonance integral as evaluated with the use of these approximations.

If $G_o$ is put equal to unity, equation (2.15) becomes

$$\int (\sigma_a)_{eff} \frac{dE}{E} \approx \int \frac{\sigma_a \sigma_p}{\sigma_a + \sigma_s + \sigma_p} \frac{dE}{E} + \frac{1}{N\bar{l}} \int \frac{\sigma_a (\sigma_a + \sigma_s)}{(\sigma_a + \sigma_s + \sigma_p)^2} \frac{dE}{E}$$

$$= A + B \frac{S}{M} \qquad (3.1)$$

where A and B depend only on the resonance structure of the fuel and not on the geometry of the fuel elements, while S/M is the surface–to–mass ratio of the fuel elements. Since $l = \frac{4.\ \text{volume}}{\text{surface}}$, we have $S/M = 4/\rho l$, where $\rho$ is the density of the fuel.

Assuming the resonance contours to be of Breit–Wigner form, we have at $0^o$ K, the resonance absorption and scattering cross sections of the $r^{th}$ resonance absorption and scattering cross sections of the $r^{th}$ resonance given by

$$\sigma_{ar} = \frac{\Gamma_\gamma \sigma_o}{\Gamma(1 + x^2)} \qquad \sigma_{sr} = \frac{\Gamma_n \sigma_o}{\Gamma(1 + x^2)}$$

where $$x = \frac{2(E - E_r)}{\Gamma}$$

$E_r$ being the neutron energy for the rth resonance, $\sigma_o$ the corresponding total microscopic cross section of the fuel, $\Gamma_\gamma$ and $\Gamma_n$ respectively the level widths for $\gamma$– and neutron emission, and $\Gamma = \Gamma_\gamma + \Gamma_n$. Hence, considering $1/E$ as constant at each resonance, since it varies only very slightly over the practical width, we have, for the contributions $A_r$, $B_r$ to A, B from the rth resonance

$$A_r = \frac{\pi \Gamma_\gamma}{2 E_r} \sigma_o \left(1 + \frac{\sigma_o}{\sigma_p}\right)^{-1/2} \qquad (3.2)$$

and

$$B_r = \frac{\rho}{4N} \frac{\pi \Gamma_\gamma}{4 E_r} \left(\frac{\sigma_o}{\sigma_p}\right)^2 \left(1 + \frac{\sigma_o}{\sigma_p}\right)^{-3/2} . \qquad (3.3)$$

We then have

$$A = \sum_r A_r \quad \text{and} \quad B = \sum_r B_r$$

where the sum is taken over all resonances.

Now using the approximations (2.16) for $G_o$, we obtain for the effective resonance integral, equation (2.18), which may be written

$$\int (\sigma_a)_{eff} \frac{dE}{E} \approx \frac{\sigma_a (1 + N\bar{l} \sigma_p)}{1 + N\bar{l}(\sigma_a + \sigma_s + \sigma_p)} \frac{dE}{E} \qquad (3.4)$$

as obtained by Chernick and Vernon (7). For Breit–Wigner resonance contours, this gives

$$\int (\sigma_a)_{eff} \frac{dE}{E} \approx \sum_r \frac{\pi \Gamma_\gamma}{2E_r} \sigma_o \left( \frac{1 + N\bar{l}\sigma_p}{1 + N\bar{l}(\sigma_o + \sigma_p)} \right)^{1/2}$$

$$= \sum_r \left\{ A_r + B_r \frac{S}{M} \delta_r(\bar{l}) \right\} \tag{3.5}$$

where $A_r$, $B_r$ are as defined by (3.2) , (3.3) and

$$\delta_r(\bar{l}) = \frac{2N\bar{l}\sigma_p{}^2}{\sigma_p} \left\{ \left( \frac{1 + N\bar{l}\sigma_p}{1 + N\bar{l}(\sigma_o + \sigma_p)} \right)^{1/2} - \left( \frac{\sigma_p}{\sigma_o + \sigma_p} \right)^{1/2} \right\} \cdot \left( \frac{\sigma_o + \sigma_p}{\sigma_p} \right)^{3/2}$$

For resonances with $\sigma_o << \sigma_p$ , $B_r$ is of the second order of small quantities, and $\delta(\bar{l}) \approx 0$. Hence we may neglect the contribution of these resonances to the part of the effective resonance integral which is dependent on the geometry of the fuel elements. Further, the resonances with $\sigma_o \approx \sigma_p$ are very few in number, with negligible contribution to this term.

On the other hand, if $\sigma_o >> \sigma_p$ ,

$$\delta_r(\bar{l}) \approx 2(N\bar{l}\sigma_p)^{1/2} \left\{ (1 + N\bar{l}\sigma_p)^{1/2} - (N\bar{l}\sigma_p)^{1/2} \right\}$$

which is independent of $\sigma_o$ and therefore constant for resonances with $\sigma_o >> \sigma_p$

Hence we may write (3.5) in the form

$$\int (\sigma_a)_{eff} \frac{dE}{E} = A + B\frac{S}{M}\delta(\bar{l}) \tag{3.6}$$

where

$$\delta(\bar{l}) = 2(N\bar{l}\sigma_p)^{1/2} \left\{ (1 + N\bar{l}\sigma_p)^{1/2} - (N\bar{l}\sigma_p)^{1/2} \right\} \tag{3.7}$$

$$= \frac{2(N\bar{l}\sigma_p)^{1/2}}{(1 + N\bar{l}\sigma_p)^{1/2} + (N\bar{l}\sigma_p)^{1/2}}$$

as obtained by Wigner et al (6).

We now consider the evaluation of the effective resonance integral from equation (2.15) following the method of Gurevich and Pomeranchouk (2), without using the above approximations.

First consider the integral

$$\int \frac{\sigma_a(\sigma_a + \sigma_s)}{(\sigma_a + \sigma_s + \sigma_p)^2} \left\{ \int (1 - e^{-\Sigma l}) f(l) dl \right\} \frac{dE}{E}$$

$$= \int f(l) dl \int \frac{\sigma_a(\sigma_a + \sigma_s)}{(\sigma_a + \sigma_s + \sigma_p)^2} (1 - e^{-\Sigma l}) \frac{dE}{E} \tag{3.8}$$

Introducing Breit-Wigner resonance contours, we have

$$\int \frac{\sigma_a(\sigma_a+\sigma_s)}{(\sigma_a+\sigma_s+\sigma_p)^2}\left\{1-e^{-Nl(\sigma_a+\sigma_s+\sigma_p)}\right\}\frac{dE}{E}$$

$$= \sum_r \frac{\Gamma\gamma}{E_r}\int_0^\infty \frac{\sigma_o^2}{(\sigma_o+\sigma_p(1+x^2))^2}\left\{1-e^{-Nl\left(\frac{\sigma_o}{1+x^2}+\sigma_p\right)}\right\}dx$$

$$= \sum_r \frac{\Gamma\gamma}{E_r}\left(\frac{\sigma_o}{\sigma_p}\right)^2\int_0^\infty \frac{1}{\left(\frac{\sigma_o}{\sigma_p}+1+x^2\right)^2}\left\{1-e^{-Nl\sigma_p\left[\frac{\sigma_o}{\sigma_p(1+x^2)}+1\right]}\right\}dx \tag{3.9}$$

For "non-blockaded" resonances, with $\sigma_o << \sigma_p$, this is approximately equal to

$$\sum_r \frac{\Gamma\gamma}{E_r}\left(\frac{\sigma_o}{\sigma_p}\right)^2\int_0^\infty \frac{1}{\left(x^2+1+\frac{\sigma_o}{\sigma_p}\right)^2}\left\{1-e^{-Nl\sigma_p}\left(1-\frac{Nl\sigma_o}{1+x^2}\right)\right\}dx$$

$$= \sum_r \frac{\Gamma\gamma}{E_r}\left(\frac{\sigma_o}{\sigma_p}\right)^2(1-e^{-Nl\sigma_p})\int_0^\infty \frac{dx}{\left(x^2+1+\frac{\sigma_o}{\sigma_p}\right)^2}$$

$$+ Nl\sigma_o\, e^{-Nl\sigma_p}\int_0^\infty \frac{dx}{\left(x^2+1+\frac{\sigma_o}{\sigma_p}\right)^2(x^2+1)}$$

$$= \sum_r \frac{\Gamma\gamma}{E_r}\left(\frac{\sigma_o}{\sigma_p}\right)^2\left\{\frac{\pi}{4}\left(1+\frac{\sigma_o}{\sigma_p}\right)^{-3/2}(1-e^{-Nl\sigma_p})\right.$$

$$\left.+ Nl\sigma_o e^{-Nl\sigma_p}\frac{\pi}{2}\left[\left(\frac{\sigma_p}{\sigma_o}\right)^2-\left(\frac{\sigma_p}{\sigma_o}\right)\left(1+\frac{\sigma_o}{\sigma_p}\right)^{-1/2}-\frac{1}{2}\frac{\sigma_p}{\sigma_o}\left(1+\frac{\sigma_o}{\sigma_p}\right)^{-3/2}\right]\right\}$$

$$\approx 0, \tag{3.10}$$

since $\sigma_o << \sigma_p$ and this condition corresponds to high energy resonances for which $\Gamma\gamma/E_r$ is small.

However, for "blockaded" resonances with $\sigma_o >> \sigma_p$, (3.9) is given approximately by

$$\sum_r \frac{\Gamma'\gamma}{E_r}\left(\frac{\sigma_o}{\sigma_p}\right)^2\cdot\int_0^\infty \frac{1}{\left(x^2+\frac{\sigma_o}{\sigma_p}\right)^2}\left\{1-e^{-Nl\sigma_p\left\{\frac{\sigma_o}{\sigma_p x^2}+1\right\}}\right\}dx$$

$$= \sum_r \frac{\Gamma_\gamma}{E_r} \left( \frac{\sigma_o}{\sigma_p} \right)^{1/2} \int_o^\infty \frac{z^2}{(1 + z^2)^2} \left\{ 1 - e^{-Nl\sigma_p (1 + z^2)} \right\} dz$$

where $z^2 = \dfrac{\sigma_o}{\sigma_p x^2}$ .

This integral may be evaluated by differentiating under the integral sign with respect to $Nl\sigma_p$ and using the results

(i) $$\int_o^\infty e^{-a(1+z^2)}\, dz = \frac{\sqrt{\pi}\, e^{-a}}{2\sqrt{a}}$$

(ii) $$\int_o^\infty \frac{e^{-a(1+z^2)}}{1+z^2}\, dz = \frac{\pi}{2} \operatorname{erfc} \sqrt{a}$$

(iii) $$\int_o^\infty \frac{e^{-t}}{\sqrt{t}}\, dt = \sqrt{\pi} \operatorname{erf} \sqrt{a}$$

(iv) $$\int_o^a \int_{\sqrt{u}}^\infty e^{-v^2}\, dv\, du = -\frac{\sqrt{a}}{z} e^{-a} + \frac{\sqrt{\pi}}{4} \operatorname{erf} \sqrt{a} - \frac{a\sqrt{\pi}}{z} \operatorname{erfc} \sqrt{a} \ .$$

Hence we obtain finally

$$\sum_r \frac{\Gamma_\gamma}{E_r} \left(\frac{\sigma_o}{\sigma_p}\right)^{1/2} \frac{\pi}{2} \left\{ \left(\frac{Nl\sigma_p}{\pi}\right)^{1/2} e^{-Nl\sigma_p} - Nl\sigma_p \operatorname{erfc} (Nl\sigma_p)^{1/2} + \tfrac{1}{2} \operatorname{erf} (Nl\sigma_p)^{1/2} \right\} \tag{3.11}$$

Thus we have the effective resonance integral given by

$$\int (\sigma_a)_{eff} \frac{dE}{E} \approx A + B \frac{S}{M} \int_{\text{all paths}} \theta(l)\, f(l)\, dl \tag{3.12}$$

where

$$\theta(l) = \left(\frac{4N\sigma_p l}{\pi}\right)^{1/2} e^{-N\sigma_p l} - 2N\sigma_p l \operatorname{erfc} (N\sigma_p l)^{1/2} + \operatorname{erf} (N\sigma_p l)^{1/2} \tag{3.13}$$

and the contribution to B of the "non–blockaded" resonances is neglected.

We could at this stage make use of the approximation

$$\int \theta(l)\, f(l)\, dl \approx \theta(\bar{l}) \tag{3.14}$$

giving as an approximate value for the effective resonance integral

$$\int (\sigma_a)_{eff} \frac{dE}{E} \approx A + B \frac{S}{M} \theta(\bar{l}) \ . \tag{3.15}$$

This approximation is closer than the approximation (3.5), but still makes no allowance for variations in the shape of the fuel elements and is, in fact, a little too high.

However, a more accurate result is obtained by calculating $\int \theta(l)\, f(l)\, dl$ for various shapes and sizes of fuel elements. We shall express this result for given values of $N\sigma_p$ as a function of $\bar{l}$ and denote it by

$$\psi(\bar{l}) = \int_{\text{all paths}} \theta(l)\, f(l)\, dl \tag{3.16}$$

so that our evaluation of the effective resonance integral may be written

$$\int (\sigma_a)_{eff} \frac{dE}{E} = A + B \frac{S}{M} \psi(\bar{l}) \quad . \tag{3.17}$$

The function $\psi(\bar{l})$ has been calculated on the UTECOM electronic computer for spherical fuel elements, for infinite slabs and for infinite cylinders as a function of $N\sigma_p\bar{l}$ over the range from 0.1 ( 0.1) 3.0. Results of these calculations are tabulated with corresponding values of $\delta(\bar{l})$ and $\theta(\bar{l})$ in Table 1, and illustrated in Figure 1. It will be seen that the shape of fuel elements has a relatively small effect on $\psi(\bar{l})$, so that the result for a cylinder would give a very good approximation for the fuel elements under present consideration, viz. clusters of rods. An outline of the evaluation of $\psi(\bar{l})$ for cylindrical fuel rods is now given.

Using the results of Case, de Hoffman and Placzek (5), we have for an infinite cylinder of radius a cm.,

$$\int_{\text{all paths}} f(l)\, dl = \frac{16a^2}{\pi} \int_0^{\infty} \int_0^{x_0} \frac{x^4}{l^3 \left(\frac{l^2}{4a^2} - x^2\right)^{1/2} (1 - x^2)^{1/2}}\, dx\, dl \tag{3.18}$$

where $x_0 = \frac{l}{2a}$ for $l < 2a$,

$= 1$ for $l > 2a$,

$$\text{i.e.} \int_{\text{all paths}} f(l)\, dl = \frac{4}{\pi} \int_0^1 \int_0^y \frac{x^4}{y^3 (y^2 - x^2)^{1/2} (1 - x^2)^{1/2}}\, dx\, dy$$

$$+ \frac{4}{\pi} \int_1^{\infty} \int_0^1 \frac{x^4}{y^3 (y^2 - x^2)^{1/2} (1 - x^2)^{1/2}}\, dx\, dy$$

$$= \frac{4}{3\pi} \int_0^1 \left\{ (2 + y^2)\, K(y) - 2(1 + y^2)\, E(y) \right\} \frac{dy}{y^3}$$

$$+ \frac{4}{3\pi} \int_1^{\infty} \left\{ \left(2 + \frac{1}{y^2}\right) K\left(\frac{1}{y}\right) - 2\left(1 + \frac{1}{y^2}\right) E\left(\frac{1}{y}\right) \right\} dy \tag{3.19}$$

where K and E are complete elliptic integrals.

Hence

$$\int_{\text{all paths}} \theta(l)\, f(l)\, dl = \frac{4}{3\pi} \left\{ \int_0^1 I_1 \, dy + \int_1^\infty I_2 \, dy \right\} \tag{3.20}$$

where

$$I_1 = \theta(2ay) \left\{ (2 + y^2)\, K(y) - 2(1 + y^2)\, E(y) \right\} \frac{1}{y^3}$$

and

$$I_2 = \theta(2ay) \left\{ (2 + \frac{1}{y^2})\, K(\frac{1}{y}) - 2(1 + \frac{1}{y^2})\, E(\frac{1}{y}) \right\}$$

It may be seen that $I_1 \to o$ as $y \to o$ and that both $I_1$ and $I_2$ are logarithmically infinite at $y = 1$. The calculation was performed by using Simpson's rule with intervals of 0.1 to evaluate $\int_0^{0.8} I_1 \, dy$ and $\int_{1.2}^{20} I_2 \, dy$, with intervals of 0.01 for

$\int_{0.8}^{0.98} I_1 \, dy$ and $\int_{1.02}^{1.2} I_2 \, dy$ and with intervals of 0.001 for

$\int_{0.98}^{0.994} I_1 \, dy$ and $\int_{1.006}^{1.02} I_2 \, dy$. Weddle's rule was used for

$\int_{0.994}^{1} I_1 \, dy$ and $\int_{1}^{1.006} I_2 \, dy$, the value of $I_1$ and $I_2$ at $y = 1$

being taken equal to their value at $y = 0.999999$ and $y = 1.000001$ respectively. The error involved in these approximations was less than 0.1% .

## TABLE I

| $N\sigma_p\bar{l}$ | $\delta(\bar{l})$ | Slab $\psi(\bar{l})$ | Slab $\frac{\psi(\bar{l})}{\delta(\bar{l})}$ | Cylinder $\psi(\bar{l})$ | Cylinder $\frac{\psi(\bar{l})}{\delta(\bar{l})}$ | Sphere $\psi(\bar{l})$ | Sphere $\frac{\psi(\bar{l})}{\delta(\bar{l})}$ | $\theta(\bar{l})$ | $\frac{\theta(\bar{l})}{\delta(\bar{l})}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.463 | 0.503 | 1.086 | 0.520 | 1.123 | 0.523 | 1.131 | 0.536 | 1.160 |
| 0.2 | 0.580 | 0.634 | 1.094 | 0.654 | 1.127 | 0.658 | 1.135 | 0.675 | 1.164 |
| 0.3 | 0.649 | 0.712 | 1.098 | 0.732 | 1.128 | 0.736 | 1.135 | 0.756 | 1.165 |
| 0.4 | 0.697 | 0.766 | 1.100 | 0.785 | 1.127 | 0.790 | 1.133 | 0.810 | 1.163 |
| 0.5 | 0.732 | 0.807 | 1.101 | 0.824 | 1.125 | 0.828 | 1.131 | 0.849 | 1.160 |
| 0.6 | 0.759 | 0.837 | 1.102 | 0.853 | 1.123 | 0.856 | 1.128 | 0.878 | 1.157 |
| 0.7 | 0.782 | 0.861 | 1.102 | 0.876 | 1.120 | 0.879 | 1.124 | 0.900 | 1.152 |
| 0.8 | 0.800 | 0.880 | 1.101 | 0.894 | 1.118 | 0.897 | 1.121 | 0.918 | 1.148 |
| 0.9 | 0.815 | 0.896 | 1.100 | 0.908 | 1.114 | 0.911 | 1.117 | 0.932 | 1.143 |
| 1.0 | 0.829 | 0.910 | 1.098 | 0.921 | 1.111 | 0.923 | 1.114 | 0.943 | 1.139 |
| 1.1 | 0.839 | 0.921 | 1.097 | 0.931 | 1.108 | 0.932 | 1.110 | 0.952 | 1.134 |
| 1.2 | 0.849 | 0.930 | 1.096 | 0.939 | 1.105 | 0.940 | 1.107 | 0.960 | 1.129 |
| 1.3 | 0.858 | 0.939 | 1.094 | 0.946 | 1.103 | 0.947 | 1.104 | 0.966 | 1.125 |
| 1.4 | 0.866 | 0.946 | 1.092 | 0.952 | 1.100 | 0.953 | 1.101 | 0.971 | 1.121 |
| 1.5 | 0.873 | 0.952 | 1.090 | 0.957 | 1.096 | 0.958 | 1.097 | 0.975 | 1.117 |
| 1.6 | 0.879 | 0.957 | 1.089 | 0.962 | 1.094 | 0.963 | 1.095 | 0.979 | 1.114 |
| 1.7 | 0.885 | 0.962 | 1.087 | 0.966 | 1.091 | 0.966 | 1.092 | 0.982 | 1.110 |
| 1.8 | 0.890 | 0.966 | 1.085 | 0.969 | 1.088 | 0.969 | 1.089 | 0.984 | 1.10[illegible] |
| 1.9 | 0.894 | 0.969 | 1.084 | 0.972 | 1.086 | 0.972 | 1.087 | 0.986 | 1.103 |
| 2.0 | 0.899 | 0.972 | 1.082 | 0.974 | 1.084 | 0.974 | 1.084 | 0.988 | 1.100 |
| 2.1 | 0.903 | 0.975 | 1.080 | 0.977 | 1.082 | 0.977 | 1.082 | 0.990 | 1.097 |
| 2.2 | 0.906 | 0.978 | 1.078 | 0.979 | 1.079 | 0.979 | 1.079 | 0.991 | 1.094 |
| 2.3 | 0.910 | 0.980 | 1.077 | 0.980 | 1.077 | 0.980 | 1.077 | 0.992 | 1.091 |
| 2.4 | 0.913 | 0.982 | 1.075 | 0.982 | 1.075 | 0.982 | 1.075 | 0.993 | 1.089 |
| 2.5 | 0.916 | 0.984 | 1.073 | 0.983 | 1.073 | 0.983 | 1.073 | 0.994 | 1.086 |
| 2.6 | 0.919 | 0.985 | 1.072 | 0.984 | 1.072 | 0.984 | 1.071 | 0.995 | 1.08[illegible] |
| 2.7 | 0.922 | 0.987 | 1.070 | 0.986 | 1.070 | 0.986 | 1.069 | 0.996 | 1.08[illegible] |
| 2.8 | 0.924 | 0.988 | 1.069 | 0.987 | 1.069 | 0.987 | 1.068 | 0.996 | 1.079 |
| 2.9 | 0.926 | 0.989 | 1.068 | 0.988 | 1.067 | 0.988 | 1.066 | 0.997 | 1.077 |
| 3.0 | 0.928 | 0.989 | 1.067 | 0.988 | 1.066 | 0.988 | 1.065 | 0.997 | 1.07[illegible] |

It will be seen from Table I that $\psi(\bar{I})$ is approximately 10% higher than $\delta(\bar{I})$ for the values of $N\sigma_p\bar{I}$ which are of practical importance.

We may apply the appropriate correction to the calculation of the effective resonance integral for a heterogeneous lattice by proceeding as follows:–

(a) Evaluate the effective resonance integral for a homogeneous mixture of various dilutions, viz.

$$\int \frac{\sigma_a \ \sigma_m}{\sigma_a + \sigma_s + s_m} \frac{dE}{E}, \text{ as a function of } \sigma_m .$$

(b) Determine the effective resonance integral for the heterogeneous case by putting $\sigma_m = \dfrac{1 + N\bar{I}\sigma_p}{N\bar{I}}$ in (a).

(c) Determine A for the heterogeneous case by putting $\sigma_m = \sigma_p$ in (a).

(d) Multiply the difference between the results of (b) and (c) by the correction factor $\psi(\bar{I}) / \delta(\bar{I})$ and add this to A to give the corrected effective resonance integral.

## 4. A CLOSE PACKED HETEROGENEOUS LATTICE

When the volume of moderator is comparable to the volume of fuel rods we must make the following allowances in the usual theory.

(1) We can no longer neglect scattering and absorption by the fuel compared with moderator scattering in the denominator of the integral in equation (2.7).

(2) The fuel rods can no longer be considered as in a uniform isotropic neutron bath, since allowance must now be made for a flux depression in the moderator at resonance energies.

(3) A correction must be made for the probability that a neutron which has traversed one fuel lump may collide with fuel in another fuel lump.

To allow for (1) and (2) we must retain equation (2.6), which may be written

$$p = \exp\left\{ -\frac{1}{\xi}\int \frac{\sigma_a \ \bar{\phi}_f/\phi_o}{(\sigma_a+\sigma_s+\sigma_p)\ \bar{\phi}_f/\phi_o + \dfrac{V_m}{V_f}\ \dfrac{\Sigma_m}{N}\ \bar{\phi}_m/\phi_o} \frac{dE}{E} \right\} \qquad (4.1)$$

where $\phi_o$ is the uniform isotropic off resonance flux which will be the same in both fuel and moderator.

To evaluate $\bar{\phi}_f$ we must equate the total collisions in fuel to the collisions in fuel due to neutrons born to lethargy u in moderator plus the collisions in fuel due to neutrons born to lethargy u in fuel. Thus

$$\bar{\phi}_f(u)\,\Sigma V_f = (1-P_1)\,V_m \sum_i \int_{u+\ln\beta_i}^{u} \bar{\phi}_m(u')\,N_i\,\sigma_{pi}\,e^{-(u-u')}\,\frac{du'}{1-\beta_i}$$

$$+\;P_o V_f \sum_i \int_{u+\ln\alpha_i}^{u} \bar{\phi}_f(u')\,N_i\left\{\sigma_{si}(u') + \sigma_{pi}\right\} e^{-(u-u')}\,\frac{du'}{1-\alpha_i} \qquad (4.2)$$

where $P_o$ is the probability that a neutron born in fuel collides in fuel, although now it is not necessary that the collision takes place in the same fuel lump as that in which the neutron originates, and $P_1$ is the similar probability for moderator. $\alpha_i$ and $\beta_i$ are the maximum fractional energy losses on collision with the various fuel and moderator nuclei respectively.

Assuming that there is only one fertile nuclear species and that its resonances are narrow compared with the energy lost by a neutron on collision with a fuel atom, the neutrons born at lethargy u originate mainly from off-resonance lethargies where $\bar{\phi}_f = \bar{\phi}_m = \phi_o$ and $\sigma_s = 0$. Thus for narrow resonances (4.2) becomes

$$\bar{\phi}_f\,\Sigma\,V_f = (1-P_1)\,V_m\,\Sigma_m\,\phi_o + P_o\,V_f\,\Sigma_p\,\phi_o$$

where $\Sigma_p = \sum_{\text{fuel}} N_i\,\sigma_{pi}$ and $\Sigma_m = \sum_{\text{moderator}} N_i\,\sigma_{pi}$

Thus

$$\frac{\bar{\phi}_f}{\phi_o} = (1-P_1)\,\frac{V_m\,\Sigma_m}{V_f\,\Sigma} + P_o\,\frac{\Sigma_p}{\Sigma} \qquad (4.3)$$

Similarly for the space-averaged moderator flux we obtain

$$\frac{\bar{\phi}_m}{\phi_o} = P_1 + (1-P_o)\,\frac{V_f\,\Sigma_p}{V_m\,\Sigma_m} \qquad (4.4)$$

Substituting from (4.3) and (4.4) into the denominator of the integrand in equation (4.1) we have

$$\Sigma\,\frac{\bar{\phi}_f}{\phi_o} + \frac{V_m}{V_f}\,\Sigma_m\,\frac{\bar{\phi}_m}{\phi_o}$$

$$= (1-P_1)\,\frac{V_m\,\Sigma_m}{V_f} + P_o\,\Sigma_p + \frac{V_m}{V_f}\,\Sigma_m P_1 + (1-P_o)\,\Sigma_p$$

$$= \Sigma_p + \frac{V_m\,\Sigma_m}{V_f}$$

so that (4.1) becomes

$$p = \exp \left\{ - \frac{1}{\Sigma \sigma_t} \int \sigma_a \frac{\bar{\phi}_f}{\phi_o} \frac{dE}{E} \right\} \tag{4.5}$$

where $$\sigma_t = \sigma_p + \frac{V_m \Sigma_m}{N V_f} \tag{4.6}$$

and is the total scattering cross-section of the system per fertile atom.

In order to determine $P_o$ and $P_1$ we follow Bell (8) and define $G_o$ as the probability that a neutron incident on a fuel lump collides with fuel on that transit and $G_1$ as the probability that a neutron incident on moderator collides with the moderator on that transit. $P^o$ is the probability that a neutron born in an isolated fuel lump collides in that fuel lump and $P_1^o$ is the probability that a neutron born in an isolated lump of moderator collides in that lump of moderator.

Thus

$$\left.\begin{aligned} 1 - P_o &= (1 - P_o^o) \left\{ G_1 + (1 - G_1)(1 - G_o) G_1 + \dots \right\} \\ &= (1 - P_o^o) \frac{G_1}{1-(1-G_o)(1-G_1)} \\ &\text{and similarly} \\ 1 - P_1 &= (1 - P_1^o) \frac{G_o}{1-(1-G_o)(1-G_1)} \end{aligned}\right\} \tag{4.7}$$

From the work of Case, de Hoffman and Placzek (5) we have

$$\left.\begin{aligned} P_o^o &= 1 - \frac{S}{4V_f \Sigma} G_o \\ P_1^o &= 1 - \frac{S}{4V_m \Sigma_m} G_1 \end{aligned}\right\} \tag{4.8}$$

Substituting from equations (4.7) and (4.8) into equation (4.3), it becomes

$$\frac{\bar{\phi}_f}{\phi_o} = \frac{\Sigma_p}{\Sigma} + \frac{S}{4V_f} \frac{1}{\Sigma} \frac{G_o G_1}{1-(1-G_o)(1-G_1)} \left\{ 1 - \frac{\Sigma_p}{\Sigma} \right\} \tag{4.9}$$

It is to be noted that for an isolated fuel lump the probability $G_1$ is unity so that (4.9) simplifies to equation (2.14), as it should in these circumstances.

The probability $G_1$ is given by

$$G_1 = \int g(l)(1 - e^{-\Sigma_m l})\, dl$$

where $g(l)$ is the distribution of neutron paths in the moderator.

Introducing the mean chord length in the moderator $\bar{L}$, which equals $\frac{V_m}{V_f}\bar{l}$ since the surface area of moderator and fuel is the same, and using the same approximation as in equation (2.16), we have

$$G_1 = \frac{\Sigma_m \bar{L}}{1+\Sigma_m \bar{L}} = \frac{\Sigma_m \frac{V_m}{V_f}\bar{l}}{1+\Sigma_m \frac{V_m}{V_f}\bar{l}} \tag{4.10}$$

The similar expression for $G_o$ was obtained earlier, as

$$G_o = \frac{\Sigma \bar{l}}{1+\Sigma \bar{l}}$$

Substituting for $G_o$ and $G_1$ into equation (4.9) we find after some manipulation that

$$\frac{\bar{\phi}_f}{\phi_o} = \frac{\Sigma_p + \Sigma_m \frac{V_m}{V_f}(1+\Sigma_p \bar{l})}{\Sigma + \Sigma_m \frac{V_m}{V_f}(1+\Sigma \bar{l})}$$

$$= \frac{1}{1+(\Sigma_a+\Sigma_s)\dfrac{1+\Sigma_m \frac{V_m}{V_f}\bar{l}}{\Sigma_p+\Sigma_m \frac{V_m}{V_f}(1+\Sigma_p \bar{l})}}$$

If we introduce $\sigma_t$ as defined by (4.6), then the effective resonance integral in equation (4.5) becomes

$$\int \sigma_a \frac{\bar{\phi}_f}{\phi_o}\frac{dE}{E} = \frac{\sigma_a}{1+(\sigma_a+\sigma_s)\dfrac{1+N\bar{l}(\sigma_t-\sigma_p)}{\sigma_t+N\bar{l}\sigma_p(\sigma_t-\sigma_p)}}\frac{dE}{E}$$

which is identical with the effective resonance integral given in equation (2.4) for a homogeneous mixture, if we take

$$\sigma_m = \frac{\sigma_t + N\bar{l}\sigma_p(\sigma_t-\sigma_p)}{1+N\bar{l}(\sigma_t-\sigma_p)} \tag{4.11}$$

Equation (4.11) gives the correct result for a true homogeneous mixture in the following three limiting cases:

(i) when $\bar{l} \to 0$ , $\sigma_m \to \sigma_t$ ;

(ii) when $\bar{l} \to \infty$ , $\sigma_m \to \sigma_p$ ;

(iii) when $\sigma_t \to \sigma_p$ , i.e. $V_m \to 0$ , $\sigma_m \to \sigma_t$ .

Again, if $\sigma_t$ is large compared with $\sigma_p$ and $N\bar{l}\ \sigma_t >> 1$, then (4.11) gives $\sigma_m = \frac{1 + N\bar{l}\sigma_p}{N\bar{l}}$, in agreement with equation (2.19) for a loosely packed heterogeneous lattice.

## 5. MEAN PATH THROUGH AND EFFECTIVE SURFACE AREA OF A CLUSTER OF FUEL LUMPS

In equation (2.10) the surface area of the fuel lump is involved in three places: (i) explicitly in the first term on the R.H.S. ; (ii) in the simplified expression for $G_o$ as given by equation (2.16) since $\bar{l} = 4V/S$; and (iii) in the formula (2.12) giving the connection between $P^o_o$ and $G_o$.

When the fuel unit is no longer a single lump or rod, but a cluster, it is natural to attempt to define effective surface areas to be used in the three situations. In this section it is shown that we can define a single effective surface area, denoted $S_{eff}$, which should be used in all three, and then methods of calculating $S_{eff}$ for a given cluster are considered.

First consider the explicit appearance of S in (2.10). Here $\frac{\phi}{4}S$ gives the number of neutrons per second in the energy element considered which cross the surface of the lump into fuel.

We must define $S_{eff}$ for the cluster by the analogous statement that $\frac{\phi}{4} S_{eff}$ gives the number of neutrons/sec. which pass through at least one lump of the cluster. Note that this implies that a neutron which passes through two or more lumps (without collision) is counted only once. It follows immediately from the definition that if $S_o$ is the surface area of the smallest convex surface enclosing the cluster, then $S_{eff} \leqslant S_o$. The equal sign applies if it is impossible for a neutron to cross $S_o$ without passing through a fuel lump.

The number of neutrons/sec which pass through at least one fuel lump in a cluster may be expressed by

$$\frac{\phi}{4} S_{eff} = \int_C \int \phi \ \hat{\underset{\sim}{\Omega}} . \hat{\underset{\sim}{n}} \ \frac{d\Omega}{4\pi} \ dS, \tag{5.1}$$

where dS is an element of surface area of a lump, $\hat{\underset{\sim}{n}}$ the inward drawn unit vector normal to dS, $d\Omega$ the element of solid angle in the direction $\hat{\underset{\sim}{\Omega}}$, and $\phi$ the neutron flux (assumed to be uniform and isotropic). If $\hat{\underset{\sim}{\Omega}}.\hat{\underset{\sim}{n}}$ is positive $\phi \frac{d\Omega}{4\pi} dS \ \hat{\underset{\sim}{\Omega}}.\hat{\underset{\sim}{n}}$ is the number of neutrons/sec with directions in $d\Omega$, entering the fuel across dS.
The range of integration is chosen so that only first passages into fuel are included. For example, integration may first be performed with respect to dS, for a fixed $\hat{\underset{\sim}{\Omega}}$ and $d\Omega$, by adding contributions from every dS for which $\hat{\underset{\sim}{\Omega}}.\hat{\underset{\sim}{n}}$ is positive and for which the direction $-\hat{\underset{\sim}{\Omega}}$ "sees" no fuel. This is followed by integrating with respect of $d\Omega$ over all directions. The symbol $\iint -- dS\,d\Omega$ will in this section always indicate integration over this range.

Secondly it is shown that the mean chord length, $\bar{l}$, through fuel in a cluster (corresponding to uniform isotropic flux) is given by $4V/S_{eff}$

$$\text{For } \bar{l} = \frac{\int_C \int l(\underset{\sim}{s}, \hat{\underset{\sim}{\Omega}})\, \hat{\underset{\sim}{\Omega}}.\hat{\underset{\sim}{n}}\ d\Omega\ dS}{\int_C \int \hat{\underset{\sim}{\Omega}}.\hat{\underset{\sim}{n}}\ d\Omega\ dS}$$

where $l(\underset{\sim}{s}, \hat{\underset{\sim}{\Omega}})$ is the total distance through fuel which must be travelled from the point $\underset{\sim}{s}$ on the surface, in the direction $\hat{\underset{\sim}{\Omega}}$, to finally emerge from the cluster.

The numerator is equal to $\int V d\Omega = 4\pi V$ where V is the total volume of fuel in the cluster, and the denominator is $\pi$ Seff from (5.1).

Thirdly it is shown that equations (2.12) continues to apply when the fuel element is a cluster, rather than a single lump, i.e. $1-P_o^o = \frac{S_{eff}}{4V\Sigma} G_o$ where $G_o$ is the probability that a neutron impinging on the cluster suffers its next collision in some fuel lump of the cluster, and $P_o^o$ is the probability that a neutron born in the cluster has its next collision in the cluster.

For, assuming uniform isotropic $\phi$,

$$G_o = \frac{\int_C \int \frac{\phi}{4\pi}(1 - e^{-\Sigma l(\underset{\sim}{s}, \hat{\underset{\sim}{\Omega}})})\ \hat{\underset{\sim}{\Omega}} . \hat{\underset{\sim}{n}}\, d\Omega\, dS}{\int_C \int \frac{\phi}{4\pi}\ \hat{\underset{\sim}{\Omega}} . \hat{\underset{\sim}{n}}\ d\Omega\ dS}$$

where $l(\underset{\sim}{s}, \hat{\underset{\sim}{\Omega}})$ is as defined above,

$$= \frac{1}{\pi S_{eff}} \int_C \int (1 - e^{-\Sigma l(\underset{\sim}{s}, \hat{\underset{\sim}{\Omega}})})\, \hat{\underset{\sim}{\Omega}} . \hat{\underset{\sim}{n}}\ d\Omega\ dS. \tag{5.2}$$

$$\text{Also } 1 - P_o^o = \frac{\int_{\text{all } \hat{\underset{\sim}{\Omega}}} \& \int_{\text{all fuel}} e^{-\Sigma l(\underset{\sim}{r}, \hat{\underset{\sim}{\Omega}})} \frac{\phi}{4\pi}\, d\Omega\, dV}{\int_{\text{all } \hat{\underset{\sim}{\Omega}}} \& \int_{\text{all fuel}} \frac{\phi}{4\pi}\ d\Omega\, dV}$$

$$= \frac{1}{4\pi V} \int_{\text{all } \underset{\sim}{\Omega}} \& \int_{\text{all fuel}} e^{-\Sigma l(\underset{\sim}{r}, \hat{\underset{\sim}{\Omega}})}\, d\Omega\, dV.$$

Here $l(\underset{\sim}{r}, \hat{\underset{\sim}{\Omega}})$ is the total distance through fuel which a neutron "born" at the position $\underset{\sim}{r}$ must travel in direction $\underset{\sim}{\Omega}$ to finally emerge from the cluster.

$$\therefore\ 1 - P_o^o = \frac{1}{4\pi V} \int_C \int \hat{\underset{\sim}{\Omega}} . \hat{\underset{\sim}{n}}\, d\Omega\ dS \int_o^{l(\underset{\sim}{s}, \hat{\underset{\sim}{\Omega}})} e^{-\Sigma l}\ dl$$

$$= \frac{1}{4\pi V \Sigma} \int_C \int (1 - e^{-\Sigma l(\underset{\sim}{s}, \hat{\underset{\sim}{\Omega}})})\, \hat{\underset{\sim}{\Omega}} . \hat{\underset{\sim}{n}}\, d\Omega\ dS$$

$$= \frac{S_{eff}}{4V\Sigma} G_o \text{ from (5.2).}$$

5.1 Calculation of $S_{eff}$ for a given cluster.

From equation (5.1) $S_{eff} = \frac{1}{\pi} \iint \hat{\underset{\sim}{\Omega}} . \hat{\underset{\sim}{n}}\ d\Omega\ dS$, and the evaluation of this integral will now be considered.

For any given $\hat{\underset{\sim}{\Omega}}$, $\hat{A} = \int \hat{\underset{\sim}{\Omega}} . \hat{n}\ dS$ is the projected area of the cluster on a plane perpendicular to $\underset{\sim}{\Omega}$ where the area of regions in which overlapping occurs is counted only once. If the mean value of A over all $\underset{\sim}{\Omega}$ is $\bar{A}$, then $S_{eff} = 4\bar{A}$. Thus an approximate method of obtaining $S_{eff}$ is to calculate the values of A for several equally spaced directions and multiply the average value by 4. This method will be quite accurate if A does not vary greatly with $\underset{\sim}{\Omega}$.

This condition is not satisfied by a cluster consisting of long cylindrical rods, in which case the integral for $S_{eff}$ may be reduced to

$$S_{eff}/\text{unit length} = \frac{1}{2}\iint \hat{\underset{\sim}{\Omega}}.\hat{\underset{\sim}{n}}\, ds\, d\phi$$

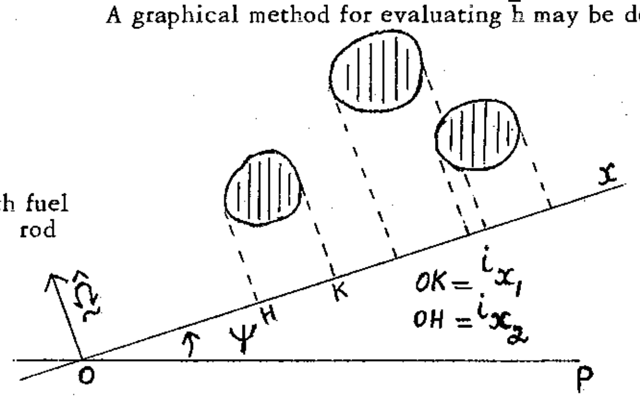
where ds is an element of length in the perimeter of the section of the cluster by a plane perpendicular to the length of the rods, $\hat{\underset{\sim}{n}}$ the inward unit vector normal to ds, $\underset{\sim}{\Omega}$ a unit vector in this plane, and $d\phi$ the element of angle in the plane about direction $\underset{\sim}{\Omega}$.

For any given $\hat{\underset{\sim}{\Omega}}$, $h = \int \underset{\sim}{\Omega}.\hat{n}\, ds$ is the length of the projection of the cross section of the bunch onto a line in the plane perpendicular to $\underset{\sim}{\Omega}$. If the mean value of h over all $\underset{\sim}{\Omega}$ is $\bar{h}$ then $S_{eff}$/unit length = $\pi\bar{h}$.

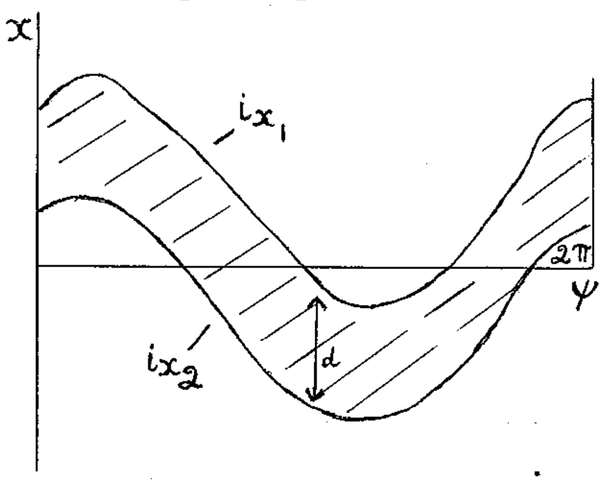
A graphical method for evaluating $\bar{h}$ may be developed as follows. A fixed point O and a fixed line OP are chosen in the plane. For any $\underset{\sim}{\Omega}$, an axis Ox perpendicular to $\underset{\sim}{\Omega}$ is taken, the positive direction being in the direction of $\hat{\underset{\sim}{k}}$ a unit vector along Ox such that $\hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{\Omega}}$ is out of the plane of the paper. Let the angle POx be $\psi$, and let the extremities of the projection of the i th fuel rod on Ox have co-ordinates ${}^ix_1$, and ${}^ix_2$ with ${}^ix_1 > {}^ix_2$

(see diagram).

i th fuel rod



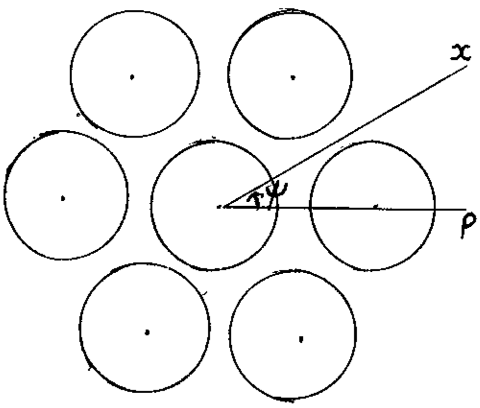

On plotting ${}^ix_1$ and ${}^ix_2$ against $\psi$ in the range $0 < \psi < 2\pi$ a band is formed similar to that in the accompanying sketch. (If the i th rod is not a circular cylinder the vertical width of the band, d, will vary. If the i th rod is circular, and has its centre at the point whose polar co-ordinates are $(\rho, \psi_0)$ relative to O and OP, the equation of the curve for ${}^ix_1$ is ${}^ix_1 = \rho\cos(\psi - \psi_0) + r$, and similarly ${}^ix_2 = \rho\cos(\psi - \psi_0) - r$, r being the radius of the rod. The vertical width of the band is then ${}^ix_1 - {}^ix_2 = 2r$ which is constant.




When bands are drawn corresponding to every fuel rod in the cluster they will partially overlap. The total area inside at least one band (overlap regions counted only once) is equal to $2\pi\bar{h}$ so that half this area gives $S_{eff}$/unit length.

If there are no "eyes" between the bands described then plainly it is impossible for neutrons to pass through the bunch without traversing fuel. In this case it is much quicker to evaluate $S_{eff}$/unit length by calculating the perimeter of the smallest convex figure enclosing the cross section of the bunch. Likewise, if the "eyes" are small, it is quicker and more accurate to draw only the sections of the graph at which they occur and subtract half their combined area from that perimeter. When the rods are circular, the area of any "eye" may be found by simple integration, without drawing the graph. If the bunch has n axes of symmetry parallel respectively to $\psi = \alpha + k2\pi/n$ ; $k = 0, 1, 2, \ldots, n-1$, then only the section of the graph for $0 < \psi < 2\pi/n$ need be drawn.

5.2 An Actual Calculation




The diagram is the cross–section of a bunch of long cylindrical rods, all of radius ¼ in. The centres of the outer circles are the points with polar co–ordinates; ( 5/8 in., $k\pi/3$ ($k = 0, 1, 2, \ldots 5$) referred to the centre of the inner circle as origin. From the diagram it is obvious that only a very small fraction of incident neutrons will pass through the bunch without traversing fuel. Hence $S_{eff}$/ unit length will be only a little less than $S_o$, viz.

($6 \times 5/8 + 2\pi \times 1/4$) sq. ins./in., so that,

$S_{eff}$/unit length < 5.3208 sq. ins./in.

The amount to be subtracted is ½ the area of the twelve "eyes" in the band diagram, all equal, of which there is one in the neighbourhood of $\psi = \frac{\pi}{6}$, $x = 1/4$ bounded by the curves $x = 5/8 \cos\psi - 1/4$

$$x = 5/8 \cos\left(\frac{\pi}{3} - \psi\right) - 1/4$$

$$x = 5/8 \cos\left(\frac{2\pi}{3} - \psi\right) + 1/4$$

and, $$x = 5/8 \cos\left(\frac{5\pi}{3} - \psi\right) + 1/4$$

Calculating the area of this "eye" by elementary integration we obtain

$$2\int_{\sin^{-1}(4/5\sqrt{3})}^{\pi/6} \left\{ \left(5/8 \cos\left(\frac{\pi}{3} - \psi\right) - 1/4\right) - \left(5/8 \cos\left(\frac{5\pi}{3} - \psi\right) + 1/4\right)\right\} d\psi$$

$$= .00187$$

and half of 12 x .00187 is to be subtracted from the 5.3208 obtained above yielding $S_{eff}$/unit length = 5.3096 sq.ins/in.

and also $\bar{l} = 4V/S_{eff} = 1.035$ ins.

## 6. THE EFFECTIVE RESONANCE INTEGRAL FOR A HOMOGENEOUS MIXTURE CONTAINING TH 232

The effective resonance integral for a homogeneous mixture as found in equation (2.4) is

$$I = \int \frac{\sigma_a}{1 + \frac{\sigma_a + \sigma_s}{\sigma_m}} \frac{dE}{E}$$

Introducing Breit–Wigner resonance contours,

$$\sigma_{ar} = \frac{\Gamma\gamma}{\Gamma} \frac{\sigma_o}{1+x^2} \quad , \quad \sigma_{sr} = \frac{\Gamma_n}{\Gamma} \frac{\sigma_o}{1+x^2}$$

where $\Gamma\gamma + \Gamma_n = \Gamma$ and $x = \frac{2}{\Gamma}(E-E_r)$, we can evaluate this integral in the form

$$I = \sum_r \pi/2 \frac{\Gamma\gamma}{E_r} \sigma_o \left(1 + \frac{\sigma_o}{\sigma_m}\right)^{-1/2} \tag{6.1}$$

where the sum is taken over all resonances.

For the resolved resonances we take the parameters tabulated by Sampson and Chernick (9)- With these parameters the sum in (6.1) for the resolved resonances has been evaluated for values of $\sigma_m$ between 100 and 1000 barns.

For the unresolved resonances, assumed to be uniformly spaced at intervals D e.v. and to have a constant radiation width, we convert the sum in (6.1) to an integral

$$I = \pi/2 \frac{\Gamma\gamma}{D} \int_{E_1}^{\infty} \sigma_o \left(1 + \frac{\sigma_o}{\sigma_m}\right)^{-1/2} \frac{dE}{E} \tag{6.2}$$

where $E_1$ is slightly greater than the energy of the last resolved resonance. Since the last resolved resonance is at 310 e.v., we shall take $E_1$ as 320 e.v. The reduced neutron width $\Gamma^o_n$ varies statistically as a $\chi^2$ distribution with two degrees of freedom but for the calculation we shall take the reduced neutron width as a constant, equal to the mean of the statistical distribution, i.e. $\Gamma^o_n = .0015(ev)^{1/2}$. The constant value of $\Gamma\gamma$ is .03 ev and D is taken as 20 e.v., as quoted by Weinberg and Wigner (10).

The integral (6.2) has been evaluated numerically for values of $\sigma_m$ between 100 and 1000 barns.

The calculated value of the effective resonance integral, showing the contributions by resolved and unresolved resonances, is given as a function of $\sigma_m$ in Table II. The values of the effective resonance integral lie very close to a straight line when plotted on log–log paper and fit the formula.

$$I = 1.40 \, (\sigma_m)^{0.45} \tag{6.3}$$

within ½% . We shall therefore use this formula for $\sigma_m$ in the range 100 to 1000 barns.

**TABLE II**

| $\sigma_m$ | 100 | 200 | 500 | 1000 |
|---|---|---|---|---|
| Resolved Resonances | 8.15 | 11.2 | 17.6 | 24.4 |
| Unresolved Resonances | 3.00 | 3.96 | 5.54 | 7.02 |
| Total E.R.I. | 11.15 | 15.2 | 23.1 | 31.4 |

The resonance integral, obtained when $\sigma_m \rightarrow \infty$, for the resolved resonances is 79.4 barns. For the unresolved resonances (6.2) becomes

$$\pi/2 \ \frac{\Gamma\gamma}{D} \int_{E1}^{\infty} \sigma_o \ \frac{dE}{E}$$

which, since

$$\sigma_o = \frac{2.6 \times 10^6 \ \bar{\Gamma}_n^o \sqrt{E}}{E(\Gamma\gamma + \bar{\Gamma}_n^o \sqrt{E})} \tag{6.4}$$

may be integrated to

$$\frac{\pi}{2} \ \frac{\Gamma\gamma}{D} \ 2.6 \times 10^6 \ \left( \frac{\bar{\Gamma}_n^o}{\Gamma\gamma} \right)^2 \left\{ \frac{\Gamma\gamma}{\bar{\Gamma}_n^o \sqrt{E1}} - \ln \left( 1 + \frac{\Gamma\gamma}{\bar{\Gamma}_n^o \sqrt{E1}} \right) \right\}$$

Thus the contribution to the resonance integral by the unresolved resonances is 11.2 barns. This gives a total resonance integral for all resonances of 90.6 barns which is not in very good agreement with the experimental value of 70 barns quoted by Macklin and Pomerance (11).

To evaluate (6.2), Dresner (12) assumed that $\Gamma\gamma << \bar{\Gamma}_n^o \sqrt{E}$, so that

$$\sigma_o = \frac{2.6 \times 10^6}{E} \tag{6.5}$$

If we make the substitution $\sigma_o / \sigma_m = u$ in (6.2) and use the approximation (6.5), we can evaluate the integral in (6.2) to obtain

$$I = \pi \frac{\Gamma\gamma}{D} \sigma_m \left\{ \sqrt{1 + \frac{2.6 \times 10^6}{\sigma_m E1}} - 1 \right\} \tag{6.6}$$

Dresner's values for the contribution by unresolved resonances to the effective resonance integral are 30 to 40% higher than those found in the present work. This is due in part to his using equation (6.6) and the remaining difference must be due to taking different values of the resonance parameters.

The numerical integration of equation (6.2) was terminated at $10^5$ e.v. The maximum error in so doing can be assessed with the aid of equation (6.6) and is found to be negligible.

Corrections should be made to the effective resonance integral in the unresolved region for the statistical variation of the reduced neutron width and also for the p–wave scattering by the fertile atoms. Chernick and Vernon (7) have incorporated these corrections in a calculation of the effective resonance integral for Th 232, taking the resonance spacing as 17 e.v. When their values are converted to a spacing of 20 e.v., it is found that for $\sigma_m$ between 100 and 1000 barns, their values of the effective resonance integral are 0.8 barns greater than those calculated in the present work. Above 30 k.e.v. the present calculations obtain a contribution of 0.2 b whereas Chernick and Vernon obtain 0.85 b. It appears, therefore, that the two corrections very nearly cancel throughout most of the range of the unresolved resonances. In view of the large uncertainty in D, quoted as 17 ± 6 e.v. by Chernick and Vernon, it is felt that no correction need be made to the values of the effective resonance integral found earlier in this section.

## 7. NUMERICAL EXAMPLES

(i) The particular lattice for which we require the effective resonance integral consists of cylindrical fuel elements two inches in diameter set in an hexagonal array with their centres three inches apart. Each fuel element is composed of a cluster of seven half inch diameter rods, one placed in the centre of the element and the other six arranged uniformly with their centres on a circle of 5/8 inch radius. For such a lattice the volume ratio of moderator to fuel $V_m/V_f = 3.385$, while in Section 5 it was found that the mean neutron path length through a fuel element is $\bar{l} = 2.629$ cms.

The fuel rods consist of a mixture of U235, Th232 and Beryllium in the atomic ratio of 1 to 30 or 40 to 1200. For the calculation we will neglect the U235 and take the microscopic scattering cross-section of Beryllium as 6 barns and the potential scattering cross section of Th232 as 12 barns. The moderating material is beryllium oxide for which the macroscopic scattering cross-section is 0.66 $cm^{-1}$.

In a mixture containing Th232 and Be atoms in the ratio of 1 to n, the number of Th232 atoms in 1 cc of the mixture is

$$N = 1 \Big/ \left( \frac{1}{N_{th}} + \frac{n}{N_{Be}} \right)$$

where $N_{th} = 0.02934 \times 10^{24}$ nuc. per cc and $N_{Be} = 0.12354 \times 10^{24}$ nuc. per cc. being respectively the number of atoms per cc. in pure thorium and beryllium.

Using the above values the entries in Table III can be computed.

**TABLE III**

| | | |
|---|---|---|
| Ratio Th/Be | 1/30 | 1/40 |
| $\sigma_p$ | 192 | 252 |
| $\sigma_t$ | 811 | 1052 |
| $\sigma_m$ | 282 | 368 |
| E.R.I. from eqn. (6.3) | 17.7 | 20.0 |
| Vol. term of E.R.I. ($\sigma_m = \sigma_p$) | 14.9 | 16.9 |
| $B\frac{S}{M}\delta$ | 2.8 | 3.1 |
| $B\frac{S}{M}\psi$ | 3.1 | 3.4 |
| Corrected E.R.I. | 18.0 | 20.3 |
| E.R.I. by Homogenising ($\sigma_m = \sigma_t$) | 28.5 | 32.1 |
| $\sigma_m$ from eqn. (2.19) | 297 | 378 |
| E.R.I. for a loose lattice | 18.2 | 20.2 |

The effective resonance integral for a close packed lattice is calculated from equation (6.3) using $\sigma_m$ as defined by equation (4.11). The volume term of the effective resonance integral is then calculated from equation (6.3) by taking $\sigma_m = \sigma_p$. We can thus decompose the effective resonance integral into a volume and a surface term, the surface term being $B\frac{S}{M}\delta$ which according to Section 3 has to be corrected by the factor $\psi/\delta$. Thus we obtain the corrected effective resonance integral. For comparison we have also entered the effective resonance integral for the mixture obtained by the usual method of homogenising the lattice, i.e. taking $\sigma_m = \sigma_t$. The final entry in the table is the effective resonance integral when the close packing correction is neglected and we take the value of $\sigma_m$ as given by equation (2.19).

It is to be noted that the close packing correction in the present problem reduces the effective resonance integral by 1 to 3%. The correction to the surface term required to allow for the approximation in equation (2.16) has only a 1% effect on the result while the rough method of homogenising gives a value of the effective resonance integral nearly 50% too high. The small effect of the close packing correction was to be expected since the distance between fuel elements is greater than the mean path for collisions in the moderator.

(ii) An interesting numerical example is to consider fuel rods of beryllium and thorium in a moderator of beryllium. The idea is to keep the ratio of thorium to beryllium in the system constant, while the rods expand by the addition of beryllium with a corresponding decrease in the volume of the moderator.

We have chosen the initial state as having 3 cm. diameter fuel rods with a thorium to beryllium ratio of 1 to 30. The initial moderator to fuel volume ratio has been taken as 4 to 1. With these values Table IV can be evaluated, giving the effective resonance integral as a function of the atomic ratio of beryllium to thorium in the fuel rods. The fifth column in Table IV gives the effective resonance integral when the surface term has been corrected in conformity with the investigation in Section 3. The first entry in Table IV which is for pure thorium rods will not be correct since equation (6.3) is not valid for such a low value of $\sigma_m$. We have included the value for completeness and to show the trend.

TABLE IV

| Ratio Be/Th in Fuel Rods | $\sigma_p$ | $\sigma_m$ | E.R.I. | Corrected E.R.I. |
|---|---|---|---|---|
| 0 | 12 | 46 | 7.7 | 8.0 |
| 10 | 72 | 128 | 12.4 | 12.7 |
| 30 | 192 | 275 | 17.6 | 17.8 |
| 50 | 312 | 411 | 21.1 | 21.3 |
| 70 | 432 | 542 | 23.8 | 24.0 |
| 90 | 552 | 666 | 26.2 | 26.4 |
| 110 | 672 | 784 | 28.1 | 28.3 |
| 130 | 792 | 891 | 29.6 | 29.7 |
| 150 | 912 | 977 | 30.8 | 30.9 |
| 166 (Homogeneous) | 1008 | 1008 | 31.5 | 31.5 |

## 8. CONCLUSION

The main feature to be noticed in the results of the present calculation is that the allowance for close-packing and the correction for the approximations in equation (2.16) give a combined correction of about 1% to the effective resonance integral in the circumstances of Table III.

The correction to the surface of the envelope of the cluster to allow for neutrons passing between the rods was found in Section 5 to be only 0.2%.

Considering the possible inaccuracy of equation (2.1) which we have taken as the starting point for the calculations, the 1% correction we have found is of no consequence. It would, therefore, be quite satisfactory when investigating a problem such as the first in Section 7 to use the simple theory of Section 2 and take equation (2.19) as giving the correspondence between heterogeneous and homogeneous systems.

Chernick and Vernon (7) have added 10% to their calculated values of the effective resonance integral to allow for the approximations in equation (2.16). However, the present investigation indicates a maximum correction of about 5% when the rods are composed of pure thorium.

Corrections still to be applied in the evaluation of the effective resonance integral include the effect due to Doppler broadening of the resonance contours with increased temperature and an allowance for the presence of a coolant between the rods in a fuel element.

## ACKNOWLEDGMENT

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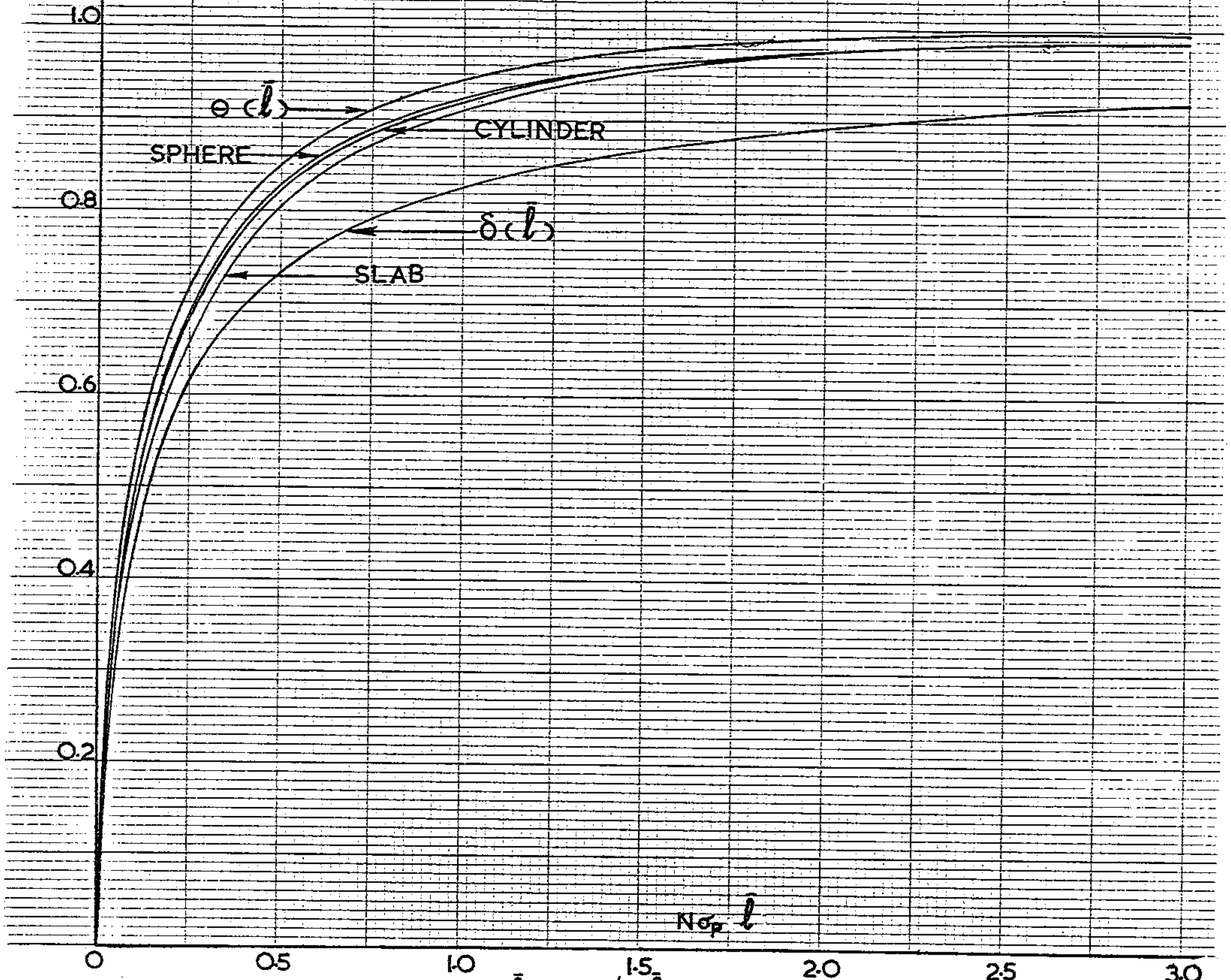
FIGURE. 1.




GRAPH AGAINST $N\sigma_p \bar{l}$ OF $\psi(\bar{l})$ FOR SLAB, CYLINDER AND SPHERE SHOWING RELATION TO $\delta(\bar{l})$ AND $\Theta(\bar{l})$