

AUSTRALIAN NUCLEAR SCIENCE AND TECHNOLOGY ORGANISATION

LUCAS HEIGHTS RESEARCH LABORATORIES

COINCIDENCE-COUNTING CORRECTIONS FOR ACCIDENTAL
COINCIDENCES, SET DEAD TIME AND INTRINSIC DEAD TIME

by

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ABSTRACT

An equation is derived for calculating the radioactivity of a source from the results of coincidence counting, taking into account dead-time losses and accidental coincidences. The corrections allow for the extension of the set dead time in the β channel by the intrinsic dead time. Experimental verification shows improvement over a previous equation [Wyllie, H.A., Appl. Radiat. Isot., 38, 385, 1987].

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COINCIDENCE METHODS; DEAD TIME; MATHEMATICAL MODELS; PROBABILITY;
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EDITORIAL NOTE

From 27 April 1987, the Australian Atomic Energy Commission (AAEC) is replaced by Australian Nuclear Science and Technology Organisation (ANSTO). Serial numbers for reports with an issue date after April 1987 have the prefix ANSTO with no change of the symbol (E, M, S or C) or numbering sequence.

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1. INTRODUCTION

1.1 The Method of Coincidence Counting

The measurement of the activity (in becquerels) of a radioactive source is beset by a number of problems. In the first place, some of the particles and photons emitted by the radionuclide are absorbed within the source; others are not detected by the equipment. The count rates measured by the detectors are thus always less than the disintegration rate, i.e. the efficiencies of the detectors are less than 100 percent. If the radionuclide decays by the emission of two coincident radiations, e.g. a β particle followed promptly by a γ photon, the disintegration rate of the source can be measured by the method of coincidence counting in which the detection efficiencies are also determined.

The $4\pi\beta$ - γ coincidence-counting equipment, shown in Figure 1, measures the β count rate in channel A, and the γ count rate in channel B. The coincidence scaler records the fact that a disintegration has been observed by both channel A and channel B. The detector in channel A is a $4\pi\beta$ proportional counter; the other items in channel A are: pre-amplifier, amplifier, discriminator, timing single-channel analyser, paralysis unit A, scaler A, and an extra scaler, I, which is used in the determination of the intrinsic dead time of the channel [Wyllie 1989b]. The detector in channel B is a thallium-activated sodium iodide [NaI (Tl)] crystal; the other items in channel B are: electron-multiplier phototube, pre-amplifier, amplifier, timing single-channel analyser, gate and delay generator, paralysis unit B and scaler B. Coincidences are detected and recorded by the coincidence mixer and scaler, respectively.

Let N be the disintegration rate of the source in becquerels, N_β the β count rate, N_γ the γ count rate, and N_c the coincidence count rate. If corrections, such as those due to dead time and resolving time, can be neglected, then

$$N = \frac{N_\beta N_\gamma}{N_c} .$$

The derivation of this equation is given in an NCRP report [NCRP 1985:76].

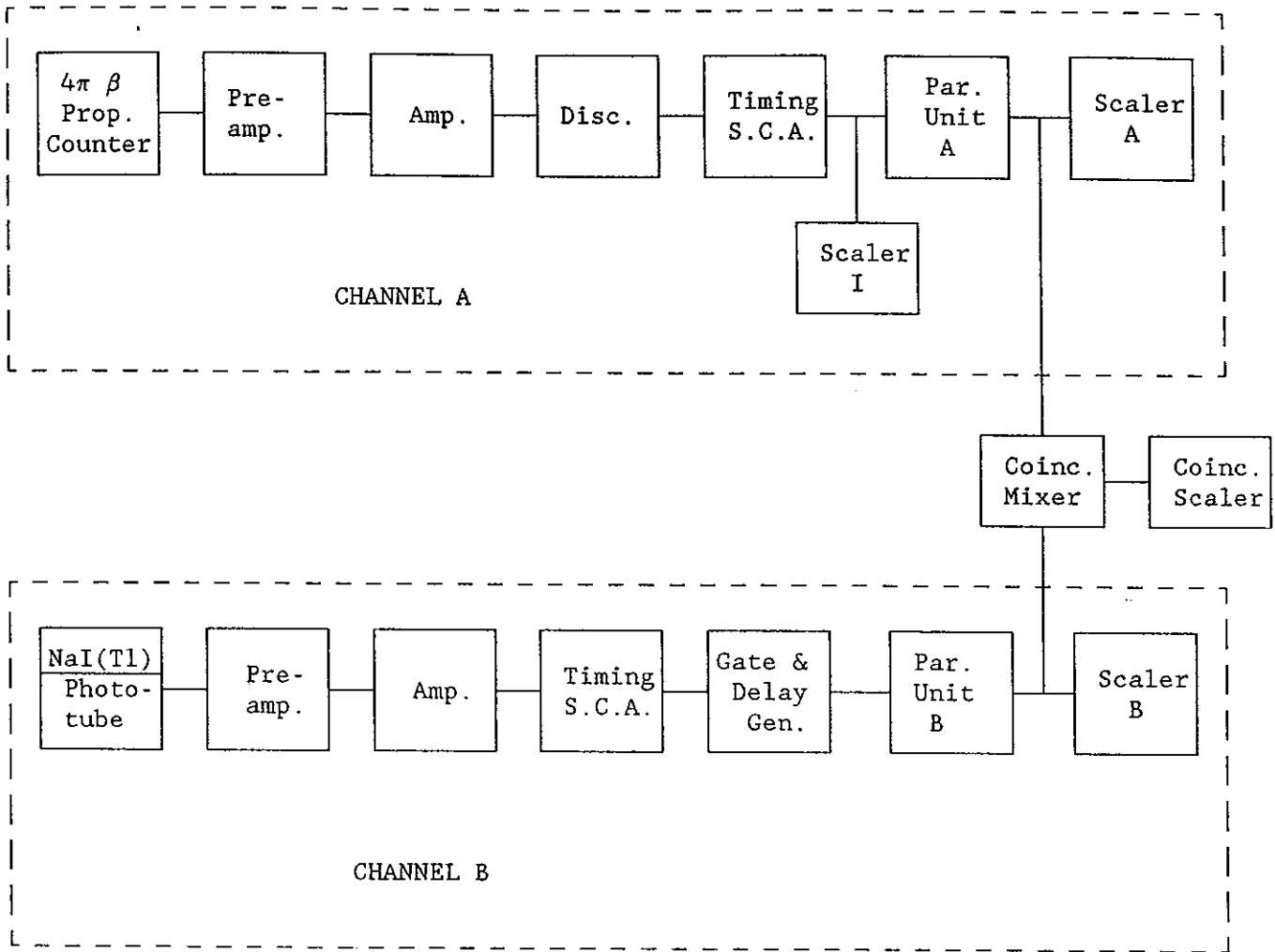


Figure 1 - Block diagram of the $4\pi\beta$ - γ coincidence-counting system used in the Radioisotope Standards Laboratory. The emission of β particles is detected by the $4\pi\beta$ proportional counter and recorded by scalers A and I. The emission of γ photons is detected by the NaI (Tl) crystal and recorded by scaler B. Coincidences are detected and recorded by the coincidence mixer and scaler, respectively.

1.2 Correction for the Set Dead Time and Accidental Coincidences

A detected disintegration gives rise to an output pulse from the detector. The pulse requires a finite time to be processed by the associated electronics. Another pulse from the detector cannot be processed during this period, which is termed the intrinsic dead time of the channel. In channel A the intrinsic dead time depends on the width of the output pulse from the β amplifier, and on the delay in the timing single-channel analyser [Wyllie 1989b]. The intrinsic dead time of that part of channel A which precedes paralysis unit A can be measured by the method of Baerg [1965] or by the method referred to in Section 1.1. The paralysis unit imposes on the channel a well-defined, set dead time which is greater than the intrinsic dead time.

In general, when a disintegration gives rise to a pulse in channel A and a pulse in channel B, i.e. a coincidence, the two pulses will not arrive at the coincidence mixer simultaneously. The maximum difference between the times of arrival is determined, and the resolving time of the coincidence mixer is set to a value which is somewhat higher than the maximum difference. This ensures that all true coincidences will be recorded. However, if a β pulse from one disintegration and a γ pulse from another disintegration arrive at the mixer within the resolving time, an accidental coincidence will be recorded.

Because of dead-time losses and accidental coincidences, the equation in Section 1.1 is only approximately true. A more elaborate equation is therefore required. In a previous report and paper [Wyllie 1987a, 1987b] a coincidence-counting correction formula was derived for the case of equal, non-extending dead times in the two detector channels. The derivation was an extension of the formula derived by J. Bryant [1963].

1.3 Additional Correction for Intrinsic Dead Time

For a more accurate correction it is necessary to take into account the extension of the set dead time in the β channel by the intrinsic dead time of the combined components preceding the paralysis unit [NCRP 1985:70]. The paralysis-unit dead time, T , is set greater than the intrinsic dead time, T_1 . At instant I_1 (Figure 2), a β particle is detected by channel A. The paralysis unit of channel A goes dead for time T , and the preceding components for time T_1 . Another β particle is detected by the 4π counter at instant I_2 . This has the effect of

extending the set dead time. Consideration of this extended dead time increases the dead-time correction in channel A [Wyllie 1989b]. In channel B, the extension of the set dead time can be neglected because of the low γ count rate which results from a low γ detection efficiency.

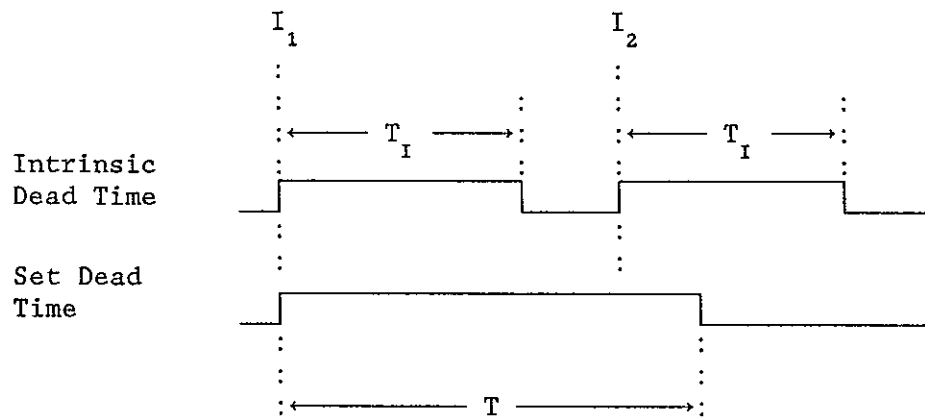


Figure 2 - Extension of the set dead time, T , by the intrinsic dead time, T_I .

1.4 Parameters of the Coincidence-counting Model

In this report, a coincidence-counting correction formula is derived for the case of equal, set dead times, T , in the two channels of the system, and for an intrinsic dead time, T_I , in channel A. The values of T_I considered are in the range

$$T/2 < T_I < T.$$

The values of the resolving time, T_R , which are considered are given by

$$T_R < T/2.$$

Imposing the latter condition prevents the occurrence of some accidental coincidences [Wyllie 1992].

The efficiency of the detector of channel A with respect to one of the coincident radiations is denoted by E_A , and the efficiency of the detector of channel B with respect to the other radiation is denoted by E_B . The efficiencies E_A and E_B are the probabilities of detection of a disintegration by the detectors of channels A and B, respectively, when these detectors are live. It is assumed that the delay in the $4\pi\beta$ -proportional counter is constant, and that the γ -channel delay is adjusted so that in both channels there is the same delay between a disintegration which is detected when the channel is live, and the start of the resultant paralysis-unit dead time.

An aid to the understanding of the following derivations can be arranged with strips of cardboard cut to represent the set dead times and the intrinsic dead time. The diagrams which show dead times and live times can then be supplemented by arranging the strips in the continuously variable juxtapositions described in the text.

2. STATES OF THE SYSTEM

2.1 Introduction

The following analyses depend on which of three portions of the system are dead, or live, at a particular moment. These are the paralysis unit A, with a characteristic dead time T ; the paralysis unit B, also with dead time T ; and that part of channel A preceding the paralysis unit, which is denoted by I and has an intrinsic dead time T_I .

The system is in state 1 if A, B and I are live. When a disintegration is detected the system changes from one state to another. The state to which the system changes depends on which of the above portions go dead, and the state which the system was in immediately prior to the change. Thus, states 2 to 12 are distinguished according to which portions were rendered dead by the last detected disintegration, and by

the state of the system immediately prior to that disintegration. A state once set up is defined to continue until A, B and I are live again, or until a further disintegration is detected.

States 2 to 4 commence when the detection of a disintegration results in one or both paralysis units going dead (Figure 3). The symbol I_1^2 indicates the instant at which the system changes from state 1 to state 2. The subscript and superscript on such symbols denote, respectively, the states immediately before and after the instant of change.

States 5A and 5B occur when the dead times of the paralysis units overlap (Figure 4). States 6 to 12 occur as a result of the extension of the set dead time of channel A by the intrinsic dead time of channel A (Figures 5 to 13).

The recent history of the system may affect the labelling of the current state if it limits the future development of that state. Thus states 5A and 12 commence with a disintegration that makes I and A dead while B is already dead. In state 5A however, B may remain dead for any time up to T, while in state 12 the recent history determines that at most $T - T_I$ remains in which B is dead. Similarly, state 11 is distinguished from state 9 by the amount of dead time remaining in the portion I.

2.2 Definitions of the States

At any instant, the system will be in one of the states defined below. In state 1, both channels are live, as at the instant I_1^1 in Figure 3.

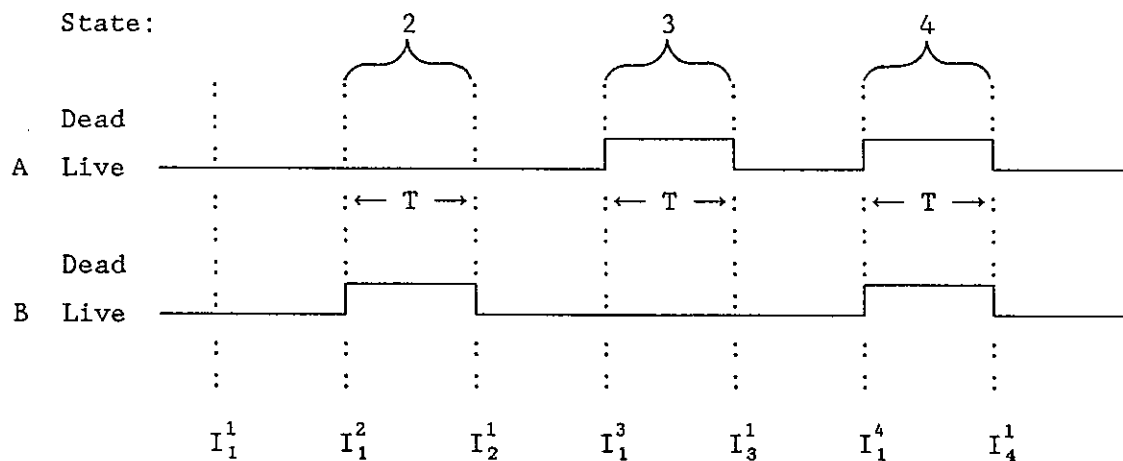


Figure 3 - Definition of states 1, 2, 3 and 4 of the channels.

At instant I_1^2 , a disintegration is detected by channel B, which goes dead for time T, and the system changes from state 1 to state 2. At instant I_2^1 , the system changes from state 2 to state 1. At instant I_1^3 , the system changes from state 1 to state 3. At instant I_1^4 , a disintegration is detected by both channels, i.e. a genuine coincidence occurs. Both channels go dead and the system changes from state 1 to state 4.

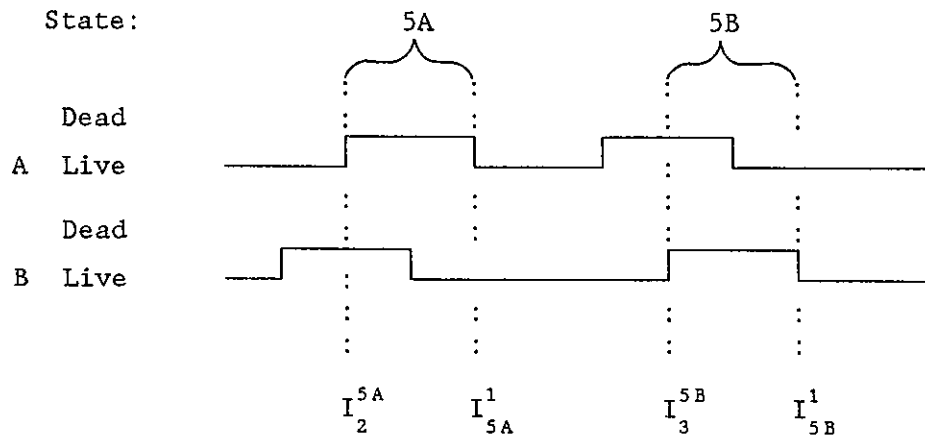


Figure 4 - Definition of states 5A and 5B

At instant I_2^{5A} (Figure 4), a disintegration is detected by channel A while B is dead, and the system changes from state 2 to state 5A. At instant I_{5A}^1 , the system changes to state 1. Similarly, at instant I_3^{5B} , a disintegration is detected by channel B while A is dead, and the system changes from state 3 to state 5B, remaining in the latter state until instant I_{5B}^1 .

The remaining states of the system involve consideration of the intrinsic dead time of channel A, denoted by T_I , as well as the set dead time, T. In the following diagrams, the line labelled by the symbol I indicates when that part of channel A preceding the paralysis unit is live or is dead for time T_I . The words "Dead" and "Live" are omitted from the remaining diagrams to make them more compact.

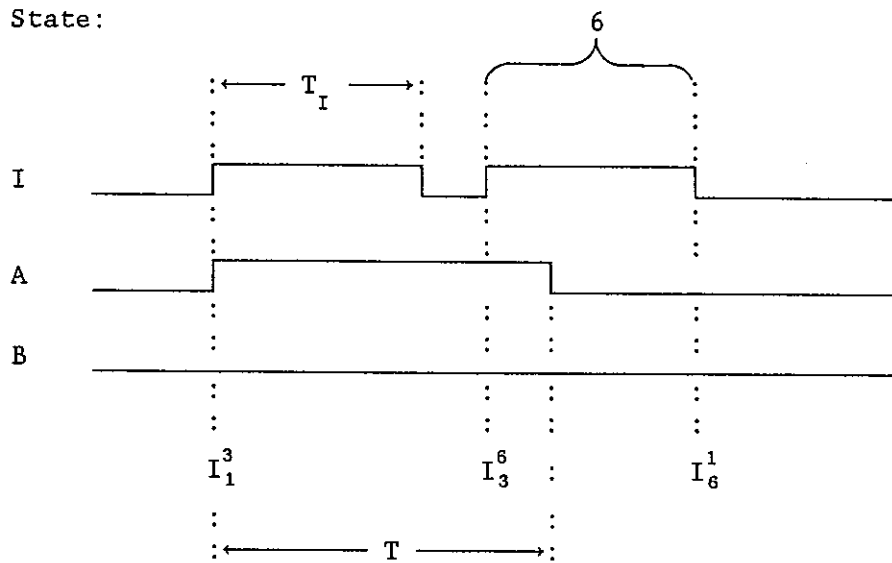


Figure 5 - Definition of state 6

At instant I_1^3 (Figure 5), a disintegration is detected by the detector of channel A and the system changes from state 1 to state 3. At instant I_3^6 , during the set dead time of channel A, another disintegration is detected by the detector of channel A, and the system changes to state 6. At instant I_6^1 the system reverts to state 1.

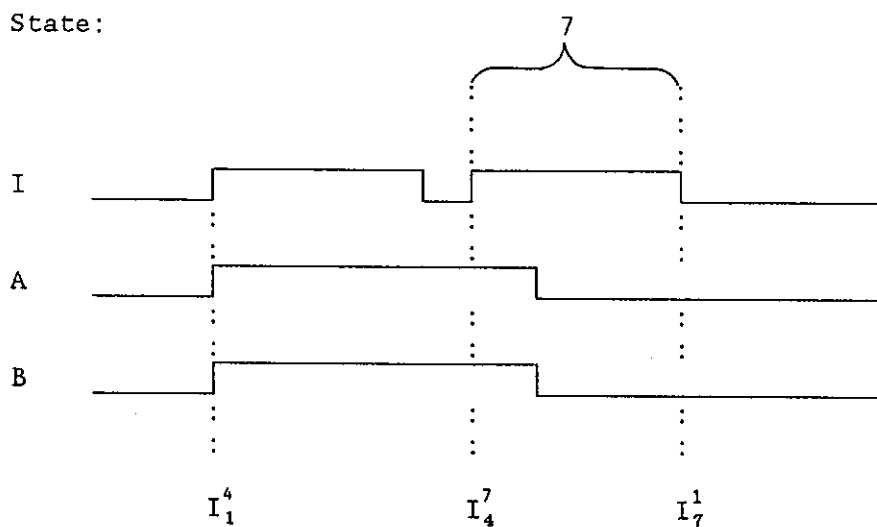


Figure 6 - Definition of state 7

At instant I_1^4 (Figure 6), channels A and B go dead as a result of a genuine coincidence. At instant I_4^7 , during the set dead times of channels A and B, a disintegration is detected by the detector of channel A, and the system changes from state 4 to state 7.

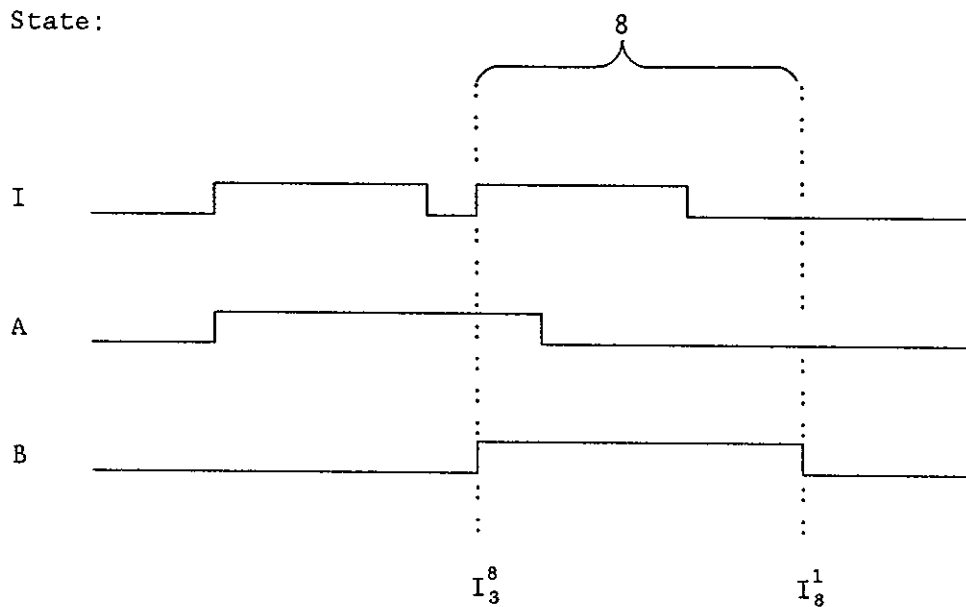


Figure 7 - Definition of state 8

At instant I_3^8 (Figure 7), when the components preceding the paralysis unit of channel A have become live again, a disintegration is detected by the detectors of channels A and B; this may be termed a "quasi-coincidence". The system changes at I_3^8 from state 3 to state 8, and reverts to state 1 at instant I_8^1 when the set dead time of channel B ends, after an interval T.

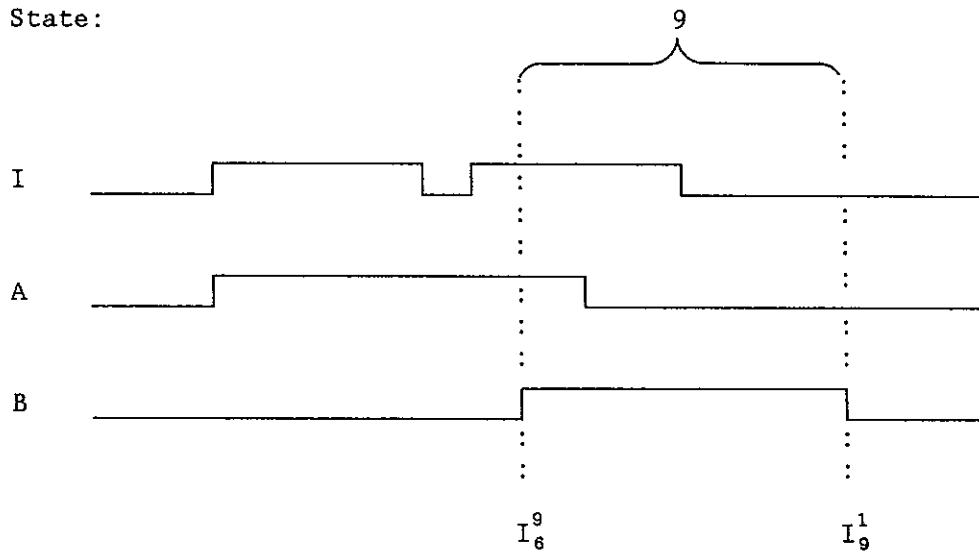


Figure 8 - Definition of state 9

At instant I_6^9 (Figure 8), after the components preceding the paralysis unit of A have gone dead for the second time, a disintegration is detected by channel B, and the system changes to state 9. The system reverts to state 1 at instant I_9^1 when the set dead time of channel B ends.

The system changes to state 10 (Figures 9, 10 and 11) when a disintegration is detected by the detector of channel A during the set dead times of channels A and B. State 10 ends when both channels have become live again.

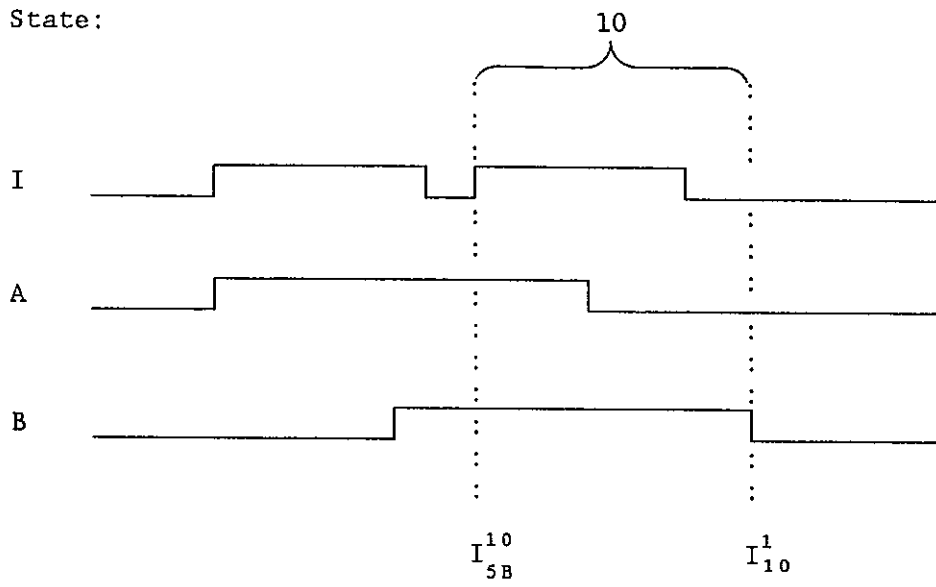


Figure 9 - Definition of state 10

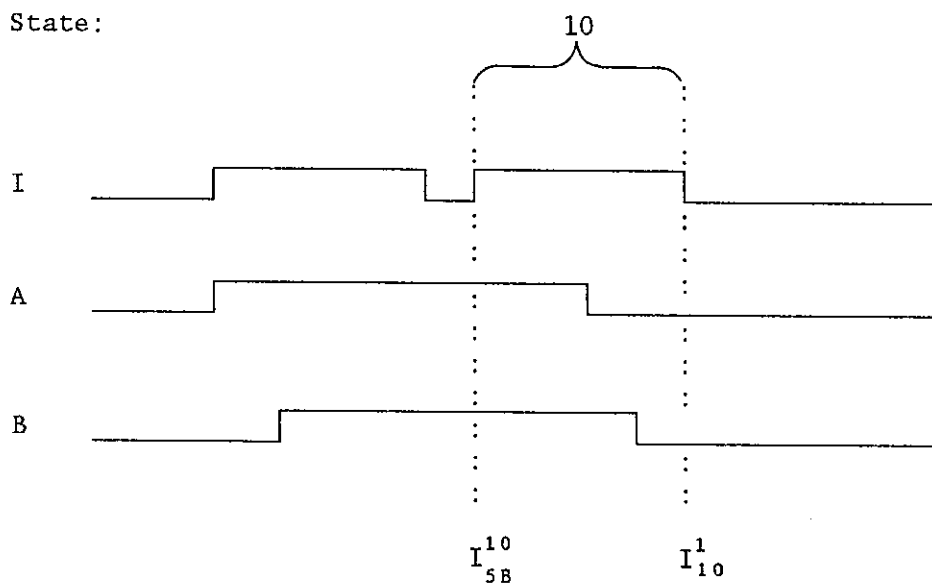


Figure 10 - Definition of state 10

At instant I_{5B}^{10} (Figures 9 and 10), channel B is dead and a disintegration is detected by the detector of channel A, thereby changing the state of the system from 5B to 10. The system remains in state 10 until it reverts to state 1 at instant I_{10}^1 . In Figure 9, channel A becomes live before instant I_{10}^1 ; in Figure 10, channel B becomes live before instant I_{10}^1 .

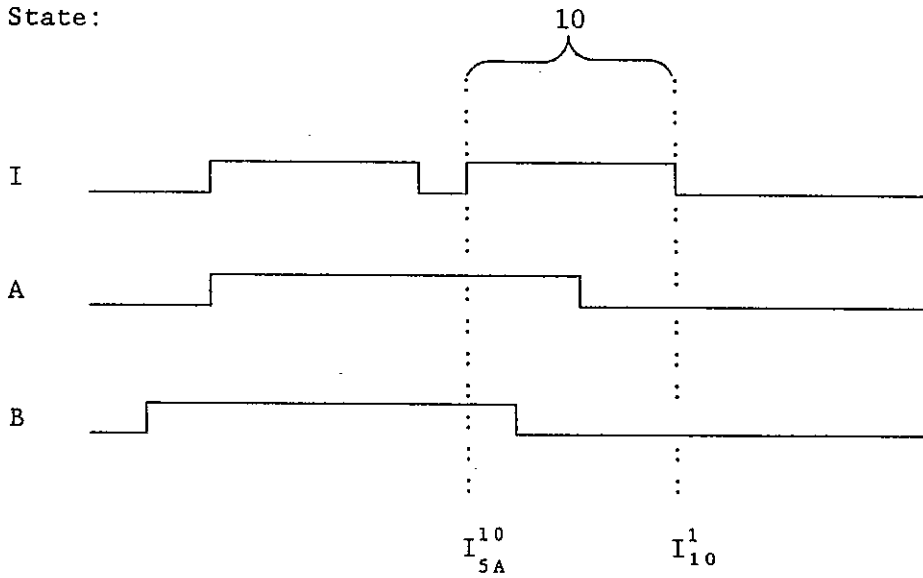


Figure 11 - Definition of state 10

In Figure 11, channel B goes dead before channel A. At instant I_{5A}^{10} , the system changes from state 5A to state 10, and back to state 1 at instant I_{10}^1 .

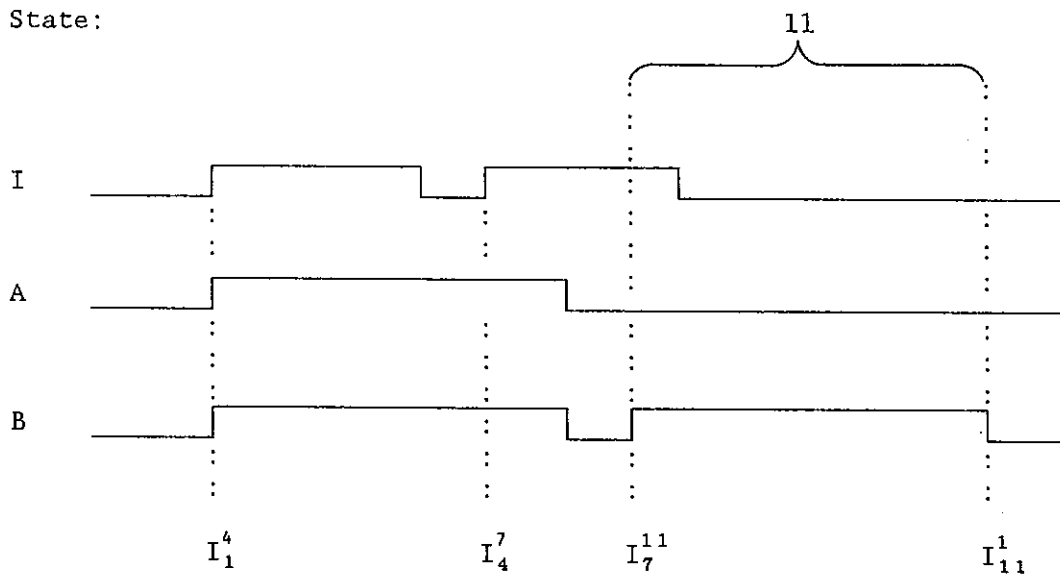


Figure 12 - Definition of state 11

At instant I_7^{11} (Figure 12), a disintegration is detected by channel B, and the system changes from state 7 to state 11. The system reverts to state 1 at instant I_{11}^1 .

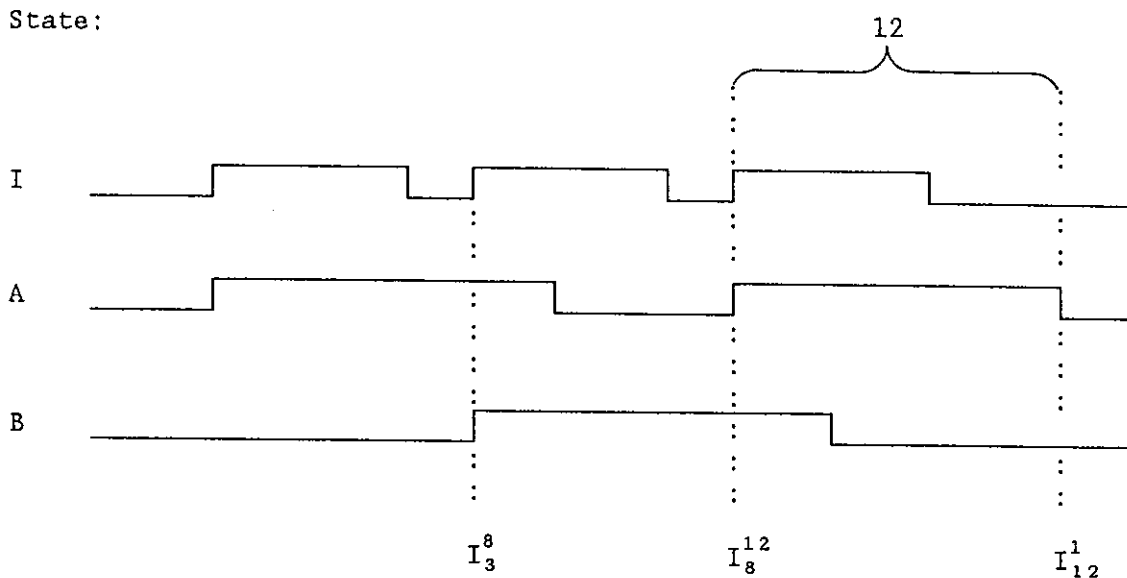


Figure 13 - Definition of state 12

At instant I_8^{12} (Figure 13), a disintegration is detected by channel A, and the system changes from state 8 to state 12. The system remains in state 12 until the end of the set dead time of channel A at instant I_{12}^1 .

3. OUTLINE OF THE METHOD FOR COMPUTING THE DISINTEGRATION RATE, N

Let the probabilities that the system is in the states 1, 2, 3, 4, 5A, 5B, 6, 7, 8, 9, 10, 11 and 12 at any arbitrary instant be $P_1, P_2, P_3, P_4, P_{5A}, P_{5B}, P_6, P_7, P_8, P_9, P_{10}, P_{11}$ and P_{12} , respectively. Let the count rates, including background, which are recorded by the scalers A, I and B be denoted by A'', I'' and B'' , respectively. Let the background rates recorded by A and B be denoted by A_b and B_b , respectively. The probabilities P_1 to P_{12} are functions of N, T, A'', I'', B'', A_b and B_b .

Let C' denote the count rate recorded by the coincidence scaler, from which the background has been subtracted. The count rate C' is the sum of the true coincidence count rate and the accidental coincidence count rate. The formula for C' is derived in Section 8. Paragraph 8.3 is a statement of the formula. The formula contains as variables the resolving time of the coincidence mixer, T_R , the channel efficiencies, E_A and E_B , and the probabilities $P_1, P_2, P_3, P_{5A}, P_{5B}, P_9, P_{11}$ and P_{12} . The expressions for E_A and E_B contain the variables N, T, A'', I'', B'', A_b and B_b . The equation for C' in paragraph 8.3 is of the form:

$$C' = f(N, T, A'', I'', B'', A_b, B_b, T_R).$$

In the above equation N is the unknown variable. Values for the other variables are obtained experimentally. The required value of N is obtained by a series of trials. The first value of N which is tried is calculated using the formula of Bryant [1963]. If this value is inserted into the right-hand side of the above equation, the equation will not balance unless the disintegration rate is low. (Low values of N calculated by Bryant's formula differ only minutely from those obtained by the equation of paragraph 8.3). A lower value of N is then tried. The difference between the left-hand and right-hand sides of the equation is thereby reduced. More trials are carried out to further reduce the difference. The equation is considered solved when a value of N is tried for which the difference between the left-hand and right-hand sides of the equation is less than 0.01 s^{-1} .

4. OUTLINE OF THE DERIVATION OF THE PROBABILITIES OF THE STATES OF THE SYSTEM

4.1 Pairs of Consecutive Disintegrations

Suppose that immediately prior to the first disintegration of a pair of consecutive disintegrations the system is in a state denoted by r , and immediately prior to the second disintegration the system is in a state denoted by s . The probability of this occurring is denoted by the symbol $(r.s)$. Formulae are derived in Section 5 for the probabilities $(r.s)$. In Table 1, zero denotes zero probability, and the $(r.s)$ symbols indicate probabilities for which formulae have been derived. The derivations make no allowance for state transitions produced by background events, which are assumed to be rare enough to be discounted.

The probability $(r.s)$ is given by

$$(r.s) = P_r F_{rs}, \quad (1)$$

where F_{rs} is the probability that the second disintegration of the pair will occur when the system is in state s . The expressions for F_{rs} contain the variables N, T, A'', I'', B'', A_b and B_b . For a particular value of r , the sum of the probabilities, F_{rs} , for all values of s is given by

$$\sum_S F_{rs} = 1. \quad (2)$$

FIRST STATE	SECOND STATE												
	1	2	3	4	5A	5B	6	7	8	9	10	11	12
1	(1.1)	(1.2)	(1.3)	(1.4)	0	0	0	0	0	0	0	0	0
2	(2.1)	(2.2)	0	0	(2.5A)	0	0	0	0	0	0	0	0
3	(3.1)	0	(3.3)	0	0	(3.5B)	(3.6)	0	(3.8)	0	0	0	0
4	(4.1)	0	0	(4.4)	0	0	0	(4.7)	0	0	0	0	0
5A	(5A.1)	0	0	0	(5A.5A)	(5A.5B)	(5A.6)	0	(5A.8)	0	(5A.10)	0	0
5B	(5B.1)	0	0	0	(5B.5A)	(5B.5B)	0	0	0	0	(5B.10)	0	0
6	(6.1)	0	0	0	0	0	(6.6)	0	0	(6.9)	0	0	0
7	(7.1)	0	0	0	0	0	0	(7.7)	0	0	0	(7.11)	0
8	(8.1)	0	0	0	0	0	0	0	(8.8)	0	0	0	(8.12)
9	(9.1)	0	0	0	(9.5A)	0	0	0	0	(9.9)	0	0	0
10	(10.1)	0	0	0	(10.5A)	0	0	0	0	(10.9)	(10.10)	0	0
11	(11.1)	0	0	0	(11.5A)	0	0	0	0	0	0	(11.11)	0
12	(12.1)	0	0	0	0	(12.5B)	(12.6)	0	(12.8)	0	0	0	(12.12)

TABLE 1

List of the probabilities (r.s). The symbol (r.s) denotes the probability that the first of a pair of consecutive disintegrations occurs when the system is in state r, and the second occurs in state s. The zeros indicate probabilities of zero. Formulae for the other probabilities are derived in Section 5.

Summing both sides of equation (1) with respect to s,

$$\sum_s (r.s) = \sum_s P_r F_{rs} = P_r \sum_s F_{rs} .$$

Substituting from equation (2),

$$\sum_s (r.s) = P_r . \tag{3}$$

4.2 Solving Thirteen Equations to Obtain the Probabilities P_1 to P_{12}

Consider arbitrarily selected pairs of consecutive disintegrations. Now consider the second disintegration of a pair. The probability of such a disintegration occurring when the system is in state s is P_s . There is a probability, (r.s), that as well as the second disintegration occurring with the system in state s, the first disintegration has occurred in state r. Thus the probability that the second disintegration occurs when the system is in state s is $\sum_r (r.s)$.

Therefore,

$$P_s = \sum_r (r.s) . \tag{4}$$

Substituting from equation (1), equation (4) can be written

$$P_s = \sum_r P_r F_{rs} . \tag{5}$$

There are 13 such equations in which the 13 probabilities, P_1 to P_{12} , are the unknowns. Only 12 of these equations are independent. To solve for the 13 unknowns, a further equation is required:

$$\sum_r P_r = 1 . \tag{6}$$

Thus the 13 equations which are used for the solution consist of equation (6) and 12 of the group of 13 equations represented by equation (5). The numerical values of the coefficients of the latter 12 equations are obtained by inserting into the formulae represented by F_{rs} the trial value of N, and the experimentally-obtained values of T, A'', I'', B'', A_b and B_b . These 12 equations and equation (6) are then solved by matrix algebra to give the numerical values of the 13 probabilities, P_1 to P_{12} .

An idea of the magnitudes of the probabilities can be obtained from Table 2 which gives the results of the determination of the activity of a ⁶⁰Co source, and lists the values of P_1 to P_{12} .

Source Mount:	Gold/Palladium - coated Mylar		
Dead Time	=	3.875 μ s	
Resolving Time	=	0.913 μ s	
Background counting period	=	2000 s	
Background β counts	=	2709	
Background γ counts	=	5124	
Background coincidence counts	=	17	
Source counting period	=	4000 s	
Source β counts, scaler A	=	129476040	
Source γ counts, scaler B	=	7972500	
Source coincidence counts	=	5995000	
Source β counts, scaler I	=	135228337	
Intrinsic dead time	=	2.651 μ s	
Right-hand side of equation (12)	=	0.87019612	
	P_1	= 0.87019612	
P_2	=	0.00113113	P_4 = 0.00550733
P_{5A}	=	0.00021056	P_6 = 0.00351169
P_7	=	0.00017093	P_8 = 0.00002641
P_{10}	=	0.00001951	P_{12} = 0.00001065
	P_3	= 0.11832680	
	P_{5B}	= 0.00065106	
	P_8	= 0.00023683	
	P_{11}	= 0.00000098	
Start of source counting period (day of the year)	=	50.465	
Reference Time (day of the year)	=	68	
Radioactivity at reference time	=	44,979.29 Bq	
β efficiency (average of 4 counting periods)	=	82.022%	
γ efficiency (average of 4 counting periods)	=	4.406%	

TABLE 2

Results from the determination of the activity of a ^{60}Co source, using the equation of paragraph 8.3. The probabilities of the states are denoted by P_1 to P_{12} .

5. DERIVATION OF THE PROBABILITIES (r.s)

5.1 The Probabilities (1.s)

5.1.1 The Probability (1.1)

The probability (1.1) is the probability that both the first and second disintegrations of a pair of successive disintegrations take place while both detectors are live. There are four cases to consider:

- (a) The first disintegration is detected by neither detector, and therefore the second disintegration must occur with both detectors live. The probability that the first disintegration occurs when the detectors are both live is P_1 . The probability that a disintegration is not detected by the live detector A is $(1 - E_A)$. Likewise, the probability that it is not detected by B is $(1 - E_B)$. Thus the probability that the first disintegration is detected by neither detector is

$$P_1 (1 - E_A)(1 - E_B).$$

- (b) The first disintegration is detected by detector A but not by B. Detector A goes dead for time T. No disintegration must occur during time T if the second disintegration is to occur when both detectors are live. The probability that no disintegration will take place in time T is e^{-NT} . A derivation of the latter expression is given in another report [Wyllie 1987a: Appendix A]. Thus the probability of case (b) is

$$P_1 E_A (1 - E_B) e^{-NT}.$$

- (c) The first disintegration is detected by detector B but not by A. The probability of this case is

$$P_1 E_B (1 - E_A) e^{-NT}.$$

- (d) The first disintegration is detected by both detectors. The second disintegration must not occur during the dead time T. The probability of this case is

$$P_1 E_A E_B e^{-NT}.$$

Summing the probabilities of the four cases,

$$(1.1) = P_1 \{ (1 - E_A)(1 - E_B) + e^{-NT} (E_A + E_B - E_A E_B) \}.$$

5.1.2 The Probability (1.2)

If the second of the pair of successive disintegrations is to occur when the system is in state 2, the first disintegration must cause the system to change from state 1 to state 2. The probability that the first disintegration will occur when the system is in state 1, and that detector B will go dead, as at instant I_1^2 in Figure 2, is $P_1 E_B$. The probability that, in addition, the disintegration is not detected by A is $P_1 E_B (1-E_A)$. The probability that one or more disintegrations will take place in time T is $(1 - e^{-NT})$. Thus the probability that the second of the pair of successive disintegrations will occur in state 2 is $(1 - e^{-NT})$. Therefore

$$(1.2) = P_1 E_B (1-E_A)(1-e^{-NT}).$$

5.1.3 The Probability (1.3)

By a derivation similar to that for (1.2),

$$(1.3) = P_1 E_A (1-E_B)(1-e^{-NT}).$$

5.1.4 The Probability (1.4)

The second of the pair of successive disintegrations occurs while A and B are dead as the result of a coincidence. The probability of the first disintegration occurring during state 1 and being detected by both A and B is $P_1 E_A E_B$. The second disintegration has to occur during the dead time T. Therefore

$$(1.4) = P_1 E_A E_B (1-e^{-NT}).$$

5.1.5 The Probabilities (1.s) which are Zero

A disintegration which occurs while the system is in state 1 cannot change the system from state 1 to any of the following states: 5A, 5B, 6, 7, 8, 9, 10, 11 or 12. The system therefore cannot be in one of the latter states when the next disintegration occurs. Thus, the following probabilities are zero:

(1.5A), (1.5B), (1.6), (1.7), (1.8), (1.9), (1.10), (1.11) and (1.12).

5.1.6 The Sum of the Probabilities F_{1s}

The probability F_{1s} is defined by equation (1). As a check on the formulae derived above for F_{1s} , it is shown below that their sum is equal to unity, in accordance with equation (2). The formulae are partly expanded. The sum of two terms with the same number above them is zero.

$$F_{11} = 1 - E_A - E_B + E_A E_B + e^{-NT} (E_A + E_B - E_A E_B)$$

$$F_{12} = E_B (1 - E_A - e^{-NT} + E_A e^{-NT})$$

$$F_{13} = E_A (1 - E_B - e^{-NT} + E_B e^{-NT})$$

$$F_{14} = E_A E_B (1 - e^{-NT})$$

5.2 The Probabilities (2.s)

5.2.1 The Probability (2.1)

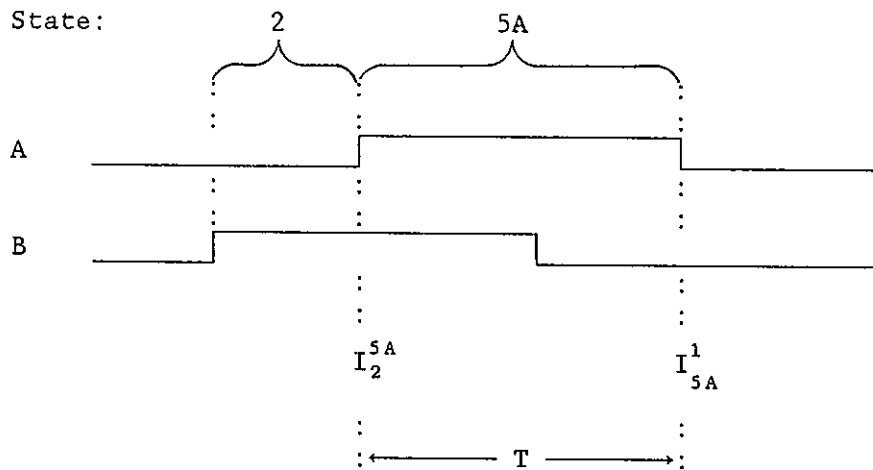


Figure 14 - The probability (2.1), case (a)

There are two cases to consider:

Case (a). In Figure 14 the first disintegration of the pair under consideration occurs at instant I_2^{5A} and is detected by channel A. For the next disintegration to take place after instant I_{5A}^1 , no disintegration shall occur during the interval T between I_2^{5A} and I_{5A}^1 . The probability of case (a) is $P_2 E_A e^{-NT}$.

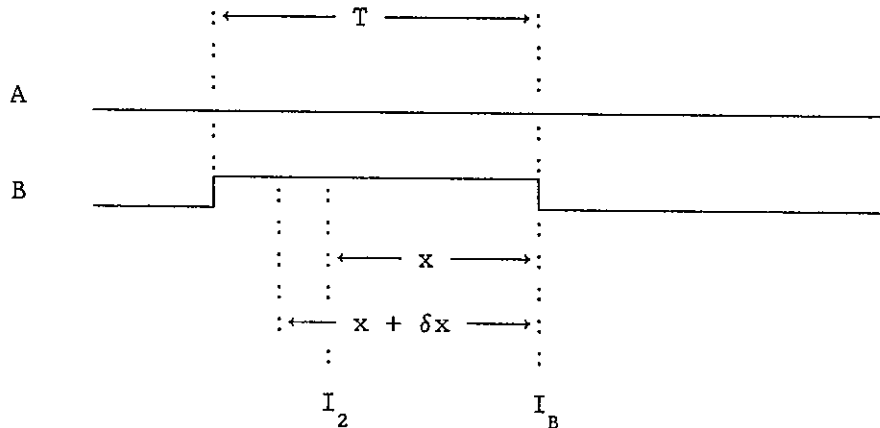


Figure 15 - The probability (2.1), case (b), and the probability (2.2)

Case (b). In Figure 15 the first disintegration at instant I_2 is not detected by channel A. For the next disintegration to take place after instant I_B , no disintegration will have occurred between I_2 and I_B . The probability of no disintegration taking place in the interval x is e^{-Nx} . It is assumed that all values of x are equally probable. The probability that the first disintegration occurs in state 2, that it is not detected by channel A and that it occurs at a time between x and $x + \delta x$ before I_B is $P_2 (1-E_A)(\delta x/T)$. The probability that, in addition, the second disintegration occurs after I_B is $P_2 (1-E_A)(\delta x/T) e^{-Nx}$.

Summing for all the intervals δx , the probability of case (b) is equal to

$$\begin{aligned}
& \int_0^T \frac{P_2 (1-E_A) e^{-Nx}}{T} dx \\
&= \frac{P_2 (1-E_A)}{T} \left[-\frac{e^{-Nx}}{N} \right]_0^T \\
&= \frac{P_2 (1-E_A)}{NT} (1-e^{-NT}) .
\end{aligned}$$

Thus

$$(2.1) = P_2 \left[E_A e^{-NT} + \frac{1}{NT} (1-E_A)(1-e^{-NT}) \right] .$$

5.2.2 The Probability (2.2)

The first disintegration occurs at instant I_2 (Figure 15) and is not detected by channel A. The second disintegration occurs before the instant I_B . The probability of a disintegration occurring during the interval x is $(1-e^{-Nx})$.

Thus

$$\begin{aligned}
(2.2) &= \frac{P_2 (1-E_A)}{T} \int_0^T (1-e^{-Nx}) dx \\
&= \frac{P_2 (1-E_A)}{T} \left[x + \frac{e^{-Nx}}{N} \right]_0^T \\
&= \frac{P_2 (1-E_A)}{T} \left(T + \frac{e^{-NT}}{N} - \frac{1}{N} \right) \\
&= P_2 (1-E_A) \left[1 - \frac{1}{NT} (1-e^{-NT}) \right] \\
&= P_2 \left[1-E_A - (1-E_A) \frac{1}{NT} (1-e^{-NT}) \right] .
\end{aligned}$$

5.2.3 The Probability (2.5A)

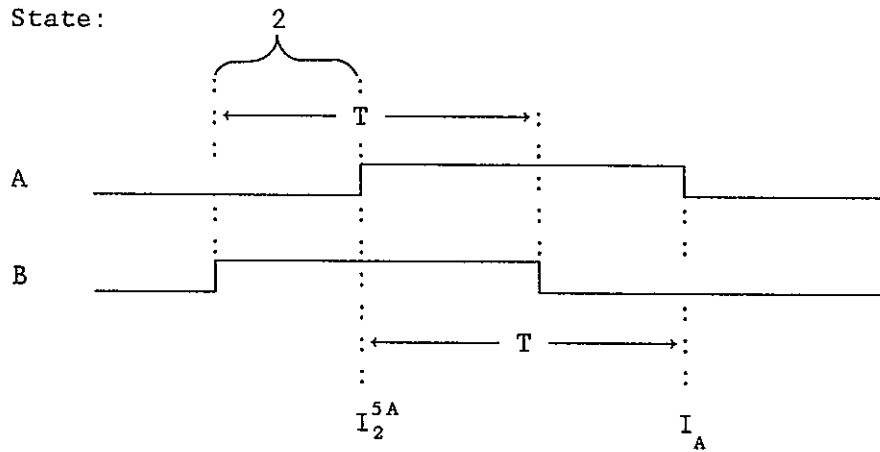


Figure 16 - The probability (2.5A)

The first disintegration of the pair occurs at instant I_2^{5A} (Figure 16) and is detected by channel A. The next disintegration occurs in state 5A, i.e. in the interval between I_2^{5A} and I_A . There is a probability $P_2 E_A$ that the first disintegration occurs in state 2 and is detected by channel A.

The probability of one or more disintegrations occurring between I_2^{5A} and I_A is $(1 - e^{-NT})$. Thus,

$$(2.5A) = P_2 E_A (1 - e^{-NT}).$$

5.2.4 The Probabilities (2.s) which are Zero

The detection of a disintegration when the system is in state 2 cannot change the system to any of the following states: 3, 4, 5B, 6, 7, 8, 9, 10, 11 or 12. Thus the following probabilities are zero:

(2.3), (2.4), (2.5B), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12).

5.2.5 The Sum of the Probabilities F_{2s}

To check the formulae derived above, it is shown below that they obey the equation: $\sum_S F_{2s} = 1$. Terms with the same superscribed number have the same magnitude but opposite sign.

$$F_{21} = E_A \overset{1}{e^{-NT}} + \frac{1}{NT} \overset{2}{(1-E_A)(1-e^{-NT})}$$

$$F_{22} = 1-E_A \overset{3}{-} (1-E_A) \overset{2}{\frac{1}{NT} (1-e^{-NT})}$$

$$F_{25A} = E_A \overset{3}{(1-e^{-NT})} \overset{1}{}$$

5.3 The Probabilities (3.s)

5.3.1 The Probability (3.1)

There are six cases to consider:

Case (a). In Figure 17 the first disintegration of the pair under consideration occurs at instant I_3 and is not detected by the detector of either channel. In case (a) the interval x is less than $(T-T_I)$. No disintegration occurs between instant I_3 and instant I_A . The probability of no disintegration occurring during the interval x is e^{-Nx} . Hence the probability of case (a) is equal to

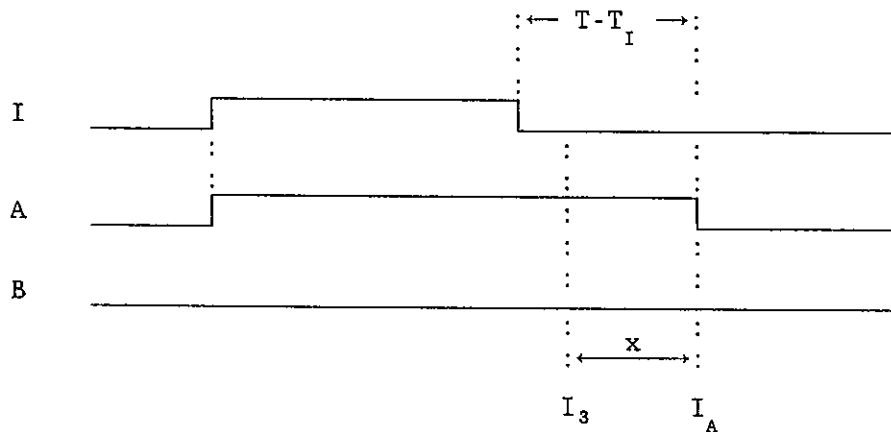


Figure 17 - The probability (3.1), case (a), and the probability (3.3), case (a)

$$\begin{aligned}
& P_3 \frac{1}{T} (1-E_A)(1-E_B) \int_0^{T-T_I} e^{-Nx} dx \\
&= P_3 \frac{1}{T} (1-E_A)(1-E_B) \left[-\frac{e^{-Nx}}{N} \right]_0^{T-T_I} \\
&= P_3 \frac{1}{NT} (1-E_A)(1-E_B) (1-e^{-N(T-T_I)})
\end{aligned}$$

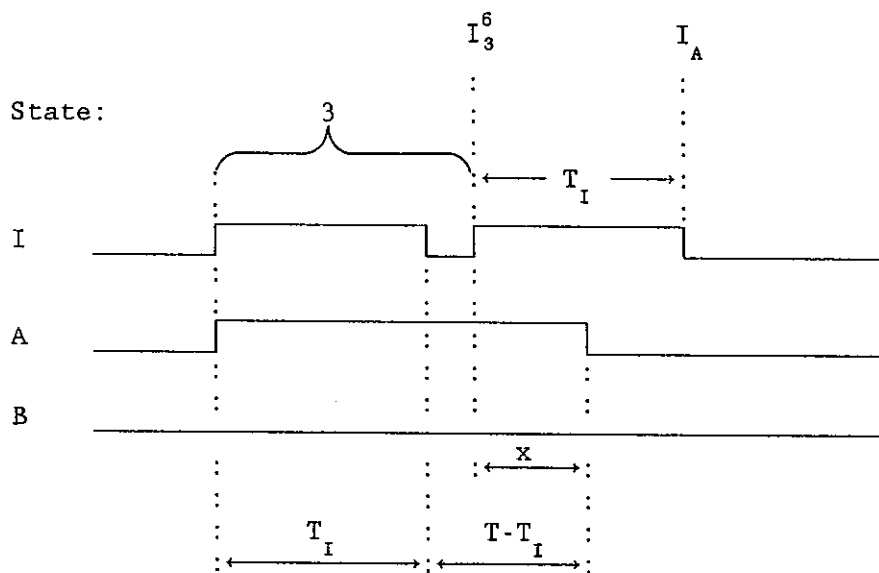


Figure 18 - The probability (3.1), case (b), and the probability (3.6)

Case (b). In Figure 18 the first disintegration of the pair occurs at instant I_3^6 and is detected by the detector of channel A, but is not detected by channel B. The interval x is less than $(T-T_I)$. No disintegration occurs in the interval T_I between instants I_3^6 and I_A . The probability of case (b) is equal to

$$P_3 \frac{1}{T} E_A (1-E_B) e^{-NT_I} \int_0^{T-T_I} dx$$

$$= P_3 \frac{T-T_I}{T} E_A (1-E_B) e^{-NT_I} .$$

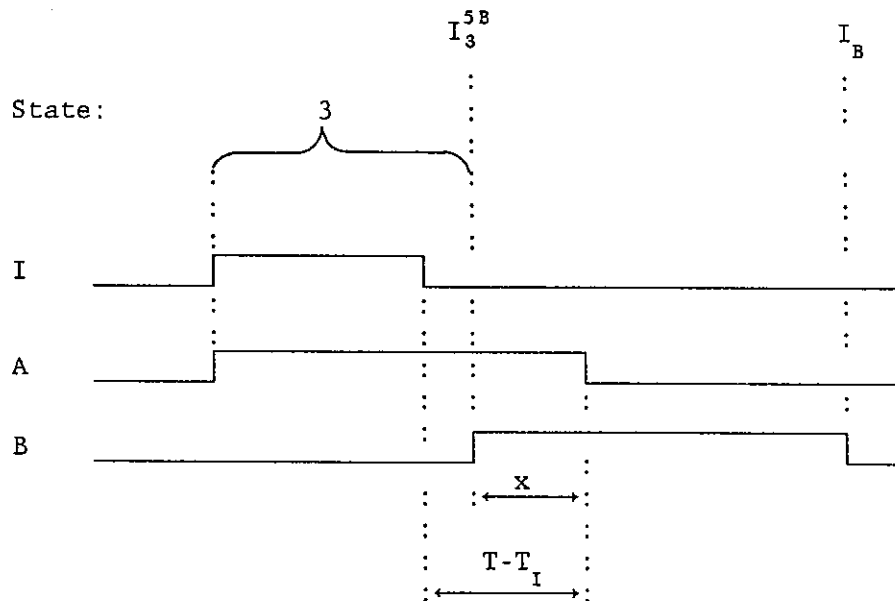


Figure 19 - The probability (3.1), case (c), and the probability (3.5B), case (a).

Case (c). In Figure 19 the first disintegration is detected by channel B but not by the detector of channel A. The interval x is less than $(T-T_I)$. No disintegration occurs in the interval T between the instants I_3^{5B} and I_B . The probability of case (c) is equal to

$$P_3 \frac{1}{T} E_B (1-E_A) e^{-NT} \int_0^{T-T_I} dx .$$

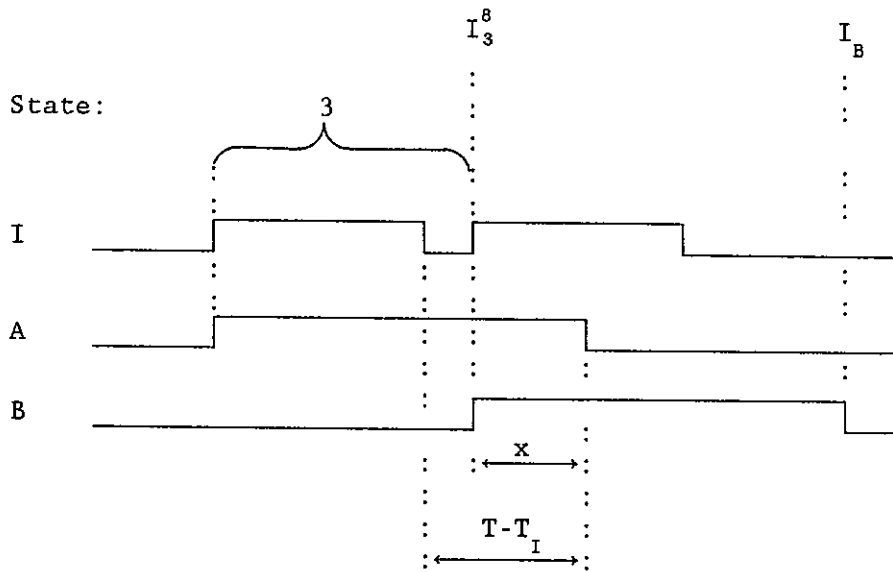


Figure 20 - The probability (3.1), case (d), and the probability (3.8)

Case (d). In Figure 20 the first disintegration is detected by both detectors. No disintegration occurs in the interval T between the instants I_3^8 and I_B . The probability of case (d) is equal to

$$P_3 \frac{1}{T} E_A E_B e^{-NT} \int_0^{T-T_I} dx .$$

The sum of the probabilities of cases (c) and (d) is equal to

$$P_3 \frac{1}{T} E_B e^{-NT} \int_0^{T-T_I} dx .$$

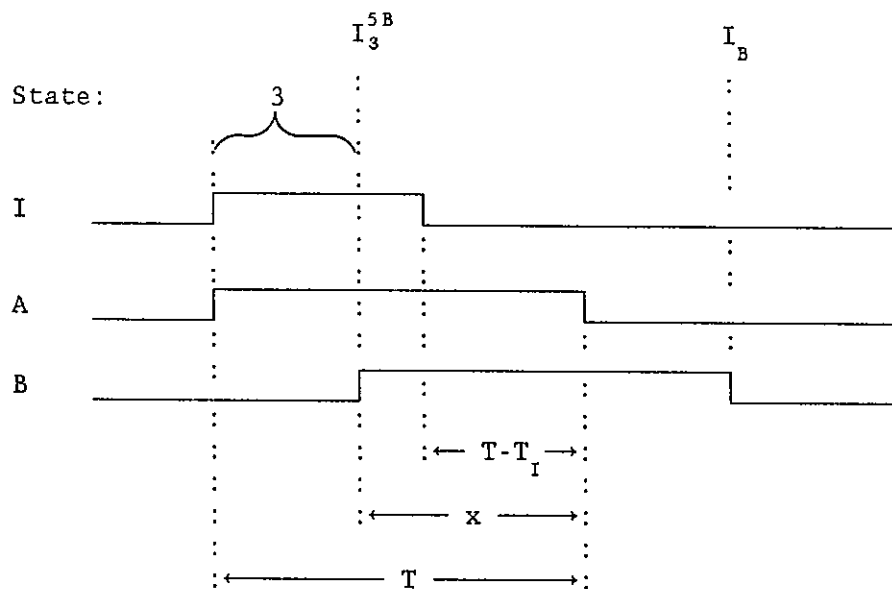


Figure 21 - The probability (3.1), case (e), and the probability (3.5B), case (b)

Case (e). In Figure 21 the first disintegration is detected by channel B at instant I_3^{5B} . The value of x lies between $(T - T_I)$ and T . No disintegration occurs in the interval T between I_3^{5B} and I_B . The probability of case (e) is equal to

$$P_3 \frac{1}{T} E_B e^{-NT} \int_{T-T_I}^T dx .$$

The sum of the probabilities of cases (c), (d) and (e) is equal to

$$P_3 \frac{1}{T} E_B e^{-NT} \int_0^T dx$$

$$= P_3 E_B e^{-NT} .$$

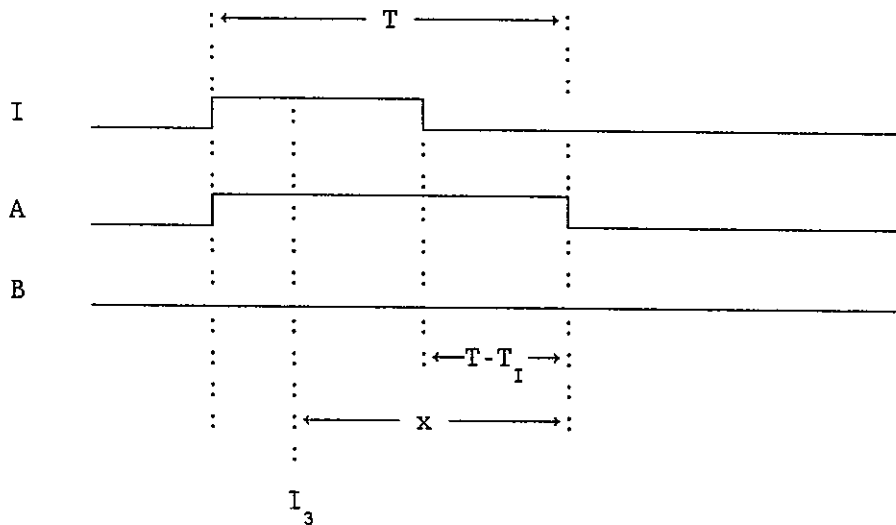


Figure 22 - The probability (3.1), case (f), and the probability (3.3), case (b)

Case (f). The first disintegration at I_3 (Figure 22) is not detected by channel B. The value of x lies between $(T - T_I)$ and T . No disintegration takes place during the interval x . The probability of case (f) is equal to

$$\begin{aligned}
 & P_3 \frac{1}{T} (1 - E_B) \int_{T - T_I}^T e^{-Nx} dx \\
 &= P_3 \frac{1}{T} (1 - E_B) \left[-\frac{e^{-Nx}}{N} \right]_{T - T_I}^T \\
 &= P_3 \frac{1}{NT} (1 - E_B) (e^{-N(T - T_I)} - e^{-NT}) .
 \end{aligned}$$

Before summing the probabilities of the six cases, the formulae for cases (a) and (f) are expanded as follows.

The probability of case (a), expanded,

$$= P_3 \left\{ \frac{1}{NT} (1 - E_B) (1 - e^{-N(T - T_I)}) - \frac{E_A}{NT} (1 - E_B) (1 - e^{-N(T - T_I)}) \right\}$$

$$= P_3 \left\{ \frac{1}{NT} (1-E_B) - \frac{1}{NT} (1-E_B) e^{-N(T-T_I)} - \frac{E_A}{NT} (1-E_B)(1-e^{-N(T-T_I)}) \right\} .$$

The probability of case (f), expanded,

$$= P_3 \left\{ \frac{1}{NT} (1-E_B) e^{-N(T-T_I)} - \frac{1}{NT} (1-E_B) e^{-NT} \right\} .$$

Summing the probabilities of the six cases, and noting that the second term of case (a) plus the first term of case (f) equals zero,

$$(3.1) = P_3 \left\{ \frac{1}{NT} (1-E_B) - \frac{E_A}{NT} (1-E_B) (1-e^{-N(T-T_I)}) \right. \\ \left. + \frac{T-T_I}{T} E_A (1-E_B) e^{-NT_I} + E_B e^{-NT} - \frac{1}{NT} (1-E_B) e^{-NT} \right\} .$$

5.3.2 The Probability (3.3)

There are two cases to consider: case (a) (Figure 17) and case (b) (Figure 22). The following table gives the limits of integration and the element of probability for each case. The derivation of the element is similar to preceding derivations.

CASE	LIMITS OF INTEGRATION wrt x	ELEMENT OF PROBABILITY
(a)	$\int_0^{T-T_I}$	$P_3 \frac{\delta x}{T} (1-E_A)(1-E_B)(1-e^{-Nx})$
(b)	$\int_{T-T_I}^T$	$P_3 \frac{\delta x}{T} (1-E_B)(1-e^{-Nx})$

Integrating with respect to x, the probabilities are:

Case (a)

$$P_3 \frac{1}{T} (1-E_A)(1-E_B) \left[x + \frac{e^{-Nx}}{N} \right]_0^{T-T_I}$$

$$= P_3 \frac{1}{T} (1-E_A)(1-E_B) \left[T-T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1) \right] .$$

Case (b)

$$P_3 \frac{1}{T} (1-E_B) \left[x + \frac{e^{-Nx}}{N} \right]_{T-T_I}^T$$

$$= P_3 \frac{1}{T} (1-E_B) \left[T_I + \frac{1}{N} (e^{-NT} - e^{-N(T-T_I)}) \right] .$$

Summing the probabilities of the two cases, and multiplying out,

$$(3.3) = P_3 \frac{1}{T} (1-E_B) \left\{ \underbrace{T-T_I + \frac{1}{N} e^{-N(T-T_I)}}_{\text{underlined}} - \frac{1}{N} - E_A (T-T_I) \right.$$

$$\left. - \frac{E_A}{N} e^{-N(T-T_I)} + \frac{E_A}{N} + \underbrace{T_I + \frac{1}{N} e^{-NT}}_{\text{underlined}} - \frac{1}{N} e^{-N(T-T_I)} \right\} . \quad (7)$$

In equation (7), the sum of the underlined terms is zero. By further regrouping

$$(3.3) = P_3 \left\{ 1-E_B + \frac{1-E_B}{T} \left[\frac{1}{N} (-1-E_A e^{-N(T-T_I)} + E_A + e^{-NT}) \right. \right.$$

$$\left. \left. - E_A (T-T_I) \right] \right\} .$$

5.3.3 The Probability (3.5B)

There are two cases to consider: case (a) (Figure 19) and case (b) (Figure 21).

CASE	LIMITS OF INTEGRATION wrt x	ELEMENT OF PROBABILITY
(a)	$\int_0^{T-T_I}$	$P_3 \frac{\delta x}{T} (1-E_A) E_B (1-e^{-NT})$
(b)	$\int_{T-T_I}^T$	$P_3 \frac{\delta x}{T} E_B (1-e^{-NT})$

Integrating with respect to x, the probabilities are as follows:

$$\text{Case (a)} \quad P_3 \frac{1}{T} (1-E_A) E_B (1-e^{-NT}) (T-T_I)$$

$$\text{Case (b)} \quad P_3 \frac{1}{T} E_B (1-e^{-NT}) T_I$$

Summing,

$$\begin{aligned} (3.5B) &= P_3 \frac{1}{T} E_B (1-e^{-NT}) [(1-E_A)(T-T_I) + T_I] \\ &= P_3 \frac{1}{T} E_B (1-e^{-NT}) [T - E_A(T-T_I)] . \end{aligned}$$

5.3.4 The Probability (3.6)

The transition from state 3 to state 6 is shown in Figure 18.

LIMITS OF INTEGRATION wrt x	ELEMENT OF PROBABILITY
$\int_0^{T-T_I}$	$P_3 \frac{\delta x}{T} E_A (1-E_B) (1-e^{-NT_I})$

Integrating with respect to x,

$$(3.6) = P_3 E_A (1-E_B) (1-e^{-NT_I}) \frac{T-T_I}{T} .$$

5.3.5 The Probability (3.8)

The transition from state 3 to state 8 is shown in Figure 20.

LIMITS OF INTEGRATION wrt x	ELEMENT OF PROBABILITY
$\int_0^{T-T_I}$	$P_3 \frac{\delta x}{T} E_A E_B (1-e^{-NT})$

Integrating with respect to x,

$$(3.8) = P_3 E_A E_B (1-e^{-NT}) \frac{T-T_I}{T} .$$

5.3.6 The Probabilities (3.s) which are Zero

The following probabilities are zero: (3.2), (3.4), (3.5A), (3.7), (3.9), (3.10), (3.11) and (3.12).

5.3.7 The Sum of the Probabilities F_{3s}

To check the formulae derived above for F_{3s} , it is shown below that they obey the equation: $\sum_s F_{3s} = 1$.

The formulae are partly expanded. The formula for F_{33} is obtained by expanding the right-hand side of equation (7). Terms with the same superscribed number have the same magnitude but opposite sign.

$$\begin{aligned}
 F_{31} &= \frac{1}{NT} (1-E_B) - \frac{E_A}{NT} (1-e^{-N(T-T_I)}) - E_B + E_B e^{-N(T-T_I)} \\
 &+ \frac{T-T_I}{T} E_A (1-E_B) e^{-NT_I} + E_B e^{-NT} - \frac{1}{NT} (1-E_B) e^{-NT} \\
 F_{33} &= 1 - \frac{1}{NT} - E_A \frac{T-T_I}{T} - \frac{E_A}{NT} e^{-N(T-T_I)} + \frac{E_A}{NT} + \frac{1}{NT} e^{-NT} \\
 &- E_B + \frac{E_B}{NT} + E_A E_B \frac{T-T_I}{T} + \frac{E_A E_B}{NT} e^{-N(T-T_I)} - \frac{E_A E_B}{NT} - \frac{E_B}{NT} e^{-NT} \\
 F_{35B} &= E_B (1-E_A \frac{T-T_I}{T} - e^{-NT} + E_A e^{-NT} \frac{T-T_I}{T}) \\
 F_{36} &= E_A (1-E_B) \frac{T-T_I}{T} - E_A (1-E_B) \frac{T-T_I}{T} e^{-NT_I} \\
 F_{38} &= E_A E_B \frac{T-T_I}{T} (1-e^{-NT})
 \end{aligned}$$

5.4 The Probabilities (4.s)

5.4.1 The Probability (4.1)

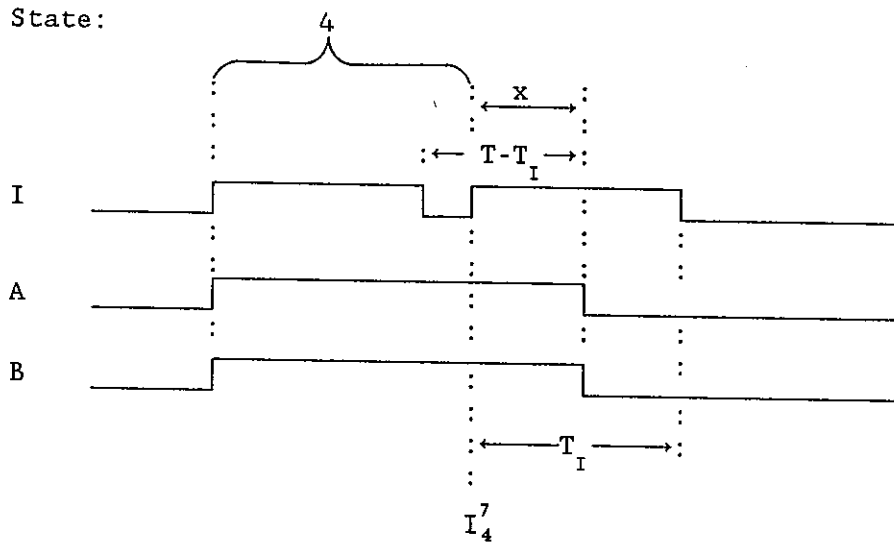


Figure 23 - The probability (4.1), case (a), and the probability (4.7)

There are 3 cases to consider. Case (a) is shown in Figure 23.

CASE	LIMITS OF INTEGRATION wrt x	ELEMENT OF PROBABILITY
(a)	$\int_0^{T-T_I}$	$P_4 \frac{\delta x}{T} E_A e^{-NT_I}$
(b)	$\int_0^{T-T_I}$	$P_4 \frac{\delta x}{T} (1-E_A) e^{-Nx}$
(c)	$\int_{T-T_I}^T$	$P_4 \frac{\delta x}{T} e^{-Nx}$

Integrating with respect to x, the probabilities are as follows:

$$\text{Case (a)} \quad P_4 \frac{E_A}{T} e^{-NT_I} (T-T_I) .$$

$$\begin{aligned} \text{Case (b)} \quad P_4 \frac{(1-E_A)}{T} \left[-\frac{e^{-Nx}}{N} \right]_0^{T-T_I} \\ = P_4 \frac{(1-E_A)}{NT} (1 - e^{-N(T-T_I)}) . \end{aligned}$$

$$\begin{aligned} \text{Case (c)} \quad P_4 \frac{1}{T} \left[-\frac{e^{-Nx}}{N} \right]_{T-T_I}^T \\ = P_4 \frac{1}{NT} (e^{-N(T-T_I)} - e^{-NT}) . \end{aligned}$$

Summing the probabilities of the 3 cases, and multiplying out,

$$\begin{aligned} (4.1) = P_4 \frac{1}{T} \left[E_A e^{-NT_I} (T-T_I) + \frac{1}{N} (1 - \underline{e^{-N(T-T_I)}}) - E_A \right. \\ \left. + E_A \underline{e^{-N(T-T_I)}} + \underline{e^{-N(T-T_I)}} - e^{-NT} \right] . \end{aligned}$$

In the above equation the sum of the underlined terms is zero.

Thus,

$$\begin{aligned} (4.1) = P_4 \frac{1}{T} \left[E_A e^{-NT_I} (T-T_I) + \frac{1}{N} (1 - E_A + E_A e^{-N(T-T_I)} \right. \\ \left. - e^{-NT}) \right] . \end{aligned}$$

5.4.2 The Probability (4.4)

CASE	LIMITS OF INTEGRATION wrt x	ELEMENT OF PROBABILITY
(a)	$\int_0^{T-T_I}$	$P_4 \frac{\delta x}{T} (1-E_A)(1-e^{-Nx})$
(b)	$\int_{T-T_I}^T$	$P_4 \frac{\delta x}{T} (1-e^{-Nx})$

Integrating with respect to x, the probabilities are:

$$\begin{aligned} \text{Case (a)} \quad & P_4 \frac{1}{T} (1-E_A) \left[x + \frac{e^{-Nx}}{N} \right]_0^{T-T_I} \\ & = P_4 \frac{1}{T} (1-E_A) \left[T-T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1) \right] . \end{aligned}$$

$$\begin{aligned} \text{Case (b)} \quad & P_4 \frac{1}{T} \left[x + \frac{e^{-Nx}}{N} \right]_{T-T_I}^T \\ & = P_4 \frac{1}{T} \left[T_I + \frac{1}{N} (e^{-NT} - e^{-N(T-T_I)}) \right] . \end{aligned}$$

Summing the probabilities of the 2 cases, and multiplying out,

$$\begin{aligned} (4.4) = & P_4 \frac{1}{T} \left\{ \overset{1}{T-T_I} + \frac{1}{N} (e^{-N(T-T_I)} - 1) - E_A T + E_A T_I \right. \\ & \left. - \frac{E_A}{N} (e^{-N(T-T_I)} - 1) + T_I + \frac{1}{N} (e^{-NT} - e^{-N(T-T_I)}) \right\} . \end{aligned}$$

In the above equation, terms with the same superscribed number add up to zero. Thus,

$$(4.4) = P_4 \left\{ 1 - E_A + \frac{1}{T} \left[E_A T_I + \frac{1}{N} (e^{-NT} + E_A - E_A e^{-N(T-T_I)} - 1) \right] \right\} .$$

5.4.3 The Probability (4.7)

The transition from state 4 to state 7 is shown in Figure 23.

LIMITS OF INTEGRATION wrt x	ELEMENT OF PROBABILITY
$\int_0^{T-T_I}$	$P_4 \frac{\delta x}{T} E_A (1 - e^{-NT_I})$

Integrating with respect to x,

$$(4.7) = P_4 \frac{1}{T} E_A (1 - e^{-NT_I})(T - T_I) .$$

5.4.4 The Probabilities (4.s) which are Zero

The following probabilities are zero: (4.2), (4.3), (4.5A), (4.5B), (4.6), (4.8), (4.9), (4.10), (4.11) and (4.12).

5.4.5 The Sum of Probabilities F_{4s}

It is shown below that the formulae for F_{4s} which have been derived above obey the equation $\sum_s F_{4s} = 1$. Terms with the same superscribed number add up to zero.

$$F_{41} = \frac{E_A}{T} e^{-NT_I} (T - T_I) + \frac{1}{NT} (1 - E_A + E_A e^{-N(T-T_I)} - e^{-NT})$$

$$F_{44} = 1 - E_A + \frac{1}{T} \left[E_A T_I + \frac{1}{N} (e^{-NT} + E_A - E_A e^{-N(T-T_I)} - 1) \right]$$

$$F_{47} = E_A - \frac{E_A T_I}{T} - \frac{E_A}{T} e^{-NT_I(T-T_I)}$$

5.5 The Probabilities (5A.s)

The derivation of (5A.5A) is given first, followed by a similar but longer derivation of (5A.1).

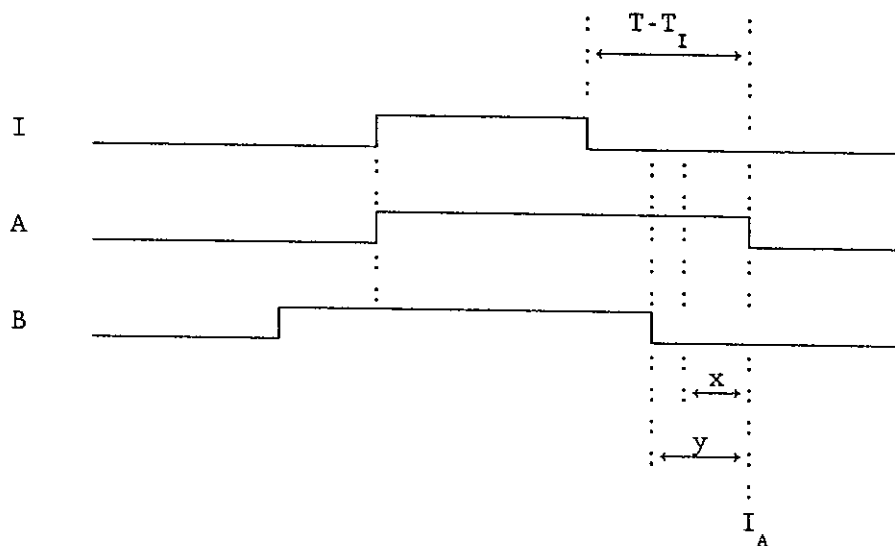


Figure 24 - The probability (5A.5A), case (a), and the probability (5A.1), case (a)

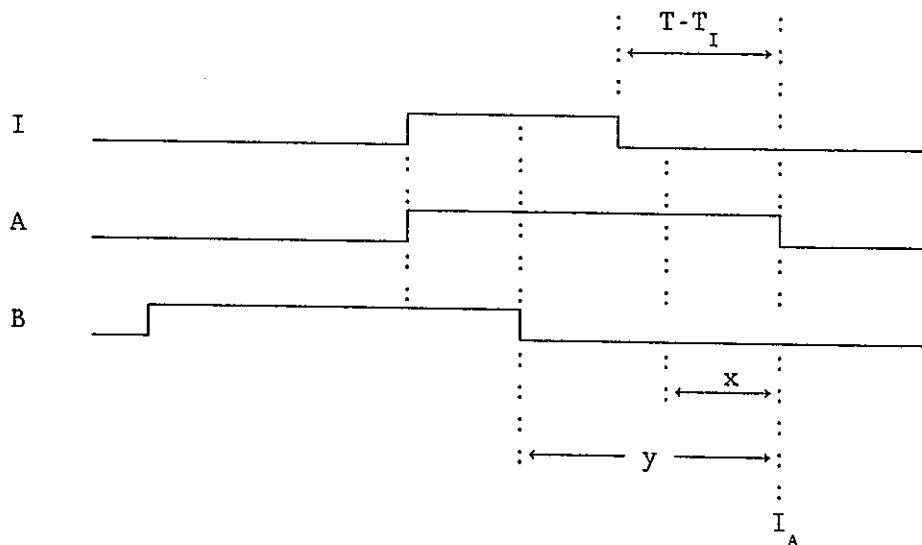


Figure 25 - The probability (5A.5A), case (d), and the probability (5A.1), case (h)

5.5.1 The Probability (5A.5A)

There are 6 cases to consider. In case (a) (Figure 24), the first of the pair of disintegrations under consideration occurs at a time x before instant I_A . Channel B becomes live at a time y before I_A . It is assumed that all values of x and y are equally probable. The probability of a disintegration occurring between x and $x + \delta x$ before I_A , and channel B becoming live between y and $y + \delta y$ before I_A is $P_{5A} \frac{\delta y}{T} \frac{\delta x}{T}$.

For the second disintegration to occur with the system in state 5A, the first disintegration must not be detected by the detector of either channel, and the second disintegration shall occur during the interval x .

Thus the element of (5A.5A) is given by

$$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T} (1-E_A)(1-E_B)(1-e^{-Nx}) .$$

For one particular increment δy , the sum of the elements is given by

$$P_{5A} \frac{\delta y}{T^2} (1-E_A)(1-E_B) \int_0^y (1-e^{-Nx}) dx .$$

The probability of case (a) is then obtained by summing for all the increments δy :

$$P_{5A} \frac{1}{T^2} (1-E_A)(1-E_B) \int_0^{T-T_I} \left[\int_0^y (1-e^{-Nx}) dx \right] dy .$$

Instead of completing the integration for this and the other cases, and then adding the integrals together, the formula for (5A.5A) is derived below more concisely.

The following table gives the element of probability for each case, and the limits of integration with respect to x and y . The product $(1-E_A)(1-E_B)$ has been expanded to $1-E_A - E_B + E_A E_B$. Case (d) is shown in Figure 25. In deriving the formula for (5A.5A), use is made of relations such as

$$\int_0^y f(x) dx + \int_y^{T-T_I} f(x) dx + \int_{T-T_I}^T f(x) dx = \int_0^T f(x) dx.$$

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE
y	x		
$\int_0^{T-T_I}$	\int_0^y	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T} (1-E_A - E_B + E_A E_B)(1-e^{-Nx})$	(a)
$\int_0^{T-T_I}$	$\int_y^{T-T_I}$	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T} (1-E_A) (1-e^{-Nx})$	(b)
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T} (1-e^{-Nx})$	(c)
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T} (1-E_A - E_B + E_A E_B)(1-e^{-Nx})$	(d)
$\int_{T-T_I}^T$	$\int_{T-T_I}^y$	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T} (1-E_B) (1-e^{-Nx})$	(e)
$\int_{T-T_I}^T$	\int_y^T	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T} (1-e^{-Nx})$	(f)

Let $1-e^{-Nx} = I$.

(5A.5A) =

$$\begin{aligned}
 P_{5A} \frac{1}{T^2} \left\{ \int_0^{T-T_I} \left[\int_0^y I \, dx - E_A \int_0^y I \, dx - E_B \int_0^y I \, dx + E_A E_B \int_0^y I \, dx \right. \right. \\
 + \left. \int_y^{T-T_I} I \, dx - E_A \int_y^{T-T_I} I \, dx + \int_{T-T_I}^T I \, dx \right] dy \\
 + \int_{T-T_I}^T \left[\int_0^{T-T_I} I \, dx - E_A \int_0^{T-T_I} I \, dx - E_B \int_0^{T-T_I} I \, dx \right. \\
 + E_A E_B \int_0^{T-T_I} I \, dx + \int_{T-T_I}^y I \, dx - E_B \int_{T-T_I}^y I \, dx \\
 \left. \left. + \int_y^T I \, dx \right] dy \right\}
 \end{aligned}$$

$$\begin{aligned}
 = P_{5A} \frac{1}{T^2} \left\{ \int_0^{T-T_I} \left[\int_0^T I \, dx - E_A \int_0^{T-T_I} I \, dx - E_B \int_0^y I \, dx \right. \right. \\
 + \left. E_A E_B \int_0^y I \, dx \right] dy \\
 + \int_{T-T_I}^T \left[\int_0^T I \, dx - E_B \int_0^y I \, dx - E_A \int_0^{T-T_I} I \, dx \right. \\
 \left. \left. + E_A E_B \int_0^{T-T_I} I \, dx \right] dy \right\}
 \end{aligned}$$

$$\begin{aligned}
&= P_{5A} \frac{1}{T^2} \left\{ \int_0^T \left[\int_0^T I \, dx - E_A \int_0^{T-T_I} I \, dx - E_B \int_0^y I \, dx \right] dy \right. \\
&\quad \left. + E_A E_B \left\langle \int_0^{T-T_I} \left(\int_0^y I \, dx \right) dy + \int_{T-T_I}^T \left(\int_0^{T-T_I} I \, dx \right) dy \right\rangle \right\}
\end{aligned}$$

The following formulae are substituted into the above equation.

$$\int_0^T I \, dx = \int_0^T (1 - e^{-Nx}) \, dx = \left[x + \frac{e^{-Nx}}{N} \right]_0^T = T + \frac{1}{N} (e^{-NT} - 1)$$

$$\int_0^{T-T_I} I \, dx = T - T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1)$$

$$\int_0^y I \, dx = y + \frac{1}{N} (e^{-Ny} - 1)$$

Thus (5A.5A) =

$$\begin{aligned}
&P_{5A} \frac{1}{T^2} \left\{ \int_0^T \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A \left\langle T - T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1) \right\rangle \right. \right. \\
&\quad \left. \left. - E_B \left(y + \frac{1}{N} e^{-Ny} \right) + \frac{E_B}{N} \right] dy \right. \\
&\quad \left. + E_A E_B \left\langle \int_0^{T-T_I} \left(y + \frac{1}{N} e^{-Ny} \right) dy - \frac{1}{N} \int_0^{T-T_I} dy \right. \right. \\
&\quad \left. \left. + \int_{T-T_I}^T \left(T - T_I + \frac{1}{N} e^{-N(T-T_I)} \right) dy - \frac{1}{N} \int_{T-T_I}^T dy \right\rangle \right\}
\end{aligned}$$

$$\begin{aligned}
&= P_{SA} \frac{1}{T^2} \left\{ \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A \left\langle T - T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1) \right\rangle \right. \right. \\
&\quad + \left. \left. \frac{E_B}{N} \right] \left[y \right]_0^T - E_B \left[\frac{y^2}{2} - \frac{1}{N^2} e^{-Ny} \right]_0^T \right. \\
&\quad + E_A E_B \left\langle \left[\frac{y^2}{2} - \frac{1}{N^2} e^{-Ny} \right]_0^{T-T_I} - \frac{1}{N} \int_0^T dy \right. \\
&\quad \left. \left. + (T - T_I + \frac{1}{N} e^{-N(T-T_I)}) \left[y \right]_{T-T_I}^T \right\rangle \right\} \\
&= P_{SA} \left\{ 1 + \frac{1}{NT} (e^{-NT} - 1) - E_A \frac{(T-T_I)}{T} - E_A \frac{1}{NT} (e^{-N(T-T_I)} - 1) \right. \\
&\quad + \frac{E_B}{NT} - \frac{E_B}{T^2} \left[\frac{T^2}{2} - \frac{1}{N^2} (e^{-NT} - 1) \right] \\
&\quad + \frac{E_A E_B}{T^2} \left\langle \frac{(T-T_I)^2}{2} - \frac{1}{N^2} (e^{-N(T-T_I)} - 1) - \frac{1}{N} T \right. \\
&\quad \left. \left. + (T-T_I) T_I + \frac{1}{N} e^{-N(T-T_I)} T_I \right\rangle \right\} \\
&= P_{SA} \left\{ 1 + \frac{E_A E_B}{T^2} \left[\frac{(T-T_I)^2}{2} + (T-T_I) T_I + \frac{T_I}{N} e^{-N(T-T_I)} \right] \right. \\
&\quad \left. + \frac{E_B}{N^2 T^2} \left[e^{-NT} - E_A e^{-N(T-T_I)} + E_A - 1 \right] \right\}
\end{aligned}$$

$$+ \frac{1}{NT} \left[E_A - 1 + E_B (1 - E_A) - E_A e^{-N(T-T_I)} + e^{-NT} \right]$$

$$- \left. \frac{E_B}{2} - E_A + E_A \frac{T_I}{T} \right\} .$$

5.5.2 The Probability (5A.1)

There are 14 cases to consider. The second of the pair of disintegrations under consideration must not occur until both channels are live again. The probability of this requirement being fulfilled is e^{-Nx} in case (a), e^{-NT_I} in case (b), e^{-NT} in case (c), and so on. In the following table the product $(1-E_A)(1-E_B)$ has been expanded to $1 - E_A - E_B + E_A E_B$.

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	\int_0^y	$(1-E_A - E_B + E_A E_B) e^{-Nx}$	(a)	24
$\int_0^{T-T_I}$	\int_0^y	$E_A (1-E_B) e^{-NT_I}$	(b)	26
$\int_0^{T-T_I}$	\int_0^y	$E_B (1 - E_A) e^{-NT}$	(c)	27
$\int_0^{T-T_I}$	\int_0^y	$E_A E_B e^{-NT}$	(d)	
$\int_0^{T-T_I}$	$\int_y^{T-T_I}$	$(1-E_A) e^{-Nx}$	(e)	

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	$\int_y^{T-T_I}$	$E_A e^{-NT_I}$	(f)	28
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	e^{-Nx}	(g)	
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	$(1-E_A - E_B + E_A E_B) e^{-Nx}$	(h)	25
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	$E_A (1-E_B) e^{-NT_I}$	(i)	
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	$E_B (1-E_A) e^{-NT}$	(j)	
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	$E_A E_B e^{-NT}$	(k)	
$\int_{T-T_I}^T$	$\int_{T-T_I}^y$	$(1-E_B) e^{-Nx}$	(l)	
$\int_{T-T_I}^T$	$\int_{T-T_I}^y$	$E_B e^{-NT}$	(m)	
$\int_{T-T_I}^T$	\int_y^T	e^{-Nx}	(n)	

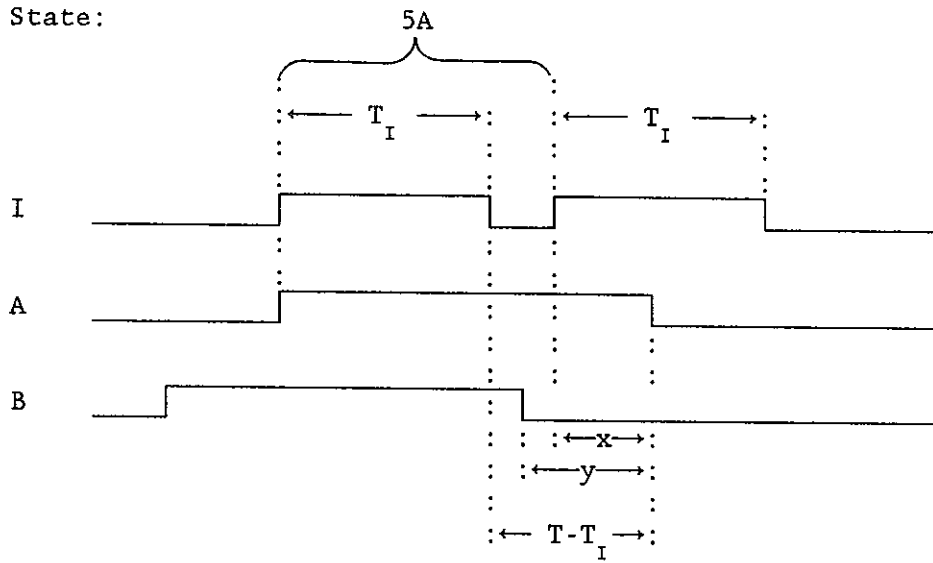


Figure 26 - The probability (5A.1), case (b), and the probability (5A.6), case (a)

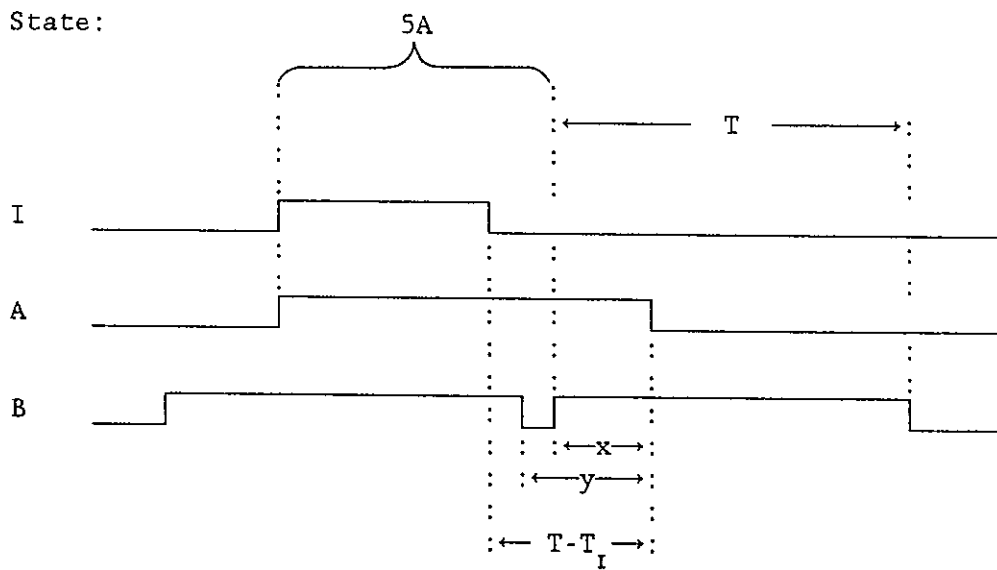


Figure 27 - The probability (5A.1), case (c), and the probability (5A.5B), case (a)

State:

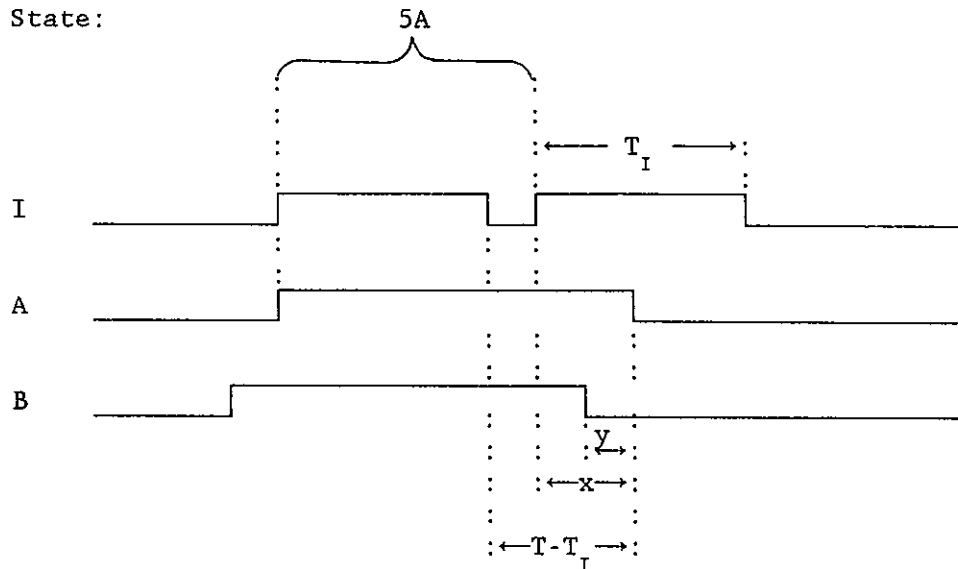


Figure 28 - The probability (5A.1), case (f), and the probability (5A.10)

(5A.1) =

$$\begin{aligned}
 P_{5A} \frac{1}{T^2} & \left\{ \int_0^{T-T_I} \left[\int_0^y e^{-Nx} dx - E_A \int_0^y e^{-Nx} dx - E_B \int_0^y e^{-Nx} dx \right. \right. \\
 & + E_A E_B \int_0^y e^{-Nx} dx + E_A e^{-NT_I} \int_0^y dx - E_A E_B e^{-NT_I} \int_0^y dx \\
 & + E_B e^{-NT_I} \int_0^y dx + \int_y^{T-T_I} e^{-Nx} dx - E_A \int_y^{T-T_I} e^{-Nx} dx \\
 & \left. \left. + E_A e^{-NT_I} \int_y^{T-T_I} dx + \int_{T-T_I}^T e^{-Nx} dx \right] dy \right. \\
 & \left. + \int_{T-T_I}^T \left[\int_0^{T-T_I} e^{-Nx} dx - E_A \int_0^{T-T_I} e^{-Nx} dx - E_B \int_0^{T-T_I} e^{-Nx} dx \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + E_A E_B \int_0^{T-T_I} e^{-Nx} dx + E_A e^{-NT_I} \int_0^{T-T_I} dx - E_A E_B e^{-NT_I} \int_0^{T-T_I} dx \\
& + E_B e^{-NT} \int_0^{T-T_I} dx + \int_{T-T_I}^y e^{-Nx} dx - E_B \int_{T-T_I}^y e^{-Nx} dx \\
& + E_B e^{-NT} \left[\int_{T-T_I}^y dx + \int_y^T e^{-Nx} dx \right] dy \} \\
= P_{SA} \frac{1}{T^2} & \left\{ \int_0^{T-T_I} \left[\int_0^T e^{-Nx} dx - E_A \int_0^{T-T_I} e^{-Nx} dx \right. \right. \\
& - E_B \int_0^y e^{-Nx} dx + E_A E_B \int_0^y e^{-Nx} dx + E_A e^{-NT_I} \int_0^{T-T_I} dx \\
& \left. \left. - E_A E_B e^{-NT_I} y + E_B e^{-NT} y \right] dy \right. \\
& + \int_{T-T_I}^T \left[\int_0^T e^{-Nx} dx - E_A \int_0^{T-T_I} e^{-Nx} dx - E_B \int_0^y e^{-Nx} dx \right. \\
& \left. + E_A E_B \int_0^{T-T_I} e^{-Nx} dx + E_A e^{-NT_I} \int_0^{T-T_I} dx - E_A E_B e^{-NT_I} (T-T_I) \right. \\
& \left. + E_B e^{-NT} \int_0^y dx \right] dy \} \\
= P_{SA} \frac{1}{T^2} & \left\{ \int_0^T \left[\int_0^T e^{-Nx} dx - E_A \int_0^{T-T_I} e^{-Nx} dx - E_B \int_0^y e^{-Nx} dx \right. \right. \\
& \left. \left. + E_A e^{-NT_I} (T-T_I) + E_B e^{-NT} y \right] dy \right\}
\end{aligned}$$

$$\begin{aligned}
& + E_A E_B \int_0^{T-T_I} \left[\int_0^y e^{-Nx} dx - e^{-NT_I} y \right] dy \\
& + E_A E_B \int_{T-T_I}^T \left[\int_0^{T-T_I} e^{-Nx} dx - e^{-NT_I} (T-T_I) \right] dy \} .
\end{aligned}$$

Since $\int_0^T e^{-Nx} dx = \left[-\frac{e^{-Nx}}{N} \right]_0^T = \frac{1}{N} (1 - e^{-NT})$,

(5A.1) =

$$\begin{aligned}
& P_{5A} \frac{1}{T^2} \left\{ \int_0^T \left[\frac{1}{N} (1 - e^{-NT}) - \frac{E_A}{N} (1 - e^{-N(T-T_I)}) \right. \right. \\
& \quad \left. \left. - \frac{E_B}{N} (1 - e^{-Ny}) + E_A e^{-NT_I} (T-T_I) + E_B e^{-NT} y \right] dy \right. \\
& \quad + E_A E_B \int_0^{T-T_I} \left[\frac{1}{N} (1 - e^{-Ny}) - e^{-NT_I} y \right] dy \\
& \quad \left. + E_A E_B \int_{T-T_I}^T \left[\frac{1}{N} (1 - e^{-N(T-T_I)}) - e^{-NT_I} (T-T_I) \right] dy \right\} \\
& = P_{5A} \frac{1}{T^2} \left\{ \left[\frac{1}{N} (1 - e^{-NT}) - \frac{E_A}{N} (1 - e^{-N(T-T_I)}) - \frac{E_B}{N} \right. \right. \\
& \quad \left. \left. + E_A e^{-NT_I} (T-T_I) \right] \left[y \right]_0^T - \frac{E_B}{N^2} \left[e^{-Ny} \right]_0^T + E_B e^{-NT} \left[\frac{y^2}{2} \right]_0^T \right. \\
& \quad \left. + E_A E_B \left\langle \frac{1}{N} \int_0^T dy + \frac{1}{N^2} \left[e^{-Ny} \right]_0^{T-T_I} - e^{-NT_I} \left[\frac{y^2}{2} \right]_0^{T-T_I} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \left[\frac{1}{N} e^{-N(T-T_I)} + e^{-NT_I} (T-T_I) \right] \left[y \right]_{T-T_I}^T \Bigg\} \\
= P_{5A} & \left\{ \frac{1}{NT} (1-e^{-NT}) - \frac{E_A}{NT} (1-e^{-N(T-T_I)}) - \frac{E_B}{NT} \right. \\
& + \frac{E_A}{N^2 T^2} e^{-NT_I} \frac{T-T_I}{T} - \frac{E_B}{N^2 T^2} (e^{-NT} - 1) + \frac{E_B}{2} \frac{e^{-NT}}{2} \\
& + \frac{E_A E_B}{T^2} \left\langle \frac{T}{N} + \frac{1}{N^2} (e^{-N(T-T_I)} - 1) - \frac{e^{-NT_I} (T-T_I)^2}{2} \right. \\
& \left. \left. - \left[\frac{1}{N} e^{-N(T-T_I)} + e^{-NT_I} (T-T_I) \right] T_I \right\rangle \right\}.
\end{aligned}$$

The sum of the underlined terms in the above equation is equal to

$$\begin{aligned}
& E_A e^{-NT_I} \frac{T-T_I}{T} \left(1 - E_B \frac{T-T_I}{2T} - E_B \frac{T_I}{T} \right) \\
& = E_A e^{-NT_I} \frac{T-T_I}{T^2} \left[T \left(1 - \frac{E_B}{2} \right) - \frac{E_B T_I}{2} \right].
\end{aligned}$$

Rearranging, (5A.1) =

$$\begin{aligned}
P_{5A} & \left\{ \frac{1}{NT} (1-E_B - E_A + E_A E_B + E_A e^{-N(T-T_I)} - e^{-NT}) \right. \\
& + \frac{E_B}{N^2 T^2} (1+E_A e^{-N(T-T_I)} - E_A - e^{-NT}) \\
& \left. + \frac{E_A}{T^2} \left\langle e^{-NT_I} (T-T_I) \left[T \left(1 - \frac{E_B}{2} \right) - \frac{E_B T_I}{2} \right] \right\rangle \right\}
\end{aligned}$$

$$- \frac{E_B}{N} T_I e^{-N(T-T_I)} \left. \right\} + \frac{E_B}{2} e^{-NT} \left. \right\} .$$

5.5.3 The Probability (5A.5B)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	\int_0^y	$(1-E_A) E_B (1 - e^{-NT})$	(a)	27
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	$(1-E_A) E_B (1 - e^{-NT})$	(b)	
$\int_{T-T_I}^T$	$\int_{T-T_I}^y$	$E_B (1 - e^{-NT})$	(c)	

Let $E_B (1 - e^{-NT}) = k$

(5A.5B) =

$$P_{5A} \frac{k}{T^2} \left\{ \int_0^{T-T_I} \left[\int_0^y (1-E_A) dx \right] dy + \int_{T-T_I}^T \left[\int_0^{T-T_I} dx - E_A \int_0^{T-T_I} dx + \int_{T-T_I}^y dx \right] dy \right\}$$

$$= P_{5A} \frac{k}{T^2} \left\{ \int_0^{T-T_I} \left[(1-E_A) y \right] dy \right.$$

$$\begin{aligned}
& + \int_{T-T_I}^T \left[\int_0^y dx - E_A (T-T_I) \right] dy \} \\
= & P_{5A} \frac{k}{T^2} \left\{ (1-E_A) \frac{1}{2} (T-T_I)^2 \right. \\
& \left. + \int_{T-T_I}^T \left[y - E_A (T-T_I) \right] dy \right\} \\
= & P_{5A} \frac{k}{T^2} \left\{ (1-E_A) \frac{1}{2} (T-T_I)^2 + \frac{1}{2} \left[T^2 - (T-T_I)^2 \right] - E_A (T-T_I) T_I \right\} \\
= & P_{5A} E_B (1-e^{-NT}) \left\{ \frac{1}{2} - \frac{E_A}{T^2} \left[\frac{1}{2} (T-T_I)^2 + (T-T_I) T_I \right] \right\} .
\end{aligned}$$

5.5.4 The Probability (5A.6)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	\int_0^y	$E_A (1-E_B) (1-e^{-NT_I})$	(a)	26
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	"	(b)	

$$\text{Let } \frac{P_{5A}}{T^2} E_A (1-E_B) (1-e^{-NT_I}) = k$$

$$\begin{aligned}
(5A.6) &= k \left\{ \int_0^{T-T_I} \left[\int_0^y dx \right] dy + \int_{T-T_I}^T \left[\int_0^{T-T_I} dx \right] dy \right\} \\
&= k \left\{ \int_0^{T-T_I} y dy + \int_{T-T_I}^T (T-T_I) dy \right\} \\
&= k \left[\frac{1}{2} (T-T_I)^2 + (T-T_I) T_I \right] \\
&= P_{5A} \frac{E_A}{2T^2} (1-E_B) (1-e^{-NT_I}) (T-T_I) (T+T_I).
\end{aligned}$$

5.5.5 The Probability (5A.8)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5A} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	\int_0^y	$E_A E_B (1-e^{-NT})$	(a)	
$\int_{T-T_I}^T$	$\int_0^{T-T_I}$	"	(b)	

The element of probability is the same as that of (5A.6) (Section 5.5.4) except that e^{-NT} replaces e^{-NT_I} , and E_B replaces $(1-E_B)$. The limits of integration are the same. Therefore

$$(5A.8) = P_{5A} \frac{E_A E_B}{2T^2} (1-e^{-NT}) (T-T_I) (T+T_I).$$

5.5.6 The Probability (5A.10)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	NO. OF FIGURE
y	x		
$\int_0^{T-T_I}$	$\int_y^{T-T_I}$	$P_{sA} \frac{\delta y}{T} \frac{\delta x}{T} E_A (1 - e^{-NT_I})$	28

$$\text{Let } P_{sA} \frac{E_A}{T^2} (1 - e^{-NT_I}) = k$$

$$(5A.10) = k \int_0^{T-T_I} \left[\int_y^{T-T_I} dx \right] dy$$

$$= k \int_0^{T-T_I} [T-T_I - y] dy$$

$$= k \left[(T-T_I) y - \frac{1}{2} y^2 \right]_0^{T-T_I}$$

$$= P_{sA} \frac{E_A}{2T^2} (1 - e^{-NT_I}) (T-T_I)^2$$

5.5.7 The Probabilities (5A.s) which are Zero

The following probabilities are zero:

(5A.2), (5A.3), (5A.4), (5A.7), (5A.9), (5A.11) and (5A.12).

5.5.8 The Sum of the Probabilities F_{5As}

It is shown below that the formulae for F_{5As} which have been derived above obey the equation $\sum_S F_{5As} = 1$. The formulae have been partly multiplied out. The sum of terms labelled with the same superscribed number is zero. An asterisked number is the label for a trio of terms. An unasterisked number is the label for a pair of terms of equal magnitude and opposite sign.

(1)

$$F_{5A1} = \frac{1}{NT} (1 - E_B - E_A + E_A E_B + E_A e^{-N(T-T_I)} - e^{-NT})$$

(2)

$$+ E_B \frac{1}{N^2 T^2} (1 + E_A e^{-N(T-T_I)} - E_A - e^{-NT})$$

(*3)

(4)

(5)

(6)

(5)

(7)

$$+ E_A \frac{1}{T^2} e^{-NT_I} (T^2 - \frac{1}{2} E_B T^2 - \frac{1}{2} E_B T T_I - T T_I + \frac{1}{2} E_B T T_I + \frac{1}{2} E_B T_I^2)$$

(8)

(9)

$$- E_A E_B \frac{1}{NT^2} T_I e^{-N(T-T_I)} + \frac{1}{2} E_B e^{-NT}$$

(10)

$$F_{5A5A} = 1 + E_A E_B \frac{1}{T^2} \left[\frac{1}{2} (T-T_I)^2 + (T-T_I) T_I \right]$$

(8)

$$+ E_A E_B \frac{1}{NT^2} T_I e^{-N(T-T_I)}$$

(2)

$$+ E_B \frac{1}{N^2 T^2} (e^{-NT} - E_A e^{-N(T-T_I)} + E_A - 1)$$

$$(1) \\ + \frac{1}{NT} \left[E_A - 1 + E_B (1 - E_A) - E_A e^{-N(T-T_I)} + e^{-NT} \right]$$

$$(11) \quad (*12) \quad (13) \\ - \frac{1}{2} E_B - E_A + E_A \frac{1}{T} T_I$$

$$(11) \quad (9) \quad (10) \\ F_{5A5B} = \frac{1}{2} E_B - \frac{1}{2} E_B e^{-NT} - E_A E_B \frac{1}{T^2} \left[\frac{1}{2} (T - T_I)^2 + (T - T_I) T_I \right]$$

$$(14) \\ + E_A E_B \frac{1}{T^2} e^{-NT} \left(\frac{1}{2} T^2 - \frac{1}{2} T_I^2 \right)$$

$$(*12) \quad (15) \quad (16) \quad (*3) \quad (17) \\ F_{5A6} = E_A \frac{1}{2T^2} \left[(T^2 - T_I^2) - E_B (T^2 - T_I^2) - e^{-NT_I} (T^2 - T_I^2) \right]$$

$$(4) \quad (7) \\ + E_B e^{-NT_I} (T^2 - T_I^2) \left. \right]$$

$$(16) \quad (14) \\ F_{5A8} = E_A E_B \frac{1}{2T^2} (1 - e^{-NT}) (T^2 - T_I^2)$$

$$(*12) \quad (13) \quad (15) \quad (*3) \quad (6) \quad (17) \\ F_{5A10} = E_A \frac{1}{2T^2} \left[(T^2 - 2TT_I + T_I^2) - e^{-NT_I} (T^2 - 2TT_I + T_I^2) \right]$$

5.6 The Probabilities (5B.s)

The derivation of these probabilities can be carried out by using strips of cardboard to represent diagrammatically the set dead times (T), the intrinsic dead time (T_I), and the limits of integration. The strips are arranged in a pattern to represent each case of the probabilities. The positioning of the strips is carried out on a sheet of graph paper. It is then possible to determine the element of probability and the expressions for the limits of integration.

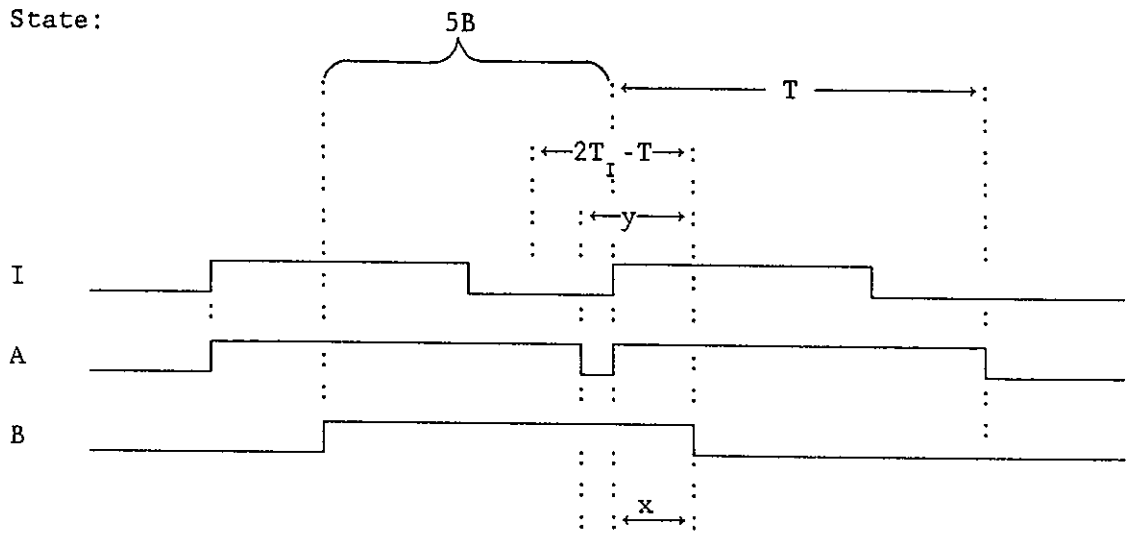


Figure 29 - The probability (5B.1), case (b), and the probability (5B.5A)

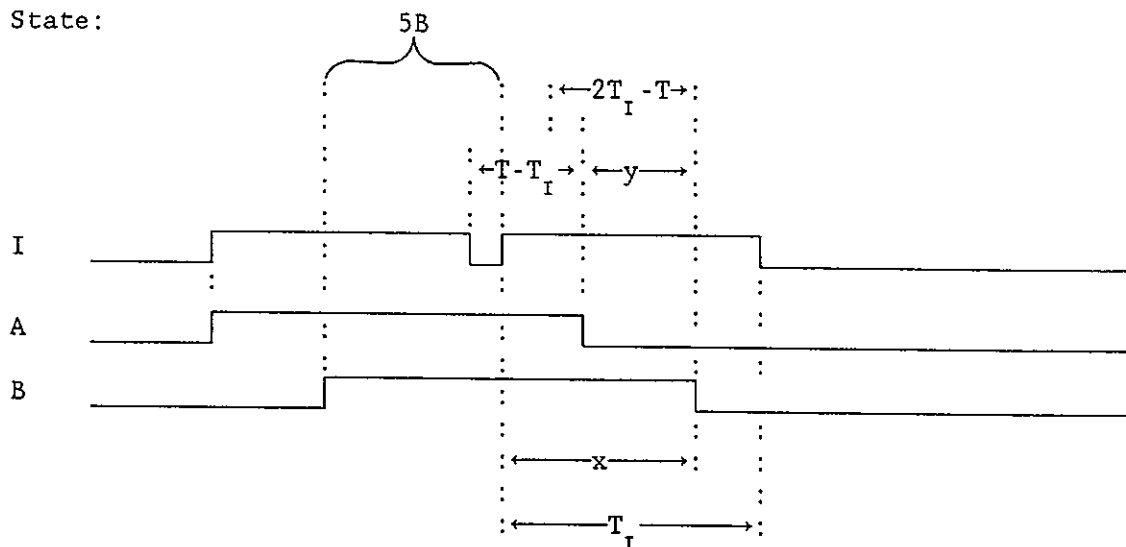


Figure 30 - The probability (5B.1), case (d), and the probability (5B.10), case (a)

5.6.1 The Probability (5B.1)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5B} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{2T_I - T}$	\int_0^y	$(1-E_A) e^{-Nx}$	(a)	29
$\int_0^{2T_I - T}$	\int_0^y	$E_A e^{-NT}$	(b)	
$\int_0^{2T_I - T}$	$\int_y^{y+T-T_I}$	$(1-E_A) e^{-Nx}$	(c)	
$\int_0^{2T_I - T}$	$\int_y^{y+T-T_I}$	$E_A e^{-NT_I}$	(d)	
$\int_0^{2T_I - T}$	$\int_{y+T-T_I}^T$	e^{-Nx}	(e)	
$\int_{2T_I - T}^{T_I}$	\int_0^y	$(1-E_A) e^{-Nx}$	(f)	
$\int_{2T_I - T}^{T_I}$	\int_0^y	$E_A e^{-NT}$	(g)	
$\int_{2T_I - T}^{T_I}$	$\int_y^{T_I}$	$(1-E_A) e^{-Nx}$	(h)	
$\int_{2T_I - T}^{T_I}$	$\int_y^{T_I}$	$E_A e^{-NT_I}$	(i)	

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5B} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_{2T_I - T}^{T_I}$	$\int_{T_I}^{y+T-T_I}$	$(1-E_A) e^{-Nx}$	(j)	
$\int_{2T_I - T}^{T_I}$	$\int_{T_I}^{y+T-T_I}$	$E_A e^{-Nx}$	(k)	
$\int_{2T_I - T}^{T_I}$	$\int_{y+T-T_I}^T$	e^{-Nx}	(l)	
$\int_{T_I}^T$	\int_0^y	$(1-E_A) e^{-Nx}$	(m)	
$\int_{T_I}^T$	\int_0^y	$E_A e^{-Nx}$	(n)	
$\int_{T_I}^T$	\int_y^T	$(1-E_A) e^{-Nx}$	(o)	
$\int_{T_I}^T$	\int_y^T	$E_A e^{-Nx}$	(p)	

The sum of the integrands of cases (j) and (k), and cases (o) and (p), is e^{-Nx} .

$$(5B.1) = P_{5B} \frac{1}{T^2} x$$

$$\left\{ \int_0^{2T_I - T} \left[\int_0^y e^{-Nx} dx - E_A \int_0^y e^{-Nx} dx + E_A e^{-NT} \int_0^y dx + \int_y^{y+T-T_I} e^{-Nx} dx \right. \right. \\ \left. \left. - E_A \int_y^{y+T-T_I} e^{-Nx} dx + E_A e^{-NT_I} \int_y^{y+T-T_I} dx + \int_{y+T-T_I}^T e^{-Nx} dx \right] dy \right. \\ \left. + \int_{2T_I - T}^{T_I} \left[\int_0^y e^{-Nx} dx - E_A \int_0^y e^{-Nx} dx + E_A e^{-NT} \int_0^y dx \right. \right. \\ \left. \left. + \int_y^{T_I} e^{-Nx} dx - E_A \int_y^{T_I} e^{-Nx} dx + E_A e^{-NT_I} \int_y^{T_I} dx \right. \right. \\ \left. \left. + \int_{T_I}^{y+T-T_I} e^{-Nx} dx + \int_{y+T-T_I}^T e^{-Nx} dx \right] dy \right. \\ \left. + \int_{T_I}^T \left[\int_0^y e^{-Nx} dx - E_A \int_0^y e^{-Nx} dx + E_A e^{-NT} \int_0^y dx \right. \right. \\ \left. \left. + \int_y^T e^{-Nx} dx \right] dy \right\}$$

$$= P_{5B} \frac{1}{T^2} x$$

$$\left\{ \int_0^{2T_I - T} \left[\int_0^T e^{-Nx} dx - E_A \int_0^{y+T-T_I} e^{-Nx} dx + E_A e^{-NT} y \right. \right. \\ \left. \left. + E_A e^{-NT_I} (T - T_I) \right] dy \right\}$$

$$\begin{aligned}
& + \int_{2T_I - T}^{T_I} \left[\int_0^T e^{-Nx} dx - E_A \int_0^{T_I} e^{-Nx} dx + E_A e^{-NT} y \right. \\
& \left. + E_A e^{-NT_I} (T_I - y) \right] dy \\
& + \int_{T_I}^T \left\langle \int_0^T e^{-Nx} dx + \frac{E_A}{N} \left[e^{-Nx} \right]_0^y + E_A e^{-NT} y \right\rangle dy \Big\} \\
= P_{5B} \frac{1}{T^2} x
\end{aligned}$$

$$\begin{aligned}
& \left\{ \int_0^T \left[\int_0^T e^{-Nx} dx \right] dy + \frac{E_A}{N} \int_0^{2T_I - T} \left\langle \left[e^{-Nx} \right]_0^{y+T-T_I} \right\rangle dy \right. \\
& + E_A e^{-NT} \int_0^T y dy + E_A e^{-NT_I} (T-T_I) (2T_I - T) \\
& + \frac{E_A}{N} \int_{2T_I - T}^{T_I} \left\langle \left[e^{-Nx} \right]_0^{T_I} \right\rangle dy + E_A e^{-NT_I} \left[T_I y - \frac{y^2}{2} \right]_{2T_I - T}^{T_I} \\
& \left. + \frac{E_A}{N} \int_{T_I}^T (e^{-Ny} - 1) dy \right\}
\end{aligned}$$

$$= P_{5B} \frac{1}{T^2} x$$

$$\begin{aligned}
& \left\{ -\frac{1}{N} \int_0^T \left\langle \left[e^{-Nx} \right]_0^T \right\rangle dy + \frac{E_A}{N} \int_0^{2T_I - T} \left\langle e^{-Ny} e^{-N(T-T_I)} - 1 \right\rangle dy \right. \\
& \left. + E_A e^{-NT} \frac{1}{2} T^2 + E_A e^{-NT_I} (T-T_I) (2T_I - T) + \frac{E_A}{N} \int_{2T_I - T}^{T_I} (e^{-NT_I} - 1) dy \right\}
\end{aligned}$$

$$\begin{aligned}
& + E_A e^{-NT_I} \left\langle T_I (T-T_I) - \frac{1}{2} \left[T_I^2 - (2T_I - T)^2 \right] \right\rangle \\
& + \frac{E_A}{N} \left[-\frac{1}{N} e^{-Ny} - y \right]_{T_I}^T \Big\} \\
= & P_{5B} \frac{1}{T^2} x
\end{aligned}$$

$$\begin{aligned}
& \left\{ -\frac{1}{N} \int_0^T (e^{-Ny} - 1) dy + \frac{E_A}{N} \left[-\frac{1}{N} e^{-N(T-T_I)} e^{-Ny} - y \right]_0^{2T_I-T} \right. \\
& + \frac{1}{2} E_A e^{-NT} T^2 + E_A e^{-NT_I} (2TT_I - T^2 - 2T_I^2 + TT_I \\
& + TT_I - T_I^2 - \frac{1}{2} T_I^2 + 2T_I^2 - 2TT_I + \frac{1}{2} T^2) \\
& \left. + \frac{E_A}{N} (e^{-NT_I} - 1) (T-T_I) + \frac{E_A}{N} \left[-\frac{1}{N} (e^{-NT} - e^{-NT_I}) + T_I - T \right] \right\} \\
= & P_{5B} \frac{1}{T^2} x
\end{aligned}$$

$$\begin{aligned}
& \left\{ -\frac{1}{N} (e^{-NT} - 1) T + \frac{E_A}{N} \left[-\frac{1}{N} e^{-N(T-T_I)} (e^{-N(2T_I-T)} - 1) + T - 2T_I \right] \right. \\
& + \frac{1}{2} E_A e^{-NT} T^2 + E_A e^{-NT_I} (2TT_I - \frac{1}{2} T^2 - \frac{3}{2} T_I^2) \\
& \left. + \frac{E_A}{N} \left[e^{-NT_I} T - e^{-NT_I} T_I - T + T_I - \frac{1}{N} (e^{-NT} - e^{-NT_I}) + T_I - T \right] \right\}
\end{aligned}$$

$$\begin{aligned}
&= P_{5B} \left\{ \frac{1}{NT} (1 - e^{-NT}) + \frac{E_A}{NT^2} \left[-\frac{1}{N} (e^{-NT_I} - e^{-N(T-T_I)}) - T \right. \right. \\
&\quad \left. \left. + e^{-NT_I} T - e^{-NT_I} T_I - \frac{1}{N} (e^{-NT} - e^{-NT_I}) \right] \right. \\
&\quad \left. + \frac{1}{2} E_A e^{-NT} + E_A e^{-NT_I} \left(2 \frac{T_I}{T} - \frac{1}{2} - \frac{3}{2} \frac{T_I^2}{T^2} \right) \right\} \\
&= P_{5B} \left\{ \frac{1}{NT} (1 - e^{-NT} - E_A + E_A e^{-NT_I} - E_A e^{-NT_I} \frac{T_I}{T}) \right. \\
&\quad \left. + \frac{E_A}{N^2 T^2} (e^{-N(T-T_I)} - e^{-NT}) \right. \\
&\quad \left. + \frac{1}{2} E_A e^{-NT} + E_A e^{-NT_I} \left(\frac{2T_I}{T} - \frac{1}{2} - \frac{3T_I^2}{2T^2} \right) \right\}.
\end{aligned}$$

* These 2 terms cancel out.

5.6.2 The Probability (5B.5A)

LIMITS OF INTEGRATION WITH RESPECT TO	ELEMENT OF PROBABILITY	NO. OF FIGURE
y x		
\int_0^T \int_0^y	$P_{5B} \frac{\delta y}{T} \frac{\delta x}{T} E_A (1 - e^{-NT})$	29

$$\begin{aligned}
(5B.5A) &= P_{5B} \frac{1}{T^2} E_A (1 - e^{-NT}) \int_0^T \left[\int_0^y dx \right] dy \\
&= P_{5B} \frac{1}{T^2} E_A (1 - e^{-NT}) \frac{1}{2} \left[y^2 \right]_0^T \\
&= \frac{1}{2} P_{5B} E_A (1 - e^{-NT}) .
\end{aligned}$$

5.6.3 The Probability (5B.5B)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE
y	x	$P_{5B} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by	
$\int_0^{T_I}$	$\int_0^{y+T-T_I}$	$(1-E_A) (1 - e^{-Nx})$	(a)
$\int_0^{T_I}$	$\int_{y+T-T_I}^T$	$(1 - e^{-Nx})$	(b)
$\int_{T_I}^T$	\int_0^T	$(1-E_A) (1 - e^{-Nx})$	(c)

$$(5B.5B) = P_{5B} \frac{1}{T^2} x$$

$$\left\{ \int_0^{T_I} \left[\int_0^{y+T-T_I} (1 - e^{-Nx}) dx - E_A \int_0^{y+T-T_I} (1 - e^{-Nx}) dx \right. \right.$$

$$\begin{aligned}
& + \int_{y+T-T_I}^T (1 - e^{-Nx}) dx \Big] dy \\
& + \int_{T_I}^T \left[\int_0^T (1 - e^{-Nx}) dx - E_A \int_0^T (1 - e^{-Nx}) dx \right] dy \\
= & P_{5B} \frac{1}{T^2} x \\
& \left\{ \int_0^{T_I} \left\langle \int_0^T (1 - e^{-Nx}) dx - E_A \left[x + \frac{1}{N} e^{-Nx} \right]_0^{y+T-T_I} \right\rangle dy \right. \\
& + \int_{T_I}^T \left\langle \int_0^T (1 - e^{-Nx}) dx - E_A \left[x + \frac{1}{N} e^{-Nx} \right]_0^T \right\rangle dy \\
= & P_{5B} \frac{1}{T^2} x \\
& \left\{ \int_0^{T_I} \left\langle \int_0^T (1 - e^{-Nx}) dx \right\rangle dy \right. \\
& - E_A \int_0^{T_I} \left\langle y + T - T_I + \frac{1}{N} (e^{-Ny} e^{-N(T-T_I)} - 1) \right\rangle dy \\
& \left. - E_A \int_{T_I}^T \left\langle T + \frac{1}{N} (e^{-NT} - 1) \right\rangle dy \right\}
\end{aligned}$$

$$= P_{5B} \frac{1}{T^2} x$$

$$\left\{ \int_0^T \left\langle \left[x + \frac{1}{N} e^{-Nx} \right]_0^T \right\rangle dy \right.$$

$$- E_A \int_0^T \left(T - \frac{1}{N} \right) dy$$

$$- E_A \left[\frac{1}{2} y^2 - T_I y - \frac{1}{N^2} e^{-Ny} e^{-N(T-T_I)} \right]_0^{T_I}$$

$$\left. - E_A \left[\frac{1}{N} e^{-NT} y \right]_{T_I}^T \right\}$$

$$= P_{5B} \frac{1}{T^2} x$$

$$\left\{ \int_0^T \left\langle T + \frac{1}{N} (e^{-NT} - 1) \right\rangle dy - E_A \left(T - \frac{1}{N} \right) T \right.$$

$$- E_A \left[\frac{1}{2} T_I^2 - T_I^2 - \frac{1}{N^2} e^{-N(T-T_I)} (e^{-NT_I} - 1) \right.$$

$$\left. \left. + \frac{1}{N} e^{-NT} (T-T_I) \right] \right\}$$

$$= P_{5B} \frac{1}{T^2} x$$

$$\left\{ \left[T + \frac{1}{N} (e^{-NT} - 1) \right] T - E_A \left[T^2 - \frac{T}{N} - \frac{1}{2} T_I^2 \right. \right.$$

$$\begin{aligned}
& \left. - \frac{1}{N^2} (e^{-NT} - e^{-N(T-T_I)}) + \frac{1}{N} e^{-NT} (T-T_I) \right] \} \\
= & P_{5B} \left\{ 1 + \frac{1}{NT} \left[e^{-NT} - 1 + E_A - E_A e^{-NT} \left(1 - \frac{T_I}{T}\right) \right] \right. \\
& \left. - E_A \left(1 - \frac{1}{2} \frac{T_I^2}{T^2}\right) + E_A \frac{1}{N^2 T^2} (e^{-NT} - e^{-N(T-T_I)}) \right\} .
\end{aligned}$$

5.6.4 The Probability (5B.10)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
y	x	$P_{5B} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{2T_I - T}$	$\int_y^{y+T-T_I}$	$E_A (1 - e^{-NT_I})$	(a)	30
$\int_{2T_I - T}^{T_I}$	$\int_y^{T_I}$	$E_A (1 - e^{-NT_I})$	(b)	
$\int_{2T_I - T}^{T_I}$	$\int_{T_I}^{y+T-T_I}$	$E_A (1 - e^{-Nx})$	(c)	
$\int_{T_I}^T$	\int_y^T	$E_A (1 - e^{-Nx})$	(d)	

$$(5B.10) = P_{5B} \frac{1}{T^2} E_A \quad x$$

$$\left\{ (1 - e^{-NT_I}) \int_0^{2T_I - T} (T - T_I) \, dy \right. \\ \left. + \int_{2T_I - T}^{T_I} \left[(1 - e^{-NT_I}) (T_I - y) + \int_{T_I}^{y+T-T_I} (1 - e^{-Nx}) \, dx \right] \, dy \right. \\ \left. + \int_{T_I}^T \left[\int_y^T (1 - e^{-Nx}) \, dx \right] \, dy \right\}$$

$$= P_{5B} \frac{1}{T^2} E_A \quad x$$

$$\left\{ (1 - e^{-NT_I}) (T - T_I) (2T_I - T) + \int_{2T_I - T}^{T_I} \left\langle T_I - y - e^{-NT_I} T_I \right. \right. \\ \left. \left. + e^{-NT_I} y + \left[x + \frac{1}{N} e^{-Nx} \right]_{T_I}^{y+T-T_I} \right\rangle \, dy \right. \\ \left. + \int_{T_I}^T \left\langle \left[x + \frac{1}{N} e^{-Nx} \right]_y^T \right\rangle \, dy \right\}$$

$$= P_{5B} \frac{1}{T^2} E_A \quad x$$

$$\left\{ (1 - e^{-NT_I}) (T - T_I) (2T_I - T) + \int_{2T_I - T}^{T_I} \left\langle T_I - y - e^{-NT_I} T_I \right. \right. \\ \left. \left. + e^{-NT_I} y + y + T - 2T_I + \frac{1}{N} (e^{-Ny} e^{-N(T-T_I)} - e^{-NT_I}) \right\rangle \, dy \right.$$

$$\begin{aligned}
& + \int_{T_I}^T \left\langle T - y + \frac{1}{N} (e^{-Ny} - e^{-NT}) \right\rangle dy \Bigg\} \\
= & P_{5B} \frac{1}{T^2} E_A \quad x \\
& \left\{ (1 - e^{-NT_I}) (T - T_I) (2T_I - T) + \left[(-T_I - e^{-NT_I} T_I + T \right. \right. \\
& \left. \left. - \frac{1}{N} e^{-NT_I} y + \frac{1}{2} e^{-NT_I} y^2 - \frac{1}{N^2} e^{-Ny} e^{-N(T-T_I)} \right]_{T_I}^{T_I} \right. \\
& \left. + \left[(T + \frac{1}{N} e^{-NT}) y - \frac{1}{2} y^2 + \frac{1}{N^2} e^{-Ny} \right]_{T_I}^T \right\} \\
= & P_{5B} \frac{1}{T^2} E_A \quad x \\
& \left\{ (T - T_I) (1 - e^{-NT_I}) (2T_I - T) + (T - T_I - e^{-NT_I} T_I - \frac{1}{N} e^{-NT_I}) (T - T_I) \right. \\
& + \frac{1}{2} e^{-NT_I} (T_I^2 - 4T_I^2 + 4TT_I - T^2) \\
& \left. - \frac{1}{N^2} e^{-N(T-T_I)} (e^{-NT_I} - e^{-N(2T_I-T)}) \right. \\
& \left. + (T + \frac{1}{N} e^{-NT}) (T - T_I) - \frac{1}{2} (T^2 - T_I^2) + \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) \right\}
\end{aligned}$$

$$\begin{aligned}
&= P_{5B} \frac{1}{T^2} E_A \times \\
&\left\{ (T-T_I) \left[2T_I - T - e^{-NT_I} (2T_I - T) + T - T_I - e^{-NT_I} T_I - \frac{1}{N} e^{-NT_I} \right. \right. \\
&\quad \left. \left. + T + \frac{1}{N} e^{-NT} - \frac{1}{2} (T+T_I) \right] \right. \\
&\quad \left. + \frac{1}{2} e^{-NT_I} \left[- (T-T_I) (T+T_I) + 4T_I (T-T_I) \right] \right. \\
&\quad \left. - \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) + \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) \right\} \\
&= P_{5B} E_A \frac{1}{T^2} (T-T_I) \left\{ \frac{1}{2} (T+T_I) + \frac{1}{N} e^{-NT} \right. \\
&\quad \left. + e^{-NT_I} \left[\frac{1}{2} (T-3T_I) - \frac{1}{N} \right] \right\} .
\end{aligned}$$

5.6.5 The Probabilities (5B.s) which are Zero

The following probabilities are zero:

(5B.2), (5B.3), (5B.4), (5B.6), (5B.7), (5B.8), (5B.9), (5B.11) and (5B.12).

5.6.6 The Sum of the Probabilities F_{5Bs}

It is shown below that the formulae for F_{5Bs} which have been derived above obey the equation $\sum_S F_{5Bs} = 1$. The formulae have been partly multiplied out. The superscribed number *9 is the label for 3 terms whose sum is zero. Each of the other superscribed numbers labels a pair of terms whose sum is zero.

(1) (2) (3) (4) (5)

$$F_{5B1} = \frac{1}{NT} (1 - e^{-NT} - E_A + E_A e^{-NT_I} - E_A e^{-NT_I} \frac{T_I}{T})$$

(6)

$$+ \frac{E_A}{N^2 T^2} (e^{-N(T-T_I)} - e^{-NT})$$

(7) (8)

$$+ \frac{1}{2} E_A e^{-NT} + E_A e^{-NT_I} \left(\frac{2T_I}{T} - \frac{1}{2} - \frac{3T_I^2}{2T^2} \right)$$

(*9) (7)

$$F_{5B5A} = \frac{1}{2} E_A (1 - e^{-NT})$$

(2) (1) (3) (10) (11)

$$F_{5B5B} = 1 + \frac{1}{NT} \left[e^{-NT} - 1 + E_A - E_A e^{-NT} + E_A e^{-NT} \frac{T_I}{T} \right]$$

(*9) (12) (6)

$$- E_A \left(1 - \frac{1}{2} \frac{T_I^2}{T^2} \right) + E_A \frac{1}{N^2 T^2} (e^{-NT} - e^{-N(T-T_I)})$$

$$F_{5B10} = E_A \left\{ \frac{1}{2T^2} (T^2 - T_I^2) + \frac{1}{NT} e^{-NT} - \frac{T_I}{NT^2} e^{-NT} \right.$$

$$\left. + \frac{1}{T^2} e^{-NT_I} \left[\frac{1}{2} (T^2 - 3TT_I - TT_I + 3T_I^2) - \frac{T}{N} + \frac{T_I}{N} \right] \right\}$$

$$\begin{aligned}
 & \quad \quad \quad (9) \quad (12) \quad \quad \quad (10) \quad \quad \quad (11) \\
 = & E_A \left\{ \frac{1}{2} - \frac{1}{2} \frac{T_I^2}{T^2} + \frac{1}{NT} e^{-NT} - \frac{T_I}{NT^2} e^{-NT} \right. \\
 & \quad \quad \quad (8) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (4) \quad (5) \\
 & \left. + e^{-NT_I} \left(\frac{1}{2} - \frac{2T_I}{T} + \frac{3T_I^2}{2T^2} \right) + e^{-NT_I} \left(-\frac{1}{NT} + \frac{T_I}{NT^2} \right) \right\}
 \end{aligned}$$

5.7 The Probabilities (6.s)

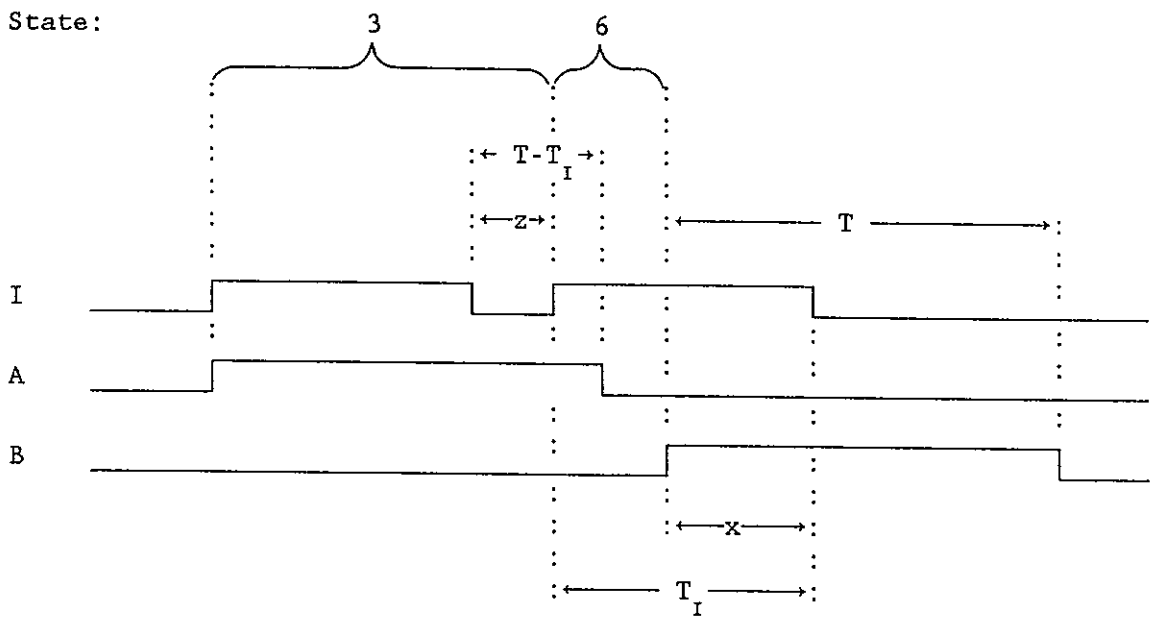


Figure 31 - The probability (6.1), case (b), and the probability (6.9)

5.7.1 The Probability (6.1)

It is assumed that all values of z (Figure 31) between 0 and $T-T_I$ are equally probable and that all values of x between 0 and T_I are equally probable. The probability that a disintegration occurs at a time between x and $x + \delta x$, and that the system changed from state 3 to state 6 at a time between z and $z + \delta z$ is $P_6 \frac{\delta x}{T_I} \frac{\delta z}{T-T_I}$.

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
z	x	$P_6 \frac{\delta z}{T-T_I} \frac{\delta x}{T_I}$ multiplied by		
$\int_0^{T-T_I}$	$\int_0^{T_I}$	$(1 - E_B) e^{-Nx}$	(a)	31
$\int_0^{T-T_I}$	$\int_0^{T_I}$	$E_B e^{-NT}$	(b)	

$$(6.1) = P_6 \frac{1}{T-T_I} \frac{1}{T_I} x$$

$$\int_0^{T-T_I} \left\langle \int_0^{T_I} \left[(1 - E_B) e^{-Nx} + E_B e^{-NT} \right] dx \right\rangle dz$$

$$= P_6 \frac{1}{T-T_I} \frac{1}{T_I} x$$

$$\int_0^{T-T_I} \left\langle \left[-\frac{1}{N} (1 - E_B) e^{-Nx} + E_B e^{-NT} x \right]_0^{T_I} \right\rangle dz$$

$$= P_6 \frac{1}{T-T_I} \frac{1}{T_I} x$$

$$\int_0^{T-T_I} \left\langle -\frac{1}{N} (1 - E_B) (e^{-NT_I} - 1) + E_B e^{-NT} T_I \right\rangle dz$$

$$= P_6 \left\{ \frac{1}{NT_I} (1 - E_B) (1 - e^{-NT_I}) + E_B e^{-NT} \right\} .$$

5.7.2 The Probability (6.6)

$$\begin{aligned}
 (6.6) &= P_6 \frac{1}{T-T_I} \frac{1}{T_I} x \\
 &\int_0^{T-T_I} \left\langle \int_0^{T_I} (1 - E_B) (1 - e^{-Nx}) dx \right\rangle dz \\
 &= P_6 \frac{1}{T-T_I} \frac{1}{T_I} (1 - E_B) \int_0^{T-T_I} \left\langle \left[x + \frac{1}{N} e^{-Nx} \right]_0^{T_I} \right\rangle dz \\
 &= P_6 \frac{1}{T-T_I} \frac{1}{T_I} (1 - E_B) \int_0^{T-T_I} \left\langle T_I + \frac{1}{N} (e^{-NT_I} - 1) \right\rangle dz \\
 &= P_6 (1 - E_B) \left[1 + \frac{1}{NT_I} (e^{-NT_I} - 1) \right].
 \end{aligned}$$

5.7.3 The Probability (6.9)

The first of the pair of consecutive disintegrations occurs when the system is in state 6. The first disintegration is detected by channel B (Figure 31). The second disintegration must occur during the dead time, T, of channel B. Thus,

$$(6.9) = P_6 E_B (1 - e^{-NT}).$$

5.7.4 The Probabilities (6.s) which are Zero

The following probabilities are zero:

(6.2), (6.3), (6.4), (6.5A), (6.5B), (6.7), (6.8), (6.10), (6.11) and (6.12).

5.7.5 The Sum of the Probabilities F_{6s}

It is shown below that $\sum_s F_{6s} = 1$. Each superscribed number labels a pair of terms whose sum is zero.

$$F_{61} = \frac{1}{NT_I} (1 - E_B) (1 - e^{-NT_I}) + E_B e^{-NT} \quad (1) \quad (2)$$

$$F_{66} = 1 - E_B + (1 - E_B) \frac{1}{NT_I} (e^{-NT_I} - 1) \quad (3) \quad (1)$$

$$F_{69} = E_B - E_B e^{-NT} \quad (3) \quad (2)$$

5.8 The Probabilities (7.s)

5.8.1 The Probability (7.1)

It is assumed that all values of y (Figure 32) are equally probable. The maximum and minimum values of y are T_I and $(2T_I - T)$, respectively, and their difference is $(T - T_I)$. Thus the probability that the interval from I_C to I_A is between y and $y + \delta y$ is $\frac{\delta y}{T - T_I}$. The probability that in addition a disintegration occurs at a time between x and $x + \delta x$ is $P_7 \frac{\delta y}{T - T_I} \frac{\delta x}{T_I}$.

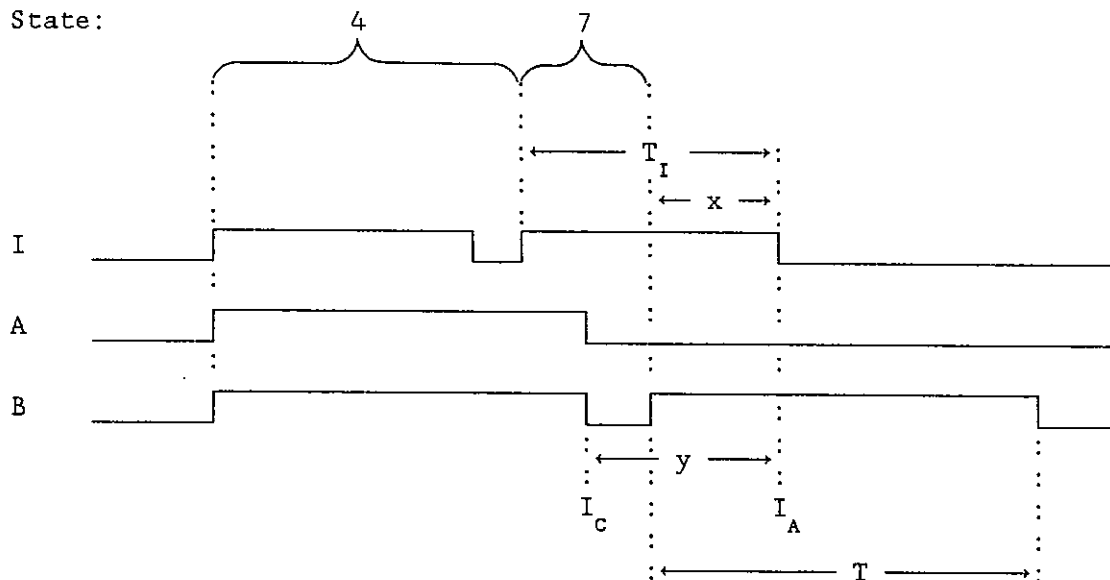


Figure 32 - The probability (7.1), case (b), and the probability (7.11)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY		CASE	NO. OF FIGURE
y	x	$P_7 \frac{\delta y}{T-T_I} \frac{\delta x}{T_I}$ multiplied by			
$\int_{2T_I-T}^{T_I}$	\int_0^y	$(1-E_B) e^{-Nx}$		(a)	32
$\int_{2T_I-T}^{T_I}$	\int_0^y	$E_B e^{-NT}$		(b)	
$\int_{2T_I-T}^{T_I}$	$\int_y^{T_I}$	e^{-Nx}		(c)	

$$(7.1) = P_7 \frac{1}{T-T_I} \frac{1}{T_I} x$$

$$\int_{2T_I-T}^{T_I} \left\langle \int_0^y e^{-Nx} dx - E_B \int_0^y e^{-Nx} dx + E_B e^{-NT} \int_0^y dx + \int_y^{T_I} e^{-Nx} dx \right\rangle dy$$

$$= P_7 \frac{1}{T-T_I} \frac{1}{T_I} x$$

$$\int_{2T_I-T}^{T_I} \left\langle \int_0^{T_I} e^{-Nx} dx + E_B \frac{1}{N} \left[e^{-Nx} \right]_0^y + E_B e^{-NT} y \right\rangle dy$$

$$= P_7 \frac{1}{T-T_I} \frac{1}{T_I} \int_{2T_I-T}^{T_I} \left\langle -\frac{1}{N} (e^{-NT_I} - 1) \right\rangle dy$$

$$\begin{aligned}
& + E_B \frac{1}{N} (e^{-Ny} - 1) + E_B e^{-NT} y \Bigg\rangle dy \\
= P_7 & \frac{1}{(T-T_I) T_I} \left[\frac{1}{N} (1 - e^{-NT_I} - E_B) y - E_B \frac{1}{N^2} e^{-Ny} \right. \\
& \left. + \frac{1}{2} E_B e^{-NT} y^2 \right]_{2T_I-T}^{T_I} \\
= P_7 & \frac{1}{(T-T_I) T_I} \left\{ \frac{1}{N} (1 - e^{-NT_I} - E_B) (T-T_I) \right. \\
& \left. - E_B \frac{1}{N^2} (e^{-NT_I} - e^{-N(2T_I-T)}) + \frac{1}{2} E_B e^{-NT} \left[T_I^2 - (2T_I-T)^2 \right] \right\} \\
= P_7 & \left\{ \frac{1}{NT_I} (1 - e^{-NT_I} - E_B) + \frac{1}{(T-T_I) T_I} E_B \left\langle \frac{1}{2} e^{-NT} \right. \right. \\
& \left. \left. \left[T_I^2 - (2T_I-T)^2 \right] - \frac{1}{N^2} (e^{-NT_I} - e^{-N(2T_I-T)}) \right\rangle \right\} .
\end{aligned}$$

5.8.2 The Probability (7.7)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE
y	x	$P_7 \frac{\delta y}{T-T_I} \frac{\delta x}{T_I}$ multiplied by	
$\int_{2T_I-T}^{T_I}$	\int_0^y	$(1 - E_B) (1 - e^{-Nx})$	(a)
$\int_{2T_I-T}^{T_I}$	$\int_y^{T_I}$	$(1 - e^{-Nx})$	(b)

$$\begin{aligned}
(7.7) &= P_7 \frac{1}{(T-T_I) T_I} \int_{2T_I-T}^{T_I} \left\langle \int_0^y (1 - e^{-Nx}) dx \right. \\
&\quad \left. - E_B \int_0^y (1 - e^{-Nx}) dx + \int_y^{T_I} (1 - e^{-Nx}) dx \right\rangle dy \\
&= P_7 \frac{1}{(T-T_I) T_I} \int_{2T_I-T}^{T_I} \left\langle \int_0^{T_I} (1 - e^{-Nx}) dx \right. \\
&\quad \left. - E_B \left[x + \frac{1}{N} e^{-Nx} \right]_0^y \right\rangle dy \\
&= P_7 \frac{1}{(T-T_I) T_I} \int_{2T_I-T}^{T_I} \left\langle T_I + \frac{1}{N} (e^{-NT_I} - 1) \right. \\
&\quad \left. - E_B \left[y + \frac{1}{N} (e^{-Ny} - 1) \right] \right\rangle dy \\
&= P_7 \frac{1}{(T-T_I) T_I} \left[\left\langle T_I + \frac{1}{N} (e^{-NT_I} - 1 + E_B) \right\rangle y \right. \\
&\quad \left. - E_B \left\langle \frac{1}{2} y^2 - \frac{1}{N^2} e^{-Ny} \right\rangle \right]_{2T_I-T}^{T_I} \\
&= P_7 \frac{1}{(T-T_I) T_I} \left[\left\langle T_I + \frac{1}{N} (e^{-NT_I} - 1 + E_B) \right\rangle (T-T_I) \right. \\
&\quad \left. - E_B \left\langle \frac{1}{2} \left\{ T_I^2 - (2T_I - T)^2 \right\} - \frac{1}{N^2} (e^{-NT_I} - e^{-N(2T_I-T)}) \right\rangle \right] \\
&= P_7 \left[1 + \frac{1}{NT_I} (e^{-NT_I} - 1 + E_B) + \frac{1}{(T-T_I) T_I} E_B \right. \\
&\quad \left. \left\langle \frac{1}{N^2} (e^{-NT_I} - e^{-N(2T_I-T)}) - \frac{1}{2} \left\{ T_I^2 - (2T_I-T)^2 \right\} \right\rangle \right].
\end{aligned}$$

5.8.3 The Probability (7.11)

The first of the pair of consecutive disintegrations occurs at an instant when x is less than y (Figure 32). The first disintegration is detected by channel B. The second disintegration must occur during the dead time, T , of channel B. Thus,

$$\begin{aligned}
 (7.11) &= P_7 \frac{1}{(T-T_I) T_I} \int_{2T_I-T}^{T_I} \left[\int_0^y E_B (1 - e^{-NT}) dx \right] dy \\
 &= P_7 \frac{1}{(T-T_I) T_I} E_B (1 - e^{-NT}) \int_{2T_I-T}^{T_I} y dy \\
 &= P_7 \frac{1}{2 (T-T_I) T_I} E_B (1 - e^{-NT}) \left[T_I^2 - (2T_I-T)^2 \right].
 \end{aligned}$$

5.8.4 The Probabilities (7.s) which are Zero

The following probabilities are zero:

(7.2), (7.3), (7.4), (7.5A), (7.5B), (7.6), (7.8), (7.9), (7.10) and (7.12).

5.8.5 The Sum of the Probabilities F_{7s}

It is shown below that $\sum_s F_{7s} = 1$. Each superscribed number labels a pair of terms whose sum is zero.

(1)

$$F_{71} = \frac{1}{NT_I} (1 - e^{-NT_I} - E_B)$$

(2)

$$+ \frac{1}{(T-T_I) T_I} E_B \left\{ \frac{1}{2} e^{-NT} \left[T_I^2 - (2T_I-T)^2 \right] \right.$$

(3)

$$\left. - \frac{1}{N^2} (e^{-NT_I} - e^{-N(2T_I-T)}) \right\}$$

(1)

$$F_{77} = 1 + \frac{1}{NT_I} (e^{-NT_I} - 1 + E_B)$$

(3)

$$+ \frac{1}{(T-T_I) T_I} E_B \left\{ \frac{1}{N^2} (e^{-NT_I} - e^{-N(2T_I-T)}) \right.$$

(4)

$$\left. - \frac{1}{2} \left[T_I^2 - (2T_I-T)^2 \right] \right\}$$

(4) (2)

$$F_{711} = \frac{1}{2 (T-T_I) T_I} E_B (1 - e^{-NT}) \left[T_I^2 - (2T_I-T)^2 \right]$$

5.9 The Probabilities (8.s)

5.9.1 The Probability (8.1)

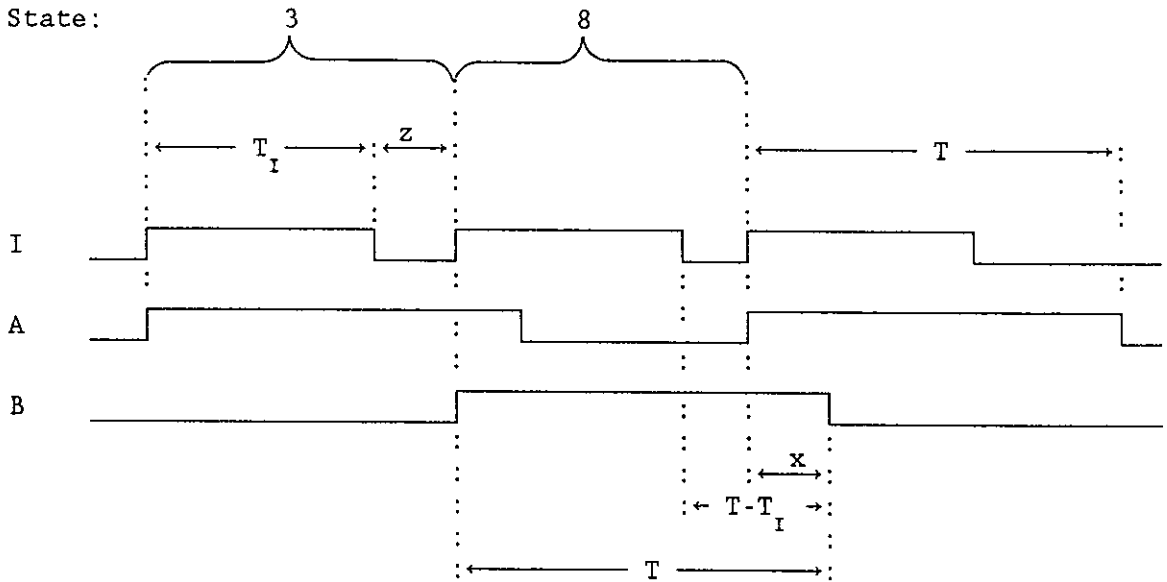


Figure 33 - The probability (8.1), case (b), and the probability (8.12)

It is assumed that all values of z (Figure 33) between 0 and $(T-T_I)$ are equally probable and that all values of x between 0 and T are equally probable.

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY		CASE	NO. OF FIGURE
z	x	$P_8 \frac{\delta z}{T-T_I}$	$\frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	$(1 - E_A)$	e^{-Nx}	(a)	33
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	E_A	e^{-NT}	(b)	
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$		e^{-Nx}	(c)	

$$\begin{aligned}
 (8.1) &= P_8 \frac{1}{(T-T_I) T} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} e^{-Nx} dx - E_A \int_0^{T-T_I} e^{-Nx} dx \right. \\
 &\quad \left. + E_A e^{-NT} \int_0^{T-T_I} dx + \int_{T-T_I}^T e^{-Nx} dx \right\} dz \\
 &= P_8 \frac{1}{(T-T_I) T} \int_0^{T-T_I} \left\{ \int_0^T e^{-Nx} dx + E_A \frac{1}{N} (e^{-N(T-T_I)} - 1) \right. \\
 &\quad \left. + E_A e^{-NT} (T-T_I) \right\} dz
 \end{aligned}$$

$$\begin{aligned}
&= P_8 \frac{1}{(T-T_I) T} \int_0^{T-T_I} \left\{ -\frac{1}{N} (e^{-NT} - 1) + E_A \frac{1}{N} (e^{-N(T-T_I)} - 1) \right. \\
&\quad \left. + E_A e^{-NT} (T-T_I) \right\} dz \\
&= P_8 \left\{ \frac{1}{NT} \left[1 - e^{-NT} + E_A (e^{-N(T-T_I)} - 1) \right] + E_A \frac{1}{T} e^{-NT} (T-T_I) \right\}.
\end{aligned}$$

5.9.2 The Probability (8.8)

LIMITS OF INTEGRATION WITH RESPECT TO		ELEMENT OF PROBABILITY	CASE
z	x	$P_8 \frac{\delta z}{T-T_I} \frac{\delta x}{T}$ multiplied by	
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	$(1 - E_A) (1 - e^{-Nx})$	(a)
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$(1 - e^{-Nx})$	(b)

$$\begin{aligned}
(8.8) &= P_8 \frac{1}{(T-T_I) T} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} (1 - e^{-Nx}) dx - E_A \int_0^{T-T_I} (1 - e^{-Nx}) dx \right. \\
&\quad \left. + \int_{T-T_I}^T (1 - e^{-Nx}) dx \right\} dz
\end{aligned}$$

$$\begin{aligned}
&= P_8 \frac{1}{(T-T_I) T} \int_0^{T-T_I} \left\{ \int_0^T (1 - e^{-Nx}) dx - E_A \int_0^{T-T_I} (1 - e^{-Nx}) dx \right\} dz \\
&= P_8 \frac{1}{(T-T_I) T} \int_0^{T-T_I} \left\{ \left[x + \frac{1}{N} e^{-Nx} \right]_0^T - E_A \left[x + \frac{1}{N} e^{-Nx} \right]_0^{T-T_I} \right\} dz \\
&= P_8 \frac{1}{(T-T_I) T} \int_0^{T-T_I} \left\{ T + \frac{1}{N} (e^{-NT} - 1) - E_A \left[T-T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1) \right] \right\} dz \\
&= P_8 \left\{ 1 - E_A \frac{1}{T} (T-T_I) + \frac{1}{NT} \left[e^{-NT} - 1 + E_A (1 - e^{-N(T-T_I)}) \right] \right\}.
\end{aligned}$$

5.9.3 The Probability (8.12)

The first of the pair of consecutive disintegrations occurs when the system is in state 8, at an instant when x is less than $(T-T_I)$ (Figure 33). The probability of this occurring is $P_8 \frac{(T-T_I)}{T}$.

The first disintegration is detected by channel A and the second disintegration occurs during the dead time, T , of channel A. Thus,

$$(8.12) = P_8 \frac{1}{T} (T-T_I) E_A (1 - e^{-NT}).$$

5.9.4 The Probabilities (8.s) which are Zero

The following probabilities are zero:

(8.2), (8.3), (8.4), (8.5A), (8.5B), (8.6), (8.7), (8.9), (8.10) and (8.11).

5.9.5 The Sum of the Probabilities F_{8s}

It is shown below that $\sum_s F_{8s} = 1$. Each superscribed number labels a pair of terms whose sum is zero.

$$F_{81} = \frac{1}{NT} \left[1 - e^{-NT} + E_A (e^{-N(T-T_I)} - 1) \right] + E_A \frac{1}{T} e^{-NT} (T-T_I) \quad (1) \quad (2)$$

$$F_{88} = 1 - E_A \frac{1}{T} (T-T_I) + \frac{1}{NT} \left[e^{-NT} - 1 + E_A (1 - e^{-N(T-T_I)}) \right] \quad (3) \quad (1)$$

$$F_{812} = \frac{1}{T} (T-T_I) E_A (1 - e^{-NT}) \quad (3) \quad (2)$$

5.10 The Probabilities (9.s)

5.10.1 The Probability (9.1)

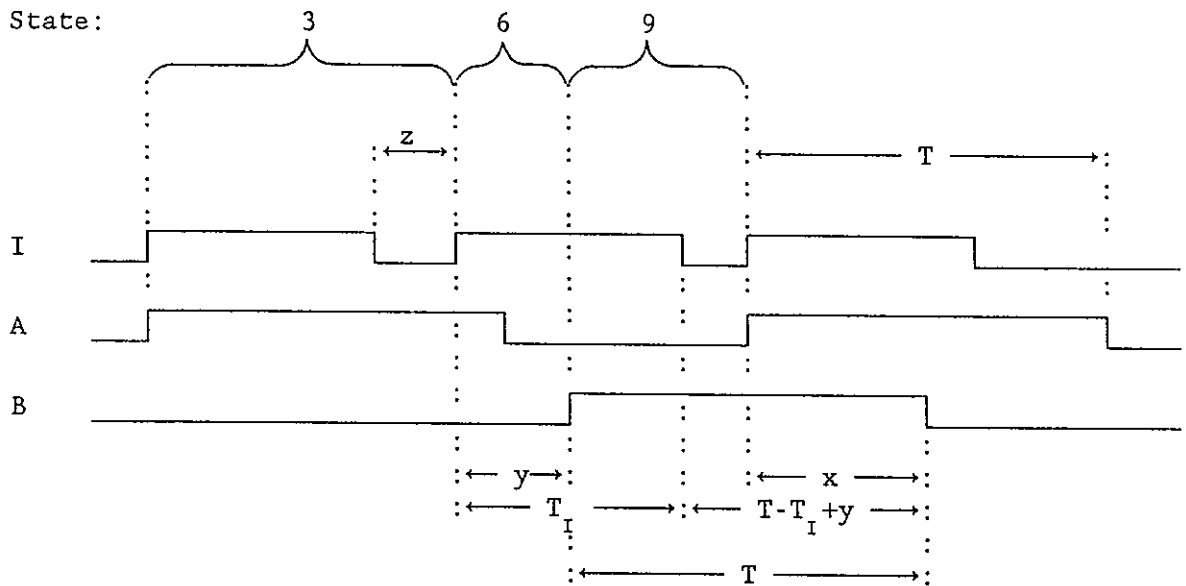


Figure 34 - The probability (9.1), case (b), and the probability (9.5A)

It is assumed that (a) all values of z (Figure 34) between 0 and $(T-T_I)$ are equally probable, (b) all values of y between 0 and T_I are equally probable, and (c) all values of x between 0 and T are equally probable.

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY			CASE	NO. OF FIGURE
			P_9	$\frac{\delta z}{T-T_I}$	$\frac{\delta y}{T_I}$		
z	y	x					
$\int_0^{T-T_I}$	$\int_0^{T_I}$	$\int_0^{T-T_I+y}$		$(1 - E_A)$	e^{-Nx}	(a)	
$\int_0^{T-T_I}$	$\int_0^{T_I}$	$\int_0^{T-T_I+y}$		E_A	e^{-NT}	(b)	34
$\int_0^{T-T_I}$	$\int_0^{T_I}$	$\int_{T-T_I+y}^T$			e^{-Nx}	(c)	

$$\begin{aligned}
(9.1) &= P_9 \frac{1}{T-T_I} \frac{1}{T_I} \frac{1}{T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle \int_0^{T-T_I+y} e^{-Nx} dx - E_A \int_0^{T-T_I+y} e^{-Nx} dx \right. \right. \\
&\quad \left. \left. + E_A e^{-NT} \int_0^{T-T_I+y} dx + \int_{T-T_I+y}^T e^{-Nx} dx \right\rangle dy \right\} dz \\
&= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle \int_0^T e^{-Nx} dx + E_A \frac{1}{N} (e^{-N(T-T_I+y)} - 1) \right. \right. \\
&\quad \left. \left. + E_A e^{-NT} (T-T_I+y) \right\rangle dy \right\} dz \\
&= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle -\frac{1}{N} (e^{-NT} - 1) + E_A \frac{1}{N} (e^{-N(T-T_I)} e^{-Ny} - 1) \right. \right. \\
&\quad \left. \left. + E_A e^{-NT} (T-T_I) + E_A e^{-NT} y \right\rangle dy \right\} dz
\end{aligned}$$

$$\begin{aligned}
&= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \left[\frac{1}{N} (1 - e^{-NT} - E_A) + E_A e^{-NT} (T-T_I) \right] T_I \right. \\
&\quad \left. - E_A \frac{1}{N^2} e^{-N(T-T_I)} (e^{-NT_I} - 1) + \frac{1}{2} E_A e^{-NT} T_I^2 \right\} dz \\
&= P_9 \left\{ \frac{1}{NT} (1 - e^{-NT} - E_A) + E_A e^{-NT} \left(1 - \frac{T_I}{T}\right) \right. \\
&\quad \left. - E_A \frac{1}{N^2} \frac{1}{T_I T} e^{-N(T-T_I)} (e^{-NT_I} - 1) + \frac{1}{2} E_A e^{-NT} \frac{T_I}{T} \right\} \\
&= P_9 \left\{ \frac{1}{NT} (1 - e^{-NT} - E_A) + E_A e^{-NT} \left(1 - \frac{T_I}{2T}\right) \right. \\
&\quad \left. - E_A \frac{1}{N^2} \frac{1}{T_I T} e^{-N(T-T_I)} (e^{-NT_I} - 1) \right\}.
\end{aligned}$$

5.10.2 The Probability (9.5A)

The first of the pair of consecutive disintegrations occurs at an instant when x is less than $(T-T_I+y)$ (Figure 34). The first disintegration is detected by channel A and the second disintegration occurs during the dead time, T , of channel A. Thus,

$$\begin{aligned}
(9.5A) &= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle \int_0^{T-T_I+y} E_A (1 - e^{-NT}) dx \right\rangle dy \right\} dz \\
&= P_9 \frac{1}{(T-T_I) T_I T} E_A (1 - e^{-NT}) \int_0^{T-T_I} \left\{ (T-T_I) T_I + \frac{1}{2} T_I^2 \right\} dz \\
&= P_9 E_A (1 - e^{-NT}) \frac{1}{T} (T-T_I + \frac{1}{2} T_I) \\
&= P_9 E_A (1 - e^{-NT}) \left(1 - \frac{T_I}{2T}\right).
\end{aligned}$$

5.10.3 The Probability (9.9)

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY		CASE
			$\frac{\delta z}{T-T_I}$	$\frac{\delta y}{T_I}$	
z	y	x			
$\int_0^{T-T_I}$	$\int_0^{T_I}$	$\int_0^{T-T_I+y}$	$(1 - E_A)$	$(1 - e^{-Nx})$	(a)
$\int_0^{T-T_I}$	$\int_0^{T_I}$	$\int_{T-T_I+y}^T$	$(1 - e^{-Nx})$		(b)

$$\begin{aligned}
 (9.9) &= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle \int_0^{T-T_I+y} (1 - e^{-Nx}) dx \right. \right. \\
 &\quad \left. \left. - E_A \int_0^{T-T_I+y} (1 - e^{-Nx}) dx + \int_{T-T_I+y}^T (1 - e^{-Nx}) dx \right\rangle dy \right\} dz \\
 &= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle \int_0^T (1 - e^{-Nx}) dx \right. \right. \\
 &\quad \left. \left. - E_A \left[x + \frac{1}{N} e^{-Nx} \right]_0^{T-T_I+y} \right\rangle dy \right\} dz \\
 &= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle \left[x + \frac{1}{N} e^{-Nx} \right]_0^T \right. \right. \\
 &\quad \left. \left. - E_A \left[T-T_I+y + \frac{1}{N} (e^{-N(T-T_I+y)} - 1) \right] \right\rangle dy \right\} dz
 \end{aligned}$$

$$\begin{aligned}
&= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{T_I} \left\langle T + \frac{1}{N} (e^{-NT} - 1) \right. \right. \\
&\quad \left. \left. - E_A \left[T - T_I + y + \frac{1}{N} e^{-N(T-T_I)} e^{-Ny} - \frac{1}{N} \right] \right\rangle dy \right\} dz \\
&= P_9 \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A \left(T - T_I - \frac{1}{N} \right) \right] T_I \right. \\
&\quad \left. - E_A \left[\frac{1}{2} T_I^2 - \frac{1}{N^2} e^{-N(T-T_I)} (e^{-NT_I} - 1) \right] \right\} dz \\
&= P_9 \left\{ 1 + \frac{1}{NT} (e^{-NT} - 1 + E_A) - E_A \left(1 - \frac{T_I}{T} \right) \right. \\
&\quad \left. - E_A \left[\frac{T_I}{2T} - \frac{1}{N^2 T_I T} e^{-N(T-T_I)} (e^{-NT_I} - 1) \right] \right\} \\
&= P_9 \left\{ 1 + \frac{1}{NT} (e^{-NT} - 1 + E_A) \right. \\
&\quad \left. - E_A \left[1 - \frac{T_I}{2T} - \frac{1}{N^2 T_I T} e^{-N(T-T_I)} (e^{-NT_I} - 1) \right] \right\} .
\end{aligned}$$

5.10.4 The Probabilities (9.s) which are Zero

The following probabilities are zero:

(9.2), (9.3), (9.4), (9.5B), (9.6), (9.7), (9.8), (9.10), (9.11) and (9.12).

5.10.5 The Sum of the Probabilities F_{g_s}

It is shown below that $\sum_S F_{g_s} = 1$. The terms are grouped in pairs, each pair being identified by a superscribed number. The sum of each pair is zero.

$$F_{91} = \frac{1}{NT} (1 - e^{-NT} - E_A) + E_A e^{-NT} \left(1 - \frac{T_I}{2T}\right) \quad (1) \quad (2)$$

$$- E_A \frac{1}{N^2 T_I T} e^{-N(T-T_I)} (e^{-NT_I} - 1) \quad (3)$$

$$F_{95A} = E_A (1 - e^{-NT}) \left(1 - \frac{T_I}{2T}\right)$$

$$= E_A - E_A \frac{T_I}{2T} - E_A e^{-NT} \left(1 - \frac{T_I}{2T}\right) \quad (4) \quad (5) \quad (2)$$

$$F_{99} = 1 + \frac{1}{NT} (e^{-NT} - 1 + E_A) \quad (1)$$

$$- E_A \left[1 - \frac{T_I}{2T} - \frac{1}{N^2 T_I T} e^{-N(T-T_I)} (e^{-NT_I} - 1) \right] \quad (4) \quad (5) \quad (3)$$

5.11 The Probabilities (10.s)

As shown in Figures 9, 10 and 11, the system changes to state 10 when the paralysis units of both channels are dead and a disintegration is detected by the detector of channel A. The system remains in state 10 until (a) both channels are again live, as shown in Figures 9, 10 and 11, or (b) a disintegration is detected by one of the channels, as discussed in Sections 5.11.3 and 5.11.4.

The derivation of the formulae for the probabilities (10.s) makes use of the probability, $P_{\bar{X}}$, that a disintegration will occur when the system is in the extended state \bar{X} which is illustrated in Figures 35 and 36.

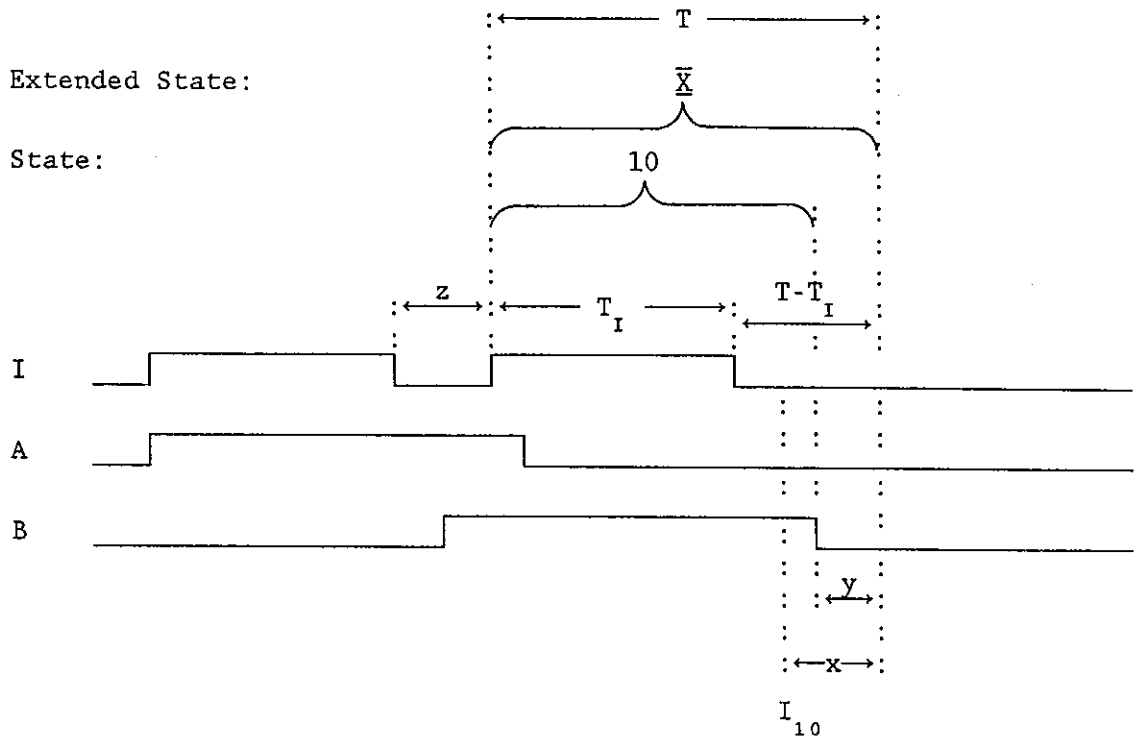


Figure 35 - The probability P_{10} , case (a), the probability (10.1), case (a), and the probability (10.10), case (a)

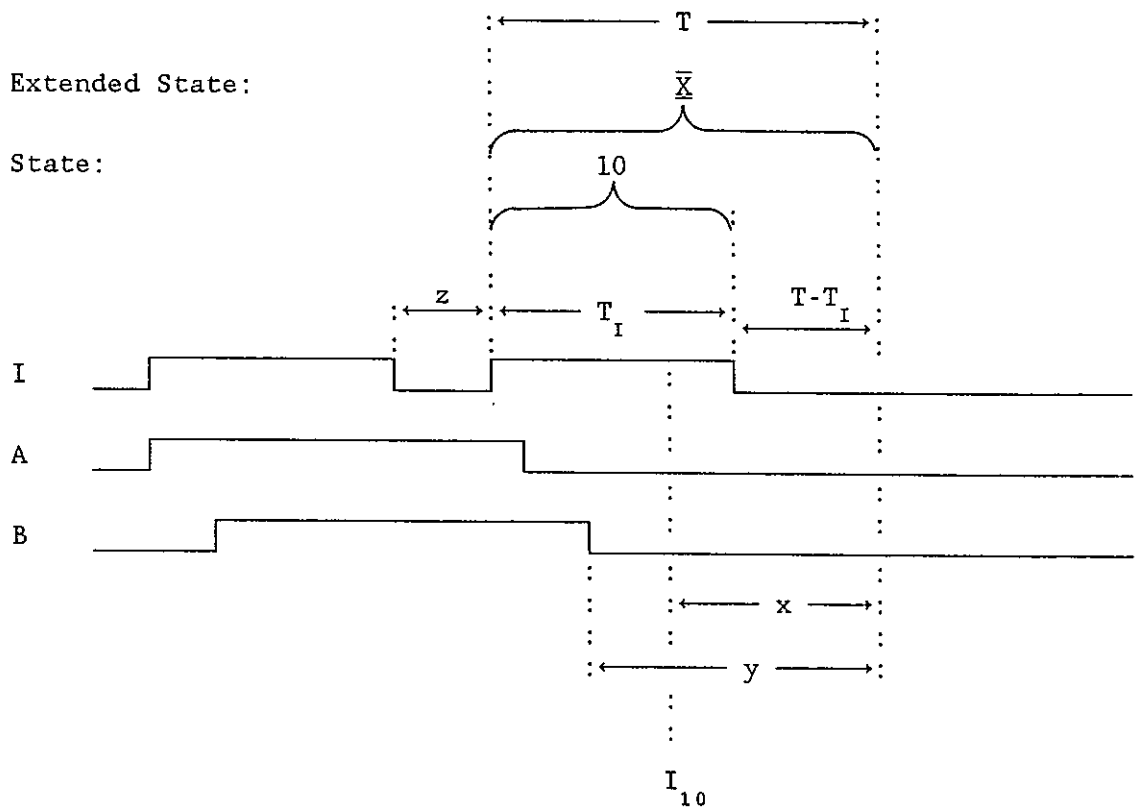


Figure 36 - The probability P_{10} , case (b), the probability (10.1), case (d), and the probability (10.10), case (c)

5.11.1 The Relationship between $P_{\bar{X}}$ and P_{10}

It is assumed that (a) all values of z between 0 and $(T-T_I)$ are equally probable, (b) all values of y between 0 and T are equally probable, and (c) all values of x between 0 and T are equally probable.

The element of the probability $P_{\bar{X}}$ is $P_{\bar{X}} \frac{\delta z}{T-T_I} \frac{\delta y}{T} \frac{\delta x}{T}$; i.e.,

$$\int_0^{T-T_I} \left\{ \int_0^T \left\langle \int_0^T P_{\bar{X}} \frac{1}{(T-T_I) T^2} dx \right\rangle dy \right\} dz = P_{\bar{X}} .$$

The element of P_{10} is also $P_{\bar{X}} \frac{\delta z}{T-T_I} \frac{\delta y}{T} \frac{\delta x}{T}$.

A formula for P_{10} is obtained by summing the elements whose increments δx are within the limits of x for state 10. The limits are given in the following table.

LIMITS OF INTEGRATION WITH RESPECT TO			CASE	NO. OF FIGURE
z	y	x		
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	\int_y^T	(a)	35
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$\int_{T-T_I}^T$	(b)	36

$$\begin{aligned}
P_{10} &= P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} \left\langle \int_y^T dx \right\rangle dy \right. \\
&\quad \left. + \int_{T-T_I}^T \left\langle \int_{T-T_I}^T dx \right\rangle dy \right\} dz \\
&= P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} (T-y) dy + \int_{T-T_I}^T T_I dy \right\} dz \\
&= P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ T (T-T_I) - \frac{1}{2} (T-T_I)^2 + T_I^2 \right\} dz \\
&= P_{\bar{X}} \frac{1}{T^2} \left\{ T^2 - T T_I - \frac{1}{2} T^2 + T T_I - \frac{1}{2} T_I^2 + T_I^2 \right\} dz \\
&= P_{\bar{X}} \frac{T^2 + T_I^2}{2T^2} .
\end{aligned}$$

Therefore $P_{\bar{X}} = P_{10} \frac{2T^2}{T^2 + T_I^2}$.

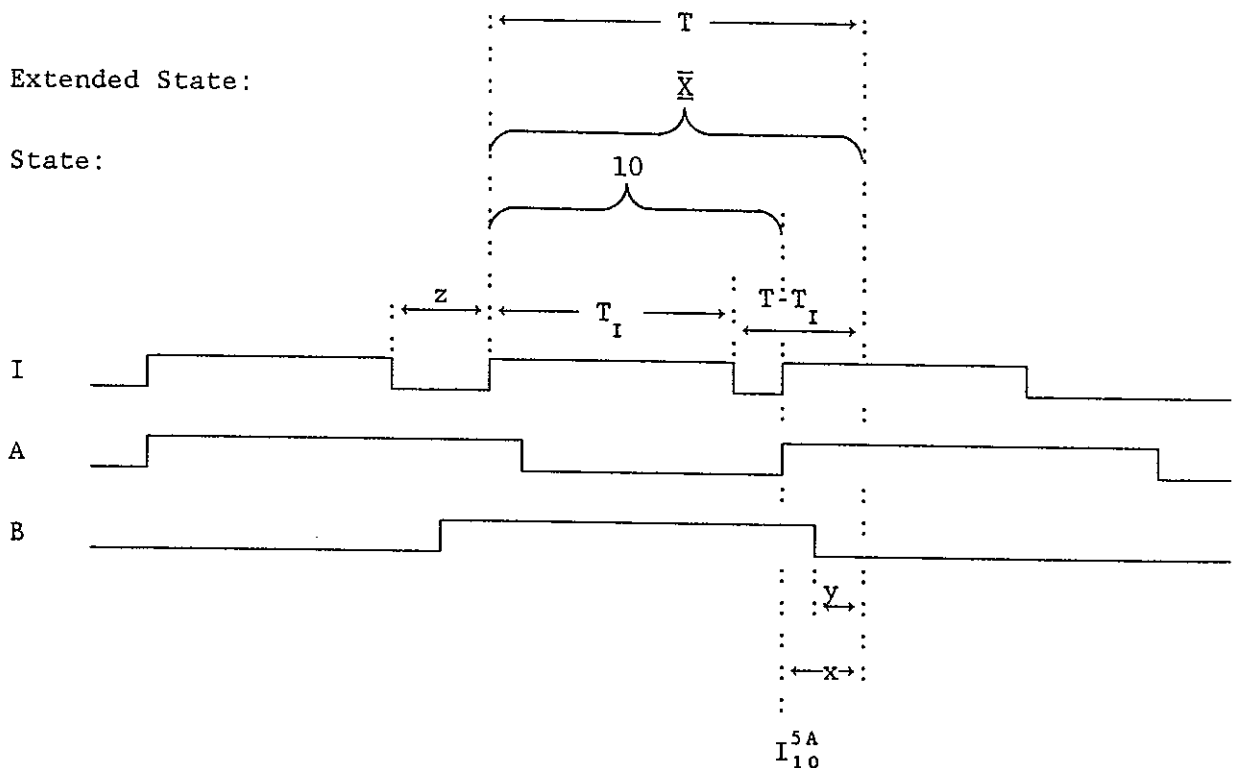


Figure 37 - The probability (10.1), case (b), and the probability (10.5A)



Figure 38 - The probability (10.1), case (e), and the probability (10.9)

5.11.2 The Probability (10.1)

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
z	y	x	$P_{\bar{X}} \frac{\delta z}{T-T_I} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	$\int_y^{T-T_I}$	$(1 - E_A) e^{-N(x-y)}$	(a)	35
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	$\int_y^{T-T_I}$	$E_A e^{-NT}$	(b)	37
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$e^{-N(x-y)}$	(c)	
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$\int_{T-T_I}^y$	$(1 - E_B) e^{-N(x-T+T_I)}$	(d)	36
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$\int_{T-T_I}^y$	$E_B e^{-NT}$	(e)	38
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	\int_y^T	$e^{-N(x-T+T_I)}$	(f)	

$$\begin{aligned}
 (10.1) = P_{\bar{X}} \frac{1}{(T-T_I) T^2} & \int_0^{T-T_I} \left\{ \int_0^{T-T_I} \left\langle \int_y^{T-T_I} e^{-N(x-y)} dx \right. \right. \\
 & - E_A \int_y^{T-T_I} e^{-N(x-y)} dx + E_A e^{-NT} \int_y^{T-T_I} dx
 \end{aligned}$$

$$\begin{aligned}
& + \int_{T-T_I}^T e^{-N(x-y)} dx \rangle dy + \int_{T-T_I}^T \left\langle \int_{T-T_I}^y e^{-N(x-T+T_I)} dx \right. \\
& - E_B \int_{T-T_I}^y e^{-N(x-T+T_I)} dx + E_B e^{-NT} \int_{T-T_I}^y dx \\
& \left. + \int_y^T e^{-N(x-T+T_I)} dx \right\rangle dy \} dz \\
= & P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} \left\langle \int_y^T e^{-Nx} e^{Ny} dx \right. \right. \\
& - E_A \int_y^{T-T_I} e^{-Nx} e^{Ny} dx + E_A e^{-NT} (T-T_I-y) \rangle dy \\
& + \int_{T-T_I}^T \left\langle \int_{T-T_I}^T e^{-Nx} e^{N(T-T_I)} dx - E_B \int_{T-T_I}^y e^{-Nx} e^{N(T-T_I)} dx \right. \\
& \left. \left. + E_B e^{-NT} (y-T+T_I) \right\rangle dy \right\} dz \\
= & P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} \left\langle -\frac{1}{N} (e^{-NT} - e^{-Ny}) e^{Ny} \right. \right. \\
& + E_A \frac{1}{N} (e^{-N(T-T_I)} - e^{-Ny}) e^{Ny} + E_A e^{-NT} (T-T_I-y) \rangle dy \\
& + \int_{T-T_I}^T \left\langle -\frac{1}{N} (e^{-NT} - e^{-N(T-T_I)}) e^{N(T-T_I)} \right. \\
& \left. \left. + E_B \frac{1}{N} (e^{-Ny} - e^{-N(T-T_I)}) e^{N(T-T_I)} + E_B e^{-NT} (y-T+T_I) \right\rangle dy \right\} dz
\end{aligned}$$

$$\begin{aligned}
&= P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ -\frac{1}{N^2} e^{-NT} (e^{N(T-T_I)} - 1) + \frac{1}{N} (T-T_I) \right. \\
&\quad + E_A \frac{1}{N^2} e^{-N(T-T_I)} (e^{N(T-T_I)} - 1) - E_A \frac{1}{N} (T-T_I) + E_A e^{-NT} (T-T_I)^2 \\
&\quad - \frac{1}{2} E_A e^{-NT} (T-T_I)^2 - \frac{1}{N} (e^{-NT_I} - 1) T_I \\
&\quad - E_B \frac{1}{N^2} (e^{-NT} - e^{-N(T-T_I)}) e^{N(T-T_I)} - E_B \frac{1}{N} T_I \\
&\quad \left. + \frac{1}{2} E_B e^{-NT} (2TT_I - T_I^2) - E_B e^{-NT} (T-T_I) T_I \right\} dz
\end{aligned}$$

(* These 2 terms cancel out.)

$$\begin{aligned}
&= P_{\bar{X}} \frac{1}{T^2} \left\{ \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) + \frac{1}{N} T + E_A \frac{1}{N^2} (1 - e^{-N(T-T_I)}) \right. \\
&\quad - E_A \frac{1}{N} (T-T_I) + \frac{1}{2} E_A e^{-NT} (T-T_I)^2 - \frac{1}{N} e^{-NT_I} T_I \\
&\quad - E_B \frac{1}{N^2} (e^{-NT_I} - 1) - E_B \frac{1}{N} T_I \\
&\quad \left. + E_B e^{-NT} (TT_I - \frac{1}{2} T_I^2 - TT_I + T_I^2) \right\} \\
&= P_{10} \frac{2}{T^2+T_I^2} \left\{ \frac{1}{N} \left[T - E_A (T-T_I) - e^{-NT_I} T_I - E_B T_I \right] \right. \\
&\quad + \frac{1}{N^2} \left[e^{-NT} - e^{-NT_I} + E_A (1 - e^{-N(T-T_I)}) + E_B (1 - e^{-NT_I}) \right] \\
&\quad \left. + \frac{1}{2} e^{-NT} \left[E_A (T-T_I)^2 + E_B T_I^2 \right] \right\} .
\end{aligned}$$

5.11.3 The Probability (10.5A)

The first of the pair of consecutive disintegrations occurs at instant I_{10}^{5A} (Figure 37) and is detected by channel A. The second disintegration occurs during the dead time, T , of channel A. Thus,

$$\begin{aligned}
 (10.5A) &= P_{\bar{X}} \frac{1}{(T-T_I) T^2} E_A (1-e^{-NT}) \int_0^{T-T_I} \left\{ \int_0^{T-T_I} \left\langle \int_y^{T-T_I} dx \right\rangle dy \right\} dz \\
 &= P_{\bar{X}} \frac{1}{(T-T_I) T^2} E_A (1 - e^{-NT}) \int_0^{T-T_I} \left\{ (T-T_I)^2 - \frac{1}{2} (T-T_I)^2 \right\} dz \\
 &= P_{10} \frac{2}{T^2+T_I^2} E_A (1 - e^{-NT}) \frac{1}{2} (T-T_I)^2 \\
 &= P_{10} E_A (1 - e^{-NT}) \frac{(T-T_I)^2}{T^2+T_I^2} .
 \end{aligned}$$

5.11.4 The Probability (10.9)

The first of the pair of consecutive disintegrations occurs at instant I_{10}^9 (Figure 38) and is detected by channel B. The second disintegration occurs during the dead time, T , of channel B. Thus,

$$\begin{aligned}
 (10.9) &= P_{\bar{X}} \frac{1}{(T-T_I) T^2} E_B (1 - e^{-NT}) \int_0^{T-T_I} \left\{ \int_{T-T_I}^T \left\langle \int_{T-T_I}^y dx \right\rangle dy \right\} dz \\
 &= P_{\bar{X}} \frac{1}{(T-T_I) T^2} E_B (1 - e^{-NT}) \int_0^{T-T_I} \left\{ \frac{1}{2} (2TT_I - T_I^2) - (T-T_I) T_I \right\} dz \\
 &= P_{10} \frac{2}{T^2+T_I^2} E_B (1 - e^{-NT}) \frac{1}{2} T_I^2 \\
 &= P_{10} E_B (1 - e^{-NT}) \frac{T_I^2}{T^2+T_I^2} .
 \end{aligned}$$

5.11.5 The Probability (10.10)

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
z	y	x	$P_{\underline{X}} \frac{\delta z}{T-T_I} \frac{\delta y}{T} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	$\int_y^{T-T_I}$	$(1 - E_A) (1 - e^{-N(x-y)})$	(a)	35
$\int_0^{T-T_I}$	$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$(1 - e^{-N(x-y)})$	(b)	
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	$\int_{T-T_I}^y$	$(1 - E_B) (1 - e^{-N(x-T+T_I)})$	(c)	36
$\int_0^{T-T_I}$	$\int_{T-T_I}^T$	\int_y^T	$(1 - e^{-N(x-T+T_I)})$	(d)	

$$\begin{aligned}
 (10.10) = & P_{\underline{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} \left\langle \int_y^T (1 - e^{-N(x-y)}) dx \right. \right. \\
 & - E_A \int_y^{T-T_I} (1 - e^{-N(x-y)}) dx \left. \right\rangle dy + \int_{T-T_I}^T \left\langle \int_{T-T_I}^T (1 - e^{-N(x-T+T_I)}) dx \right. \\
 & \left. - E_B \int_{T-T_I}^y (1 - e^{-N(x-T+T_I)}) dx \right\rangle dy \left. \right\} dz
 \end{aligned}$$

$$\begin{aligned}
&= P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ \int_0^{T-T_I} \left\langle T - y + \frac{1}{N} (e^{-Ny} - e^{-NT}) e^{Ny} \right. \right. \\
&\quad \left. \left. - E_A \left[T - y + \frac{1}{N} (e^{-N(T-T_I)} - e^{-Ny}) e^{Ny} \right] \right\rangle dy \right. \\
&\quad \left. + \int_{T-T_I}^T \left\langle T_I + \frac{1}{N} (e^{-NT} - e^{-N(T-T_I)}) e^{N(T-T_I)} \right. \right. \\
&\quad \left. \left. - E_B \left[y - T + T_I + \frac{1}{N} (e^{-Ny} - e^{-N(T-T_I)}) e^{N(T-T_I)} \right] \right\rangle dy \right\} dz \\
&= P_{\bar{X}} \frac{1}{(T-T_I) T^2} \int_0^{T-T_I} \left\{ \left[T - \frac{1}{N} - E_A \left(T - T_I - \frac{1}{N} \right) \right] (T-T_I) \right. \\
&\quad \left. + \frac{1}{2} (E_A - 1) (T-T_I)^2 + \frac{1}{N^2} (e^{-NT} - E_A e^{-N(T-T_I)}) (e^{N(T-T_I)} - 1) \right. \\
&\quad \left. + \left[T_I + \frac{1}{N} (e^{-NT_I} - 1) - E_B \left(-T + T_I - \frac{1}{N} \right) \right] T_I - \frac{1}{2} E_B (2TT_I - T_I^2) \right. \\
&\quad \left. + E_B \frac{1}{N^2} (e^{-NT} - e^{-N(T-T_I)}) e^{N(T-T_I)} \right\} dz \\
&= P_{10} \frac{2}{T^2 + T_I^2} \left\{ T^2 - T T_I - \frac{1}{N} T + \frac{1}{N} T_I - E_A \left[(T-T_I)^2 - \frac{1}{N} (T-T_I) \right] \right. \\
&\quad \left. + \frac{1}{2} E_A (T-T_I)^2 - \frac{1}{2} (T^2 - 2 T T_I + T_I^2) + \frac{1}{N^2} (e^{-NT_I} - e^{-NT}) \right. \\
&\quad \left. - E_A \frac{1}{N^2} (1 - e^{-N(T-T_I)}) + T_I^2 + \frac{1}{N} e^{-NT_I} T_I - \frac{1}{N} T_I - E_B \left[-T T_I + T_I^2 \right. \right. \\
&\quad \left. \left. - \frac{1}{N} T_I + T T_I - \frac{1}{2} T_I^2 + \frac{1}{N^2} (1 - e^{-NT_I}) \right] \right\}
\end{aligned}$$

* These terms cancel out.

$$\begin{aligned}
&= P_{10} \left\{ 1 + \frac{2}{T^2 + T_I^2} \left\langle \frac{1}{N} (e^{-NT_I} T_I - T) + \frac{1}{N^2} (e^{-NT_I} - e^{-NT}) \right. \right. \\
&\quad - E_A \left[\frac{1}{2} (T - T_I)^2 - \frac{1}{N} (T - T_I) + \frac{1}{N^2} (1 - e^{-N(T - T_I)}) \right] \\
&\quad \left. \left. - E_B \left[\frac{1}{2} T_I^2 - \frac{1}{N} T_I + \frac{1}{N^2} (1 - e^{-NT_I}) \right] \right\rangle \right\} .
\end{aligned}$$

5.11.6 The Probabilities (10.s) which are Zero

The following probabilities are zero:

(10.2), (10.3), (10.4), (10.5B), (10.6), (10.7), (10.8), (10.11) and (10.12).

5.11.7 The Sum of the Probabilities F_{10s}

It is shown below that $\sum_S F_{10s} = 1$.

$$\begin{aligned}
F_{101} &= \frac{2}{T^2 + T_I^2} \left\{ \frac{1}{N} \left[T - E_A (T - T_I) - e^{-NT_I} T_I - E_B T_I \right] \right. \\
&\quad + \frac{1}{N^2} \left[e^{-NT} - e^{-NT_I} + E_A (1 - e^{-N(T - T_I)}) + E_B (1 - e^{-NT_I}) \right] \left. \right\} \\
&\quad + \frac{1}{T^2 + T_I^2} e^{-NT} \left[E_A (T - T_I)^2 + E_B T_I^2 \right] \\
F_{105A} &= E_A (1 - e^{-NT}) \frac{(T - T_I)^2}{T^2 + T_I^2} \\
F_{109} &= E_B (1 - e^{-NT}) \frac{T_I^2}{T^2 + T_I^2}
\end{aligned}$$

$$\begin{aligned}
F_{1010} = & 1 + \frac{2}{T^2 + T_I^2} \left\langle \frac{1}{N} (e^{-NT_I} T_I - T) + \frac{1}{N^2} (e^{-NT_I} - e^{-NT}) \right\rangle \\
& - \frac{1}{T^2 + T_I^2} \left[E_A (T - T_I)^2 + E_B T_I^2 \right] \\
& + \frac{2}{T^2 + T_I^2} \left\langle E_A \left[\frac{1}{N} (T - T_I) - \frac{1}{N^2} (1 - e^{-N(T - T_I)}) \right] \right. \\
& \left. + E_B \left[\frac{1}{N} T_I - \frac{1}{N^2} (1 - e^{-NT_I}) \right] \right\rangle .
\end{aligned}$$

State:

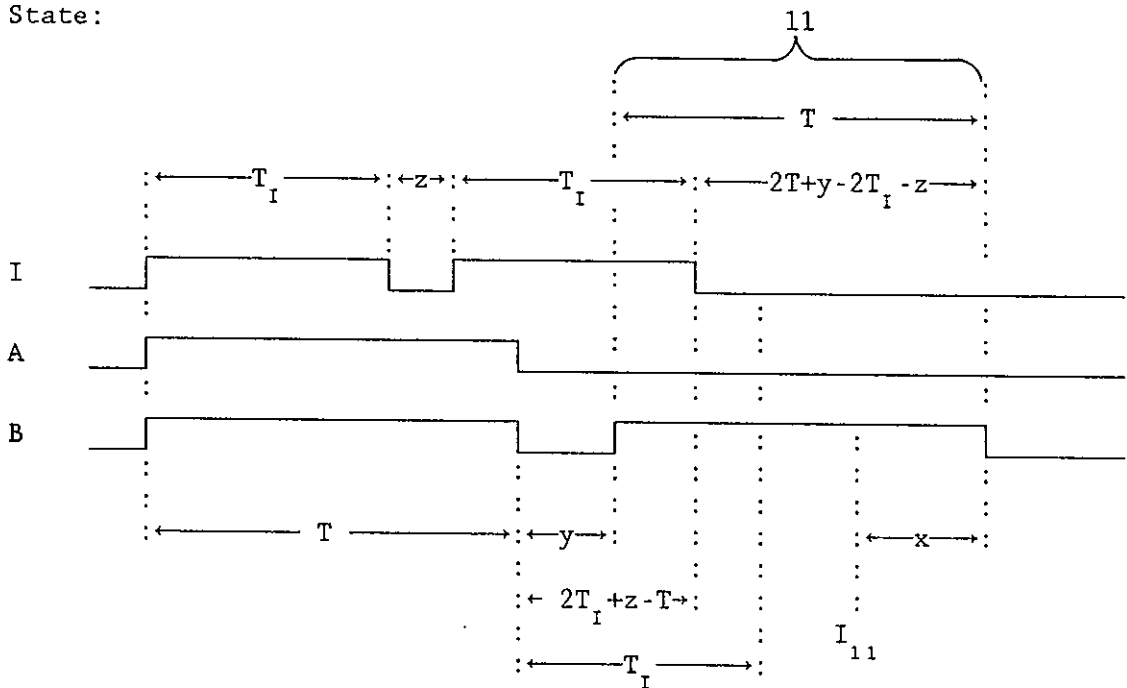


Figure 39 - The probability P_{11} , the probability (11.1), case (a), and the probability (11.11), case (a)

State:

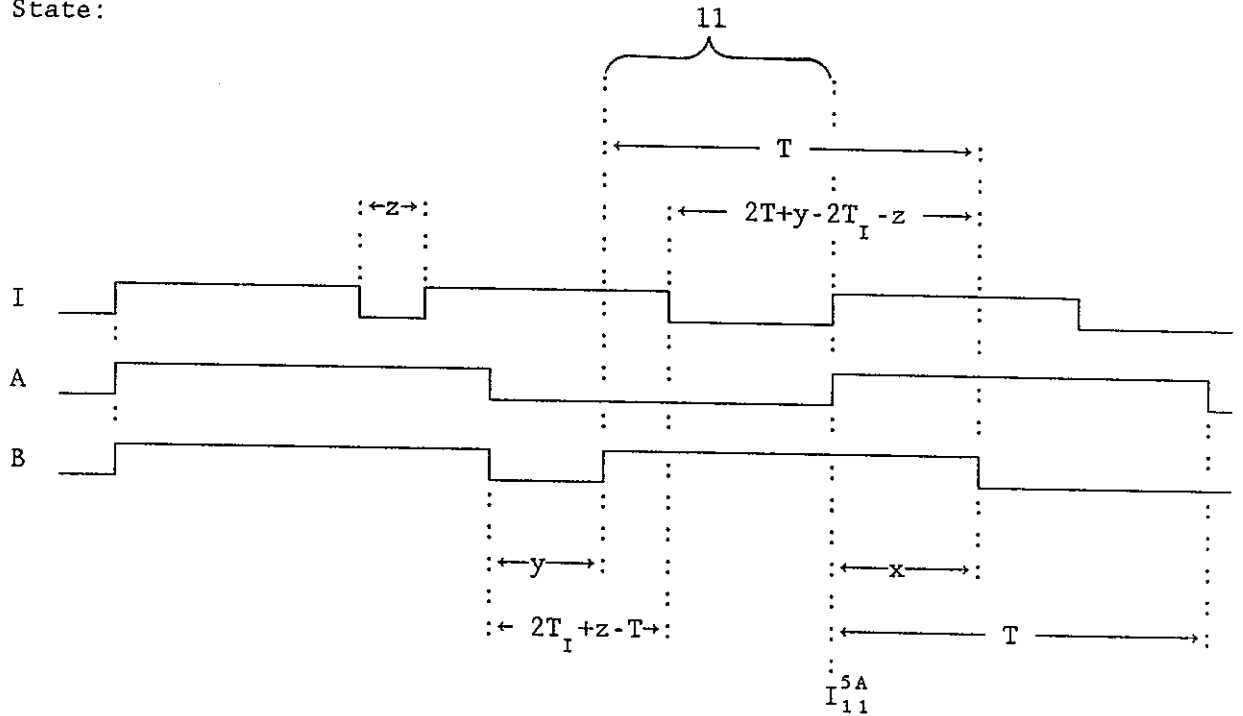


Figure 40 - The probability (11.1), case (b),
and the probability (11.5A)

5.12 The Probabilities (11.s)

The derivation of the formulae for the probabilities (11.s) makes use of the probability, $P_{\overline{XI}}$, that a disintegration will occur when the system is in the extended state \overline{XI} . Consider a disintegration which occurs at instant I_{11} (Figure 39) when the system is in state 11. The interval x can vary from 0 to T , the interval z from 0 to $(T - T_I)$, and the interval y from 0 to $(2T_I + z - T)$. At instant I_{11} , the system is also in extended state \overline{XI} , in which x can vary from 0 to T , z from 0 to $(T - T_I)$, and y from 0 to T_I . When y is greater than $(2T_I + z - T)$, state 11 does not exist. It is assumed that (a) all values of x between 0 and T are equally probable, (b) all values of y between 0 and T_I are equally probable, and (c) all values of z between 0 and $(T - T_I)$ are equally probable.

5.12.1 The Relationship Between $P_{\overline{XI}}$ and P_{11}

The element of the probability $P_{\overline{XI}}$ is $P_{\overline{XI}} \frac{\delta z}{T - T_I} \frac{\delta y}{T_I} \frac{\delta x}{T}$, which is also the element of the probability P_{11} . A formula for P_{11} is obtained by summing the elements whose increments δy are within the limits of y for state 11. Thus,

$$\begin{aligned}
P_{11} &= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{2T_I-T+z} \left[\int_0^T dx \right] dy \right\} dz \\
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I} \int_0^{T-T_I} \left\{ 2T_I - T + z \right\} dz \\
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I} \left\{ (2T_I - T) (T-T_I) + \frac{1}{2} (T-T_I)^2 \right\} \\
&= P_{\underline{XI}} \frac{1}{T_I} \left(\frac{3T_I - T}{2} \right) .
\end{aligned}$$

Therefore $P_{\underline{XI}} = P_{11} \frac{2T_I}{3T_I - T}$.

5.12.2 The Probability (11.1)

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY		CASE	NO. OF FIGURE
z	y	x	$P_{\underline{XI}} \frac{\delta z}{T-T_I} \frac{\delta y}{T_I} \frac{\delta x}{T}$ multiplied by			
$\int_0^{T-T_I}$	$\int_0^{2T_I-T+z}$	$\int_0^{2T-2T_I-z+y}$	$(1 - E_A) e^{-Nx}$		(a)	39
$\int_0^{T-T_I}$	$\int_0^{2T_I-T+z}$	$\int_0^{2T-2T_I-z+y}$	$E_A e^{-Nx}$		(b)	40
$\int_0^{T-T_I}$	$\int_0^{2T_I-T+z}$	$\int_{2T-2T_I-z+y}^T$	e^{-Nx}		(c)	

$$\begin{aligned}
(11.1) &= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{2T_I-T+z} \left\langle \int_0^T e^{-Nx} dx \right. \right. \\
&\quad \left. \left. - E_A \int_0^{2T-2T_I-z+y} e^{-Nx} dx + E_A e^{-NT} (2T-2T_I-z+y) \right\rangle dy \right\} dz \\
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{2T_I-T+z} \left\langle -\frac{1}{N} (e^{-NT} - 1) \right. \right. \\
&\quad \left. \left. + E_A \frac{1}{N} (e^{-N(2T-2T_I-z)} e^{-Ny} - 1) + E_A e^{-NT} (2T-2T_I-z+y) \right\rangle dy \right\} dz \\
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \times \\
&\int_0^{T-T_I} \left\{ \left[\frac{1}{N} (1 - e^{-NT} - E_A) + E_A e^{-NT} (2T-2T_I-z) \right] (2T_I-T+z) \right. \\
&\quad \left. - E_A \frac{1}{N^2} e^{-N(2T-2T_I-z)} (e^{-N(2T_I-T+z)} - 1) \right. \\
&\quad \left. + E_A e^{-NT} \frac{1}{2} \left[(2T_I-T)^2 + 2(2T_I-T)z + z^2 \right] \right\} dz \\
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \frac{1}{N} (1 - e^{-NT} - E_A) (2T_I-T+z) \right. \\
&\quad \left. + E_A e^{-NT} \left[(2T-2T_I) (2T_I-T) + z (2T-2T_I-2T_I+T) - z^2 \right] \right. \\
&\quad \left. - E_A \frac{1}{N^2} e^{-NT} + E_A \frac{1}{N^2} e^{-N(2T-2T_I)} e^{Nz} \right. \\
&\quad \left. + E_A e^{-NT} (2T_I-T) \left(T_I - \frac{1}{2} T + z \right) + E_A e^{-NT} \frac{1}{2} z^2 \right\} dz
\end{aligned}$$

$$\begin{aligned}
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \frac{1}{N} (1 - e^{-NT} - E_A) (2T_I - T) \right. \\
&\quad + E_A e^{-NT} \left[(2T_I - T) \left(\frac{3}{2} T - T_I \right) - \frac{1}{N^2} \right] \\
&\quad + z \left[\frac{1}{N} (1 - e^{-NT} - E_A) + E_A e^{-NT} (2T - 2T_I) \right] \\
&\quad \left. - \frac{1}{2} E_A e^{-NT} z^2 + E_A \frac{1}{N^2} e^{-2N(T-T_I)} e^{Nz} \right\} dz \\
&= P_{11} \frac{2T_I}{3T_I - T} \times \frac{1}{(T-T_I) T_I T} \left\{ \left[\frac{1}{N} (1 - e^{-NT} - E_A) (2T_I - T) \right. \right. \\
&\quad \left. \left. + E_A e^{-NT} (4TT_I - 2T_I^2 - \frac{3}{2} T^2 - \frac{1}{N^2}) \right] (T-T_I) \right. \\
&\quad \left. + \frac{1}{2} (T-T_I)^2 \left[\frac{1}{N} (1 - e^{-NT} - E_A) + 2 E_A e^{-NT} (T-T_I) \right] \right. \\
&\quad \left. - \frac{1}{6} E_A e^{-NT} (T-T_I)^3 + E_A \frac{1}{N^3} e^{-2N(T-T_I)} (e^{N(T-T_I)} - 1) \right\} \\
&= P_{11} \frac{2}{3T_I - T} \times \frac{1}{T} \left\{ \frac{1}{N} (1 - e^{-NT} - E_A) (2T_I - T + \frac{1}{2} T - \frac{1}{2} T_I) \right. \\
&\quad \left. + E_A e^{-NT} \left[4TT_I - 2T_I^2 - \frac{3}{2} T^2 - \frac{1}{N^2} + (T-T_I)^2 - \frac{1}{6} (T-T_I)^2 \right] \right. \\
&\quad \left. + E_A \frac{1}{N^3} (e^{-N(T-T_I)} - e^{-2N(T-T_I)}) \right\}
\end{aligned}$$

$$\begin{aligned}
&= P_{11} \frac{2}{3T_I - T} \times \frac{1}{T} \left\{ \frac{1}{N} (1 - e^{-NT} - E_A) \left(\frac{3T_I - T}{2} \right) \right. \\
&\quad + E_A \left[e^{-NT} \left(4TT_I - 2T_I^2 - \frac{3}{2} T^2 - \frac{1}{N^2} + \frac{5}{6} T^2 - \frac{5}{3} TT_I + \frac{5}{6} T_I^2 \right) \right. \\
&\quad \left. \left. + \frac{1}{N^3 (T - T_I)} (e^{-N(T - T_I)} - e^{-2N(T - T_I)}) \right] \right\} \\
&= P_{11} \left\{ \frac{1}{NT} (1 - e^{-NT} - E_A) \right. \\
&\quad + \frac{2E_A}{(3T_I - T) T} \left[e^{-NT} \left(\frac{7}{3} TT_I - \frac{7}{6} T_I^2 - \frac{2}{3} T^2 - \frac{1}{N^2} \right) \right. \\
&\quad \left. \left. + \frac{1}{N^3 (T - T_I)} (e^{-N(T - T_I)} - e^{-2N(T - T_I)}) \right] \right\}.
\end{aligned}$$

5.12.3 The Probability (11.5A)

The first of the pair of consecutive disintegrations occurs at instant I_{11}^{5A} (Figure 40) and is detected by channel A. The second disintegration occurs during the dead time, T , of channel A. Thus,

$$\begin{aligned}
(11.5A) &= P_{\underline{XI}} \frac{1}{(T - T_I) T_I T} E_A (1 - e^{-NT}) \times \\
&\quad \int_0^{T - T_I} \left\{ \int_0^{2T_I - T + z} \left[\int_0^{2T - 2T_I - z + y} dx \right] dy \right\} dz \\
&= P_{\underline{XI}} \frac{1}{(T - T_I) T_I T} E_A (1 - e^{-NT}) \times \\
&\quad \int_0^{T - T_I} \left\{ (2T - 2T_I - z) (2T_I - T + z) + \frac{1}{2} (2T_I - T + z)^2 \right\} dz
\end{aligned}$$

$$= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} E_A (1 - e^{-NT}) x$$

$$\int_0^{T-T_I} (2T_I - T + z) \left(\frac{3}{2} T - T_I - \frac{1}{2} z \right) dz$$

$$= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} E_A (1 - e^{-NT}) x$$

$$\int_0^{T-T_I} (3T_I T - 2T_I^2 - T_I z - \frac{3}{2} T^2 + TT_I + \frac{1}{2} Tz + \frac{3}{2} Tz - T_I z - \frac{1}{2} z^2) dz$$

$$= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} E_A (1 - e^{-NT}) \left[(4T_I T - 2T_I^2 - \frac{3}{2} T^2) (T-T_I) + \frac{1}{2} (T-T_I)^2 (-2T_I + 2T) - \frac{1}{6} (T-T_I)^3 \right]$$

$$= P_{\underline{XI}} \frac{1}{T_I T} E_A (1 - e^{-NT}) (4T_I T - 2T_I^2 - \frac{3}{2} T^2 - TT_I + T^2 + T_I^2 - TT_I - \frac{1}{6} T^2 + \frac{1}{3} TT_I - \frac{1}{6} T_I^2)$$

$$= P_{11} \frac{2}{(3T_I - T) T} E_A (1 - e^{-NT}) \left(\frac{7}{3} TT_I - \frac{7}{6} T_I^2 - \frac{2}{3} T^2 \right) .$$

5.12.4 The Probability (11.11)

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
z	y	x	$P_{\underline{XI}} \frac{\delta z}{T-T_I} \frac{\delta y}{T_I} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	$\int_0^{2T_I-T+z}$	$\int_0^{2T-2T_I-z+y}$	$(1 - E_A) (1 - e^{-Nx})$	(a)	39
$\int_0^{T-T_I}$	$\int_0^{2T_I-T+z}$	$\int_{2T-2T_I-z+y}^T$	$(1 - e^{-Nx})$	(b)	

$$\begin{aligned}
 (11.11) &= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{2T_I-T+z} \left\langle \int_0^T (1 - e^{-Nx}) dx \right. \right. \\
 &\quad \left. \left. - E_A \int_0^{2T-2T_I-z+y} (1 - e^{-Nx}) dx \right\rangle dy \right\} dz \\
 &= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \int_0^{2T_I-T+z} \left\langle T + \frac{1}{N} (e^{-NT} - 1) \right. \right. \\
 &\quad \left. \left. - E_A \left[2T-2T_I-z+y + \frac{1}{N} (e^{-N(2T-2T_I-z)} e^{-Ny} - 1) \right] \right\rangle dy \right\} dz \\
 &= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A (2T-2T_I-z - \frac{1}{N}) \right] \right. \\
 &\quad \left. (2T_I-T+z) - \frac{1}{2} E_A \left[(2T_I-T)^2 + 2(2T_I-T)z + z^2 \right] \right. \\
 &\quad \left. + E_A \frac{1}{N^2} e^{-N(2T-2T_I-z)} (e^{-N(2T_I-T+z)} - 1) \right\} dz
 \end{aligned}$$

$$\begin{aligned}
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \int_0^{T-T_I} \left\{ \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A (2T-2T_I - \frac{1}{N}) \right] (2T_I - T) \right. \\
&\quad + z \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A (2T-2T_I - \frac{1}{N}) + E_A (2T_I - T) \right] + E_A z^2 \\
&\quad - \frac{1}{2} E_A (2T_I - T)^2 - E_A (2T_I - T) z - \frac{1}{2} E_A z^2 \\
&\quad \left. + E_A \frac{1}{N^2} (e^{-NT} - e^{-N(2T-2T_I)} e^{Nz}) \right\} dz \\
&= P_{\underline{XI}} \frac{1}{(T-T_I) T_I T} \left\{ \left\langle \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A (2T-2T_I - \frac{1}{N}) \right] (2T_I - T) \right. \right. \\
&\quad \left. \left. - \frac{1}{2} E_A (2T_I - T)^2 + E_A \frac{1}{N^2} e^{-NT} \right\rangle (T-T_I) \right. \\
&\quad + \frac{1}{2} (T-T_I)^2 \left[T + \frac{1}{N} (e^{-NT} - 1) - E_A (2T-2T_I - \frac{1}{N}) + E_A (2T_I - T) \right. \\
&\quad \left. \left. - E_A (2T_I - T) \right] + \frac{1}{6} E_A (T-T_I)^3 - E_A \frac{1}{N^3} e^{-2N(T-T_I)} (e^{N(T-T_I)} - 1) \right\} \\
&= P_{11} \frac{2T_I}{3T_I - T} \times \frac{1}{T_I T} \left\{ T(2T_I - T) + \frac{1}{N} (e^{-NT} - 1 + E_A) (2T_I - T) \right. \\
&\quad \left. - E_A (2T-2T_I) (2T_I - T) - \frac{1}{2} E_A (2T_I - T)^2 + E_A \frac{1}{N^2} e^{-NT} \right. \\
&\quad \left. + \frac{1}{2} (T-T_I) \left[T + \frac{1}{N} (e^{-NT} - 1 + E_A) - E_A (2T-2T_I) \right] + \frac{1}{6} E_A (T-T_I)^2 \right. \\
&\quad \left. - E_A \frac{1}{N^3} (T-T_I) (e^{-N(T-T_I)} - e^{-2N(T-T_I)}) \right\}
\end{aligned}$$

$$\begin{aligned}
&= P_{11} \frac{2}{(3T_I - T) T} \left\{ T(2T_I - T) + \frac{1}{N} (e^{-NT} - 1 + E_A) \left(\frac{3}{2} T_I - \frac{1}{2} T \right) \right. \\
&\quad + E_A \left(-4TT_I + 2T^2 + 4T_I^2 - 2TT_I - 2T_I^2 - \frac{1}{2} T^2 + 2TT_I + \frac{1}{N^2} e^{-NT} \right) \\
&\quad + \left(\frac{1}{2} T - \frac{1}{2} T_I \right) T - E_A (T^2 - 2TT_I + T_I^2) + E_A \left(\frac{1}{6} T^2 - \frac{1}{3} TT_I + \frac{1}{6} T_I^2 \right) \\
&\quad \left. - E_A \frac{1}{N^3 (T - T_I)} (e^{-N(T - T_I)} - e^{-2N(T - T_I)}) \right\} \\
&= P_{11} \frac{2}{(3T_I - T) T} \left\{ \frac{T(3T_I - T)}{2} + \frac{1}{N} (e^{-NT} - 1 + E_A) \left(\frac{3T_I - T}{2} \right) \right. \\
&\quad + E_A \left(-\frac{7}{3} TT_I + \frac{2}{3} T^2 + \frac{7}{6} T_I^2 + \frac{1}{N^2} e^{-NT} \right) \\
&\quad \left. - E_A \frac{1}{N^3 (T - T_I)} (e^{-N(T - T_I)} - e^{-2N(T - T_I)}) \right\} \\
&= P_{11} \left\{ 1 + \frac{1}{NT} (e^{-NT} - 1 + E_A) \right. \\
&\quad + \frac{2E_A}{(3T_I - T) T} \left[\frac{1}{3} (2T^2 + \frac{7}{2} T_I^2 - 7TT_I) + \frac{1}{N^2} e^{-NT} \right. \\
&\quad \left. \left. + \frac{1}{N^3 (T - T_I)} (e^{-2N(T - T_I)} - e^{-N(T - T_I)}) \right] \right\} .
\end{aligned}$$

5.12.5 The Probabilities (11.s) which are Zero

The following probabilities are zero:

(11.2), (11.3), (11.4), (11.5B), (11.6), (11.7), (11.8), (11.9), (11.10) and (11.12).

5.12.6 The Sum of the Probabilities F_{11s}

It is shown below that $\sum_s F_{11s} = 1$.

(1)

$$F_{111} = \frac{1}{NT} (1 - e^{-NT} - E_A)$$

$$+ \frac{2E_A}{(3T_I - T) T} \left[e^{-NT} \left(\frac{7}{3} TT_I - \frac{7}{6} T_I^2 - \frac{2}{3} T^2 \right) - e^{-NT} \frac{1}{N^2} \right]$$

(4)

$$+ \frac{1}{N^3 (T - T_I)} (e^{-N(T - T_I)} - e^{-2N(T - T_I)})$$

(5) (2)

$$F_{115A} = \frac{2E_A}{(3T_I - T) T} (1 - e^{-NT}) \left(\frac{7}{3} TT_I - \frac{7}{6} T_I^2 - \frac{2}{3} T^2 \right)$$

(1)

$$F_{1111} = 1 + \frac{1}{NT} (e^{-NT} - 1 + E_A)$$

(5)

(3)

$$+ \frac{2E_A}{(3T_I - T) T} \left[\frac{1}{3} (2T^2 + \frac{7}{2} T_I^2 - 7TT_I) + \frac{1}{N^2} e^{-NT} \right]$$

(4)

$$+ \frac{1}{N^3 (T - T_I)} (e^{-2N(T - T_I)} - e^{-N(T - T_I)})$$

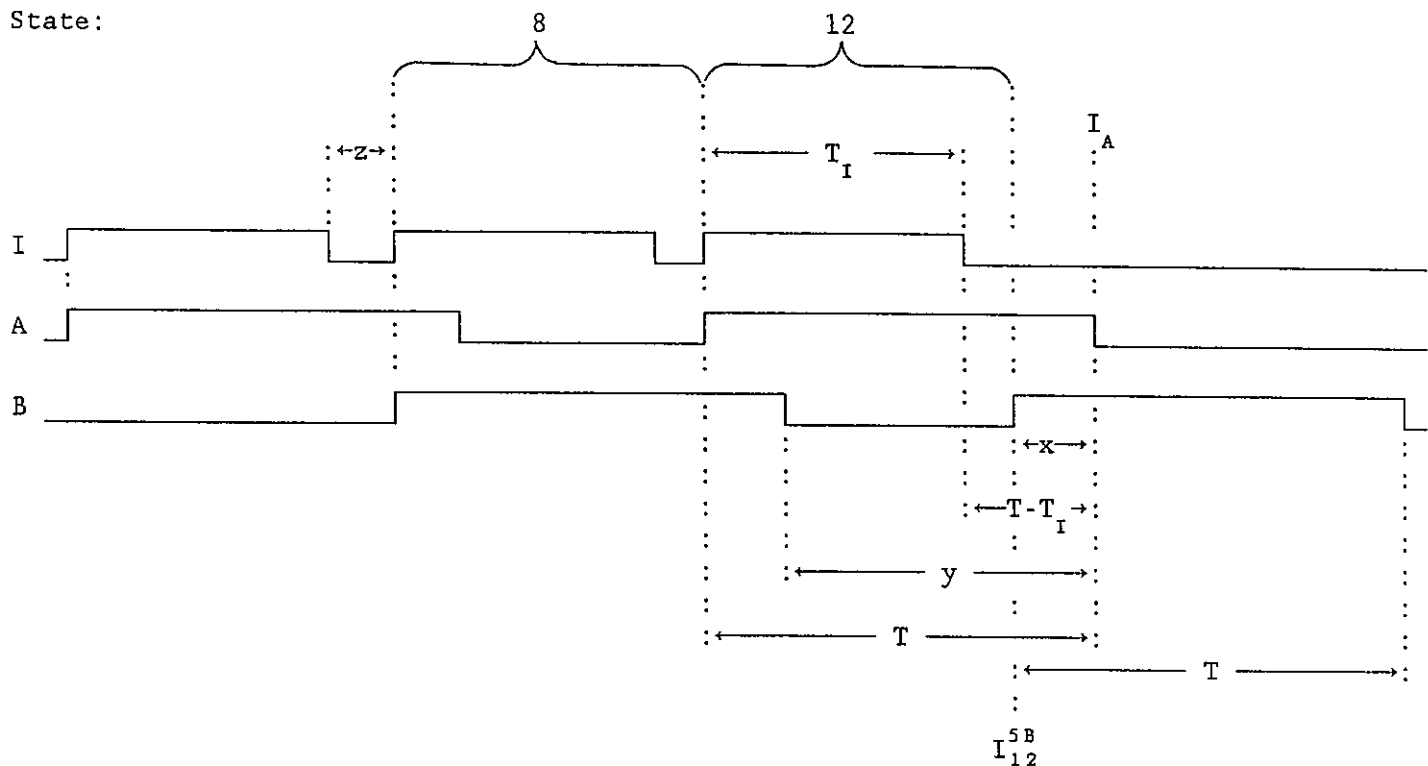


Figure 41 - The probability (12.1), case (b),
and the probability (12.5B), case (a)

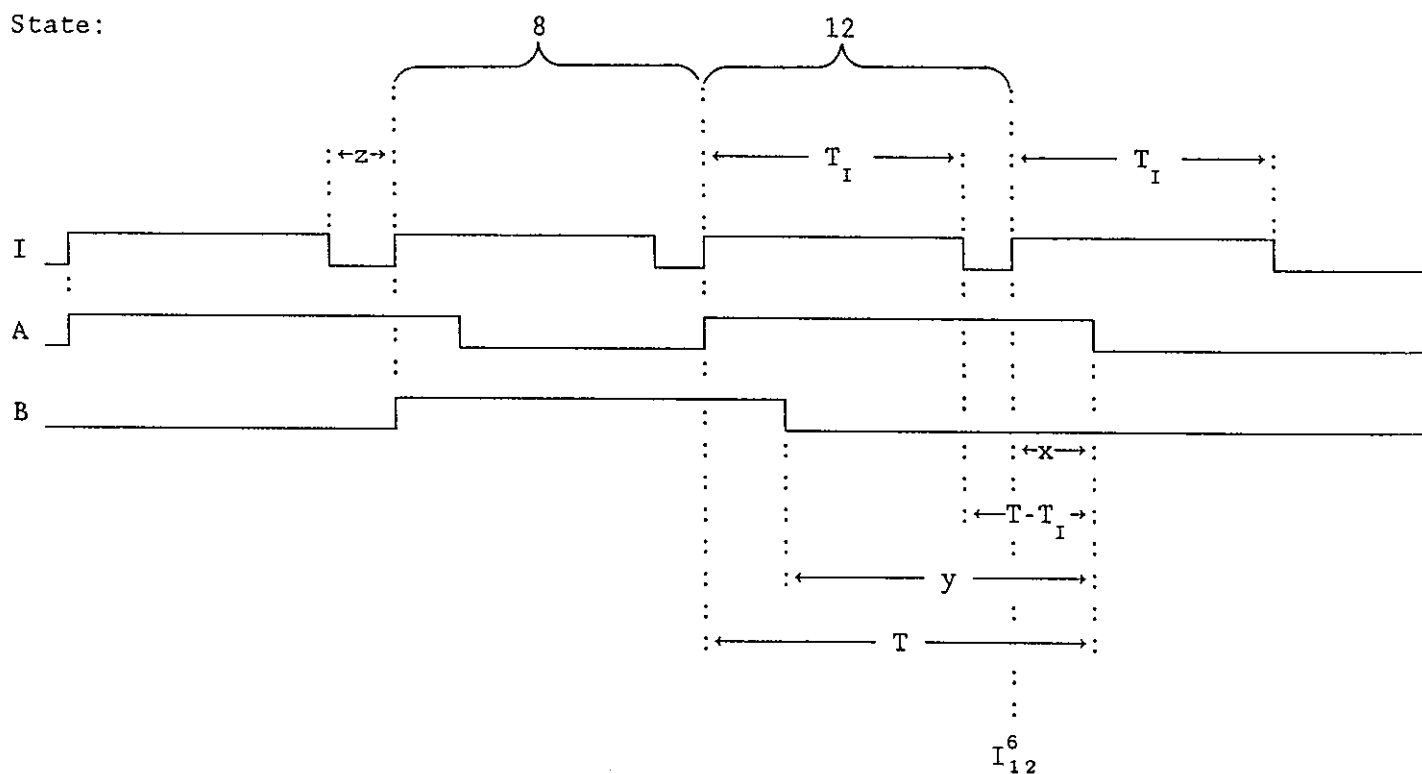


Figure 42 - The probability (12.1), case (c),
and the probability (12.6)

5.13 The Probabilities (12.s)

In Figure 41 a disintegration occurs at instant I_{12}^{5B} and is detected by channel B. The system changes from state 12 to state 5B. Considering now any disintegration which occurs when the system is in state 12, x denotes the interval between the instant of the disintegration and the instant I_A when channel A becomes live again. The interval x can vary from 0 to T , and it is assumed that all such values of x are equally probable. The minimum and maximum values of the interval y are T_I and T , respectively. It is assumed that all values of y between T_I and T are equally probable, and that all values of z between 0 and $(T-T_I)$ are equally probable.

5.13.1 The Probability (12.1)

In the following table, $(1-E_A)(1-E_B)$ is expanded to $[1-E_B - E_A(1-E_B)]$.

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
z	y	x	$P_{12} \frac{\delta z}{T-T_I} \frac{\delta y}{T-T_I} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_0^{T-T_I}$	$[1-E_B - E_A(1-E_B)] e^{-Nx}$	(a)	
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_0^{T-T_I}$	$(1-E_A) E_B e^{-NT}$	(b)	41
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_0^{T-T_I}$	$(1-E_B) E_A e^{-NT_I}$	(c)	42
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_0^{T-T_I}$	$E_A E_B e^{-NT}$	(d)	
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_{T-T_I}^y$	$(1-E_B) e^{-Nx}$	(e)	
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_{T-T_I}^y$	$E_B e^{-NT}$	(f)	
$\int_0^{T-T_I}$	$\int_{T_I}^T$	\int_y^T	e^{-Nx}	(g)	

$$\begin{aligned}
(12.1) &= P_{12} \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle \int_0^T e^{-Nx} dx - E_B \int_0^y e^{-Nx} dx \right. \right. \\
&\quad - E_A (1 - E_B) \int_0^{T-T_I} e^{-Nx} dx + E_B e^{-NT} \int_0^y dx \\
&\quad \left. \left. + (1 - E_B) E_A e^{-NT_I} \int_0^{T-T_I} dx \right\rangle dy \right\} dz \\
&= P_{12} \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle -\frac{1}{N} (e^{-NT} - 1) + E_B \frac{1}{N} (e^{-Ny} - 1) \right. \right. \\
&\quad + E_A (1 - E_B) \frac{1}{N} (e^{-N(T-T_I)} - 1) + E_B e^{-NT} y \\
&\quad \left. \left. + (1 - E_B) E_A e^{-NT_I} (T-T_I) \right\rangle dy \right\} dz \\
&= P_{12} \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ \left\langle \frac{1}{N} \left[1 - e^{-NT} - E_B + E_A (1-E_B) (e^{-N(T-T_I)} - 1) \right] \right. \right. \\
&\quad + (1-E_B) E_A e^{-NT_I} (T-T_I) \left. \right\rangle (T-T_I) - E_B \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) \\
&\quad \left. + \frac{1}{2} E_B e^{-NT} (T^2 - T_I^2) \right\} dz \\
&= P_{12} \frac{1}{(T-T_I)^2 T} \left\{ \left\langle \frac{1}{N} \left[1 - e^{-NT} - E_B - E_A (1-E_B) (1 - e^{-N(T-T_I)}) \right] \right. \right. \\
&\quad + E_A (1-E_B) e^{-NT_I} (T-T_I) \left. \right\rangle (T-T_I) \\
&\quad \left. + E_B \left[\frac{1}{2} e^{-NT} (T+T_I) (T-T_I) - \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) \right] \right\} (T-T_I)
\end{aligned}$$

$$\begin{aligned}
&= P_{12} \left\{ \frac{1}{NT} \left[1 - e^{-NT} - E_B - E_A (1-E_B) (1 - e^{-N(T-T_I)}) \right] \right. \\
&\quad + E_A (1-E_B) e^{-NT_I} \frac{T-T_I}{T} \\
&\quad \left. + E_B \left[\frac{1}{2} e^{-NT} \frac{T+T_I}{T} - \frac{1}{N^2 (T-T_I) T} (e^{-NT} - e^{-NT_I}) \right] \right\} .
\end{aligned}$$

5.13.2 The Probability (12.5B)

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY	CASE	NO. OF FIGURE
z	y	x	$P_{12} \frac{\delta z}{T-T_I} \frac{\delta y}{T-T_I} \frac{\delta x}{T}$ multiplied by		
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_0^{T-T_I}$	$(1-E_A) E_B (1 - e^{-NT})$	(a)	41
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_{T-T_I}^y$	$E_B (1 - e^{-NT})$	(b)	

$$\begin{aligned}
(12.5B) &= P_{12} \frac{1}{(T-T_I)^2 T} E_B (1 - e^{-NT}) \int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle \int_0^y dx \right. \right. \\
&\quad \left. \left. - E_A \int_0^{T-T_I} dx \right\rangle dy \right\} dz \\
&= P_{12} \frac{1}{(T-T_I)^2 T} E_B (1 - e^{-NT}) \int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle y - E_A (T-T_I) \right\rangle dy \right\} dz \\
&= P_{12} \frac{1}{(T-T_I)^2 T} E_B (1 - e^{-NT}) \int_0^{T-T_I} \left\{ \frac{1}{2} (T^2 - T_I^2) - E_A (T-T_I)^2 \right\} dz \\
&= P_{12} \frac{1}{T} E_B (1 - e^{-NT}) \left[\frac{1}{2} (T+T_I) - E_A (T-T_I) \right] .
\end{aligned}$$

5.13.3 The Probability (12.6)

The first of the pair of consecutive disintegrations occurs at instant I_{12}^6 (Figure 42) and is detected by the detector of channel A. The second disintegration occurs in the interval, T_I , between the instant I_{12}^6 and the instant at which channel A becomes live again. Thus,

$$(12.6) = P_{12} \frac{1}{(T-T_I)^2 T} E_A (1-E_B) (1 - e^{-NT_I}) x$$

$$\int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle \int_0^{T-T_I} dx \right\rangle dy \right\} dz$$

$$= P_{12} \frac{1}{(T-T_I)^2 T} E_A (1-E_B) (1 - e^{-NT_I}) (T-T_I)^3$$

$$= P_{12} E_A (1-E_B) (1 - e^{-NT_I}) \frac{T-T_I}{T} .$$

5.13.4 The Probability (12.8)

The first of the pair of consecutive disintegrations occurs at an instant when x is less than $(T-T_I)$. The disintegration is detected by both detectors. The second disintegration occurs during the dead time, T , of channel B. Thus,

$$(12.8) = P_{12} \frac{1}{(T-T_I)^2 T} E_A E_B (1 - e^{-NT}) x$$

$$\int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle \int_0^{T-T_I} dx \right\rangle dy \right\} dz$$

$$= P_{12} E_A E_B (1 - e^{-NT}) \frac{T-T_I}{T} .$$

5.13.5 The Probability (12.12)

LIMITS OF INTEGRATION WITH RESPECT TO			ELEMENT OF PROBABILITY	CASE
z	y	x	$P_{12} \frac{\delta z}{T-T_I} \frac{\delta y}{T-T_I} \frac{\delta x}{T}$ multiplied by	
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_0^{T-T_I}$	$[1-E_B - E_A (1-E_B)] (1 - e^{-Nx})$	(a)
$\int_0^{T-T_I}$	$\int_{T_I}^T$	$\int_{T-T_I}^y$	$(1-E_B) (1 - e^{-Nx})$	(b)
$\int_0^{T-T_I}$	$\int_{T_I}^T$	\int_y^T	$(1 - e^{-Nx})$	(c)

$$\begin{aligned}
 (12.12) &= P_{12} \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle \int_0^T (1 - e^{-Nx}) dx \right. \right. \\
 &\quad \left. \left. - E_B \int_0^y (1 - e^{-Nx}) dx - E_A (1-E_B) \int_0^{T-T_I} (1 - e^{-Nx}) dx \right\rangle dy \right\} dz \\
 &= P_{12} \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ \int_{T_I}^T \left\langle T + \frac{1}{N} (e^{-NT} - 1) \right. \right. \\
 &\quad \left. \left. - E_B \left[y + \frac{1}{N} (e^{-Ny} - 1) \right] \right. \right. \\
 &\quad \left. \left. - E_A (1-E_B) \left[T-T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1) \right] \right\rangle dy \right\} dz
 \end{aligned}$$

$$\begin{aligned}
&= P_{12} \frac{1}{(T-T_I)^2} \int_0^{T-T_I} \left\{ \left\langle T + \frac{1}{N} (e^{-NT} - 1 + E_B) \right. \right. \\
&\quad - E_A (1-E_B) \left[T-T_I + \frac{1}{N} (e^{-N(T-T_I)} - 1) \right] \left. \right\rangle (T-T_I) \\
&\quad - E_B \left[\frac{1}{2} (T^2 - T_I^2) - \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) \right] \left. \right\} dz \\
&= P_{12} \frac{1}{(T-T_I)^2} \left\{ \left\langle T + \frac{1}{N} (e^{-NT} - 1 + E_B) \right. \right. \\
&\quad - E_A (1-E_B) \left[T-T_I - \frac{1}{N} (1 - e^{-N(T-T_I)}) \right] \left. \right\rangle (T-T_I) \\
&\quad - E_B \left[\frac{1}{2} (T+T_I) (T-T_I) - \frac{1}{N^2} (e^{-NT} - e^{-NT_I}) \right] \left. \right\} (T-T_I) \\
&= P_{12} \left\{ 1 + \frac{1}{NT} (e^{-NT} - 1 + E_B) \right. \\
&\quad - \frac{E_A (1-E_B)}{T} \left[T-T_I - \frac{1}{N} (1 - e^{-N(T-T_I)}) \right] \\
&\quad \left. - E_B \left[\frac{1}{2} \frac{T+T_I}{T} - \frac{1}{N^2 (T-T_I) T} (e^{-NT} - e^{-NT_I}) \right] \right\} .
\end{aligned}$$

5.13.6 The Probabilities (12.s) which are Zero

The following probabilities are zero:

(12.2), (12.3), (12.4), (12.5A), (12.7), (12.9), (12.10) and (12.11).

5.13.7 The Sum of the Probabilities F_{12s}

It is shown below that $\sum_S F_{12s} = 1$. Terms with the same superscribed number cancel out.

$$F_{121} = \frac{1}{NT} \left[\begin{matrix} (1) & (2) & (3) & (4) \\ 1 - e^{-NT} & - E_B & - E_A & (1 - E_B) (1 - e^{-N(T-T_I)}) \end{matrix} \right]$$

$$+ E_A \begin{matrix} (5) & (6) \\ (1 - E_B) & e^{-NT_I} \frac{T-T_I}{T} \end{matrix}$$

$$+ E_B \left[\begin{matrix} (7) & (8) \\ \frac{1}{2} e^{-NT} \frac{T+T_I}{T} & - \frac{1}{N^2 (T-T_I) T} (e^{-NT} - e^{-NT_I}) \end{matrix} \right]$$

$$F_{125B} = \frac{1}{T} E_B \left[\begin{matrix} (9) & (10) & (7) & (11) \\ \frac{1}{2} (T+T_I) & - E_A (T-T_I) & - \frac{1}{2} e^{-NT} (T+T_I) & + E_A e^{-NT} (T-T_I) \end{matrix} \right]$$

$$F_{126} = E_A \begin{matrix} (12) & (13) & (5) & (6) \\ (1 - E_B - e^{-NT_I} + E_B e^{-NT_I}) & \frac{T-T_I}{T} \end{matrix}$$

$$F_{128} = E_A E_B \begin{matrix} (10) & (11) \\ (1 - e^{-NT}) & \frac{T-T_I}{T} \end{matrix}$$

$$F_{1212} = 1 + \frac{1}{NT} \begin{matrix} (2) & (1) & (3) & (12) & (13) \\ (e^{-NT} - 1 + E_B) & - E_A & (1 - E_B) & \frac{T-T_I}{T} \end{matrix}$$

$$+ E_A \begin{matrix} (4) \\ (1 - E_B) \frac{1}{NT} (1 - e^{-N(T-T_I)}) \end{matrix}$$

$$- E_B \left[\begin{matrix} (9) & (8) \\ \frac{1}{2} \frac{T+T_I}{T} & - \frac{1}{N^2 (T-T_I) T} (e^{-NT} - e^{-NT_I}) \end{matrix} \right]$$

6. THE FORMULAE FOR THE CHANNEL EFFICIENCIES AND THE INTRINSIC DEAD TIME

In the derivation given below of the formula for the efficiency, E_A , of the detector of channel A, account has been taken of the extension of the set dead time by the intrinsic dead time of the combined components preceding the paralysis unit (Figures 1 and 43). In channel B, the efficiency of the γ detector is low. Because, in consequence, the count rate is low, the extension of the set dead time can be neglected. The derivation of the formula for E_B is simpler and is given first.

6.1 The Formula for the Efficiency E_B

The count rate, including background, which is recorded by the scaler of channel B is denoted by B'' , and the background rate is denoted by B_b . Let the unit of time used be the second. The total dead time per second is $B''T$. If the dead time were zero, the number of events which would be recorded per second would be $NE_B + B_b$. Suppose the average disintegration rate during the dead time $B''T$ is N . If the channel were live during $B''T$ the number of events recorded in that time would be $(NE_B + B_b) B''T$. This is the number of events per second which are not recorded because of the channel being dead. The latter quantity is also equal to $NE_B + B_b - B''$. Thus,

$$NE_B + B_b - B'' = (NE_B + B_b) B''T.$$

Rearranging,

$$(NE_B + B_b) (1 - B''T) = B''$$

$$NE_B + B_b = \frac{B''}{1 - B''T}$$

$$E_B = \frac{B''}{N (1 - B''T)} - \frac{B_b}{N} \quad (7)$$

6.2 The Intrinsic Dead Time T_1 and the Efficiency E_A

The count rates, including backgrounds, which are recorded by scalers I and A of channel A (Figure 1) are denoted by I'' and A'' , respectively. The background rate recorded by A is denoted by A_b . Consider an event which is detected by the proportional counter at instant I_1 (Figure 43).

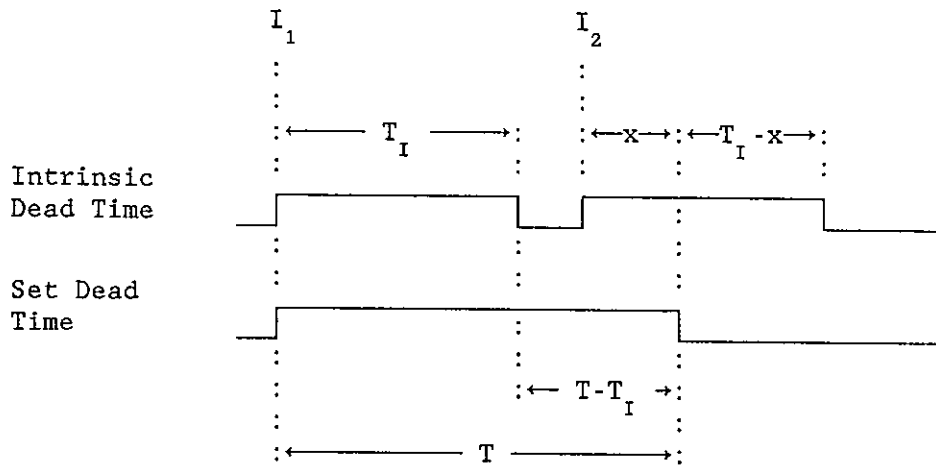


Figure 43 - Extension ($T_I - x$) of the set dead time (T) by the intrinsic dead time (T_I)

The paralysis unit A goes dead for time T , and the preceding components for time T_I . An event which is detected at instant I_2 extends the set dead time by an interval $T_I - x$. The event detected at instant I_1 is recorded by scalers I and A, but the event detected at instant I_2 is recorded by scaler I only. In other words, all the events which are detected in all the intervals $T - T_I$ are recorded by scaler I but not by scaler A. The number of such events per second is equal to $I'' - A''$. It is assumed that all values of x between 0 and $T - T_I$ are approximately equally probable. Thus the number per second of such events which occur at instants between x and $x + \delta x$ is $(I'' - A'') \frac{\delta x}{T - T_I}$.

In one second, the resulting extension of the dead time intervals T is

$$(I'' - A'') \frac{\delta x}{T - T_I} (T_I - x) .$$

Let the total extension of the intervals T during one second be denoted by X . The following derivation is for the case $T_I > T/2$.

$$\begin{aligned} X &= \frac{I'' - A''}{T - T_I} \int_0^{T - T_I} (T_I - x) dx \\ &= \frac{I'' - A''}{T - T_I} \left[T_I (T - T_I) - \frac{1}{2} (T - T_I)^2 \right] \end{aligned}$$

$$\begin{aligned}
&= (I'' - A'') \left(T_I - \frac{1}{2} T + \frac{1}{2} T_I \right) \\
&= \frac{1}{2} (I'' - A'') (3T_I - T) .
\end{aligned}$$

If the dead times T and T_I were zero, the number of events which would be recorded per second by scalers I and A would be $NE_A + A_b$. The number of events per second which are not recorded by scaler I because of the preceding components being dead is $(NE_A + A_b) I''T_I$. This is also equal to $NE_A + A_b - I''$. Thus,

$$NE_A + A_b = \frac{I''}{1 - I''T_I} . \quad (8)$$

The total dead time per second of the combined components preceding scaler A is $A''T + X$. The number of events per second which are not recorded by scaler A because of the channel being dead is equal to $(NE_A + A_b) (A''T + X)$, and it is also equal to $NE_A + A_b - A''$. Thus,

$$NE_A + A_b - A'' = (NE_A + A_b) (A''T + X).$$

Rearranging, and then substituting for X ,

$$(NE_A + A_b) (1 - A''T - X) = A''$$

$$NE_A + A_b = \frac{A''}{1 - A''T - (I'' - A'') (3T_I - T)/2} . \quad (9)$$

The simultaneous equations (8) and (9), in which the two unknowns are (NE_A) and T_I , are solved by first equating the right-hand sides of the equations.

$$\frac{I''}{1 - I''T_I} = \frac{A''}{1 - A''T - (I'' - A'') (3T_I - T)/2}$$

Multiplying out,

$$I'' - I''A''T - \frac{1}{2} I'' (3T_I I'' - TI'' - 3T_I A'' + TA'') = A'' - A''I''T_I$$

$$I''T_I \left(-\frac{3}{2} I'' + \frac{3}{2} A'' + A'' \right) = A'' - I'' + I''A''T - \frac{1}{2} TI''^2 + \frac{1}{2} I''TA''$$

$$I''T_I \frac{5A'' - 3I''}{2} = A'' + I'' \left(-1 + \frac{3}{2} A''T - \frac{1}{2} TI'' \right)$$

$$T_I = \frac{2A'' + I'' [T(3A'' - I'') - 2]}{I'' (5A'' - 3I'')} \quad (10)$$

Rearranging equation (8),

$$NE_A = \frac{I''}{1 - I''T_I} - A_b,$$

$$E_A = \frac{I''}{N(1 - I''T_I)} - \frac{A_b}{N}, \quad (11)$$

where T_I is given by equation (10).

6.3 Dead Time Correction for Count Rate A''

A dead-time correction for the count rate A'' , taking into account the intrinsic dead time, may be made by using either of the simultaneous equations (8) and (9), and substituting for T_I from equation (10). If the counting of a radioactive source is to be carried out with one channel only, such a correction is more accurate than that obtained by using the usual equation

$$NE_A + A_b = \frac{A''}{1 - A''T}$$

7. SOLUTION OF SIMULTANEOUS EQUATIONS TO OBTAIN THE PROBABILITIES P_1 TO P_{12}

As stated in Section 4.2, the numerical values of the 13 probabilities, P_1 to P_{12} , are obtained by solving 13 simultaneous equations in which those probabilities are the unknowns. One of the equations is equation (6):

$$\sum_r P_r = 1.$$

The others are 12 of the 13 represented by equation (5):

$$P_s = \sum_r P_r F_{rs}.$$

The right-hand side of equation (5) represents the summation of the probabilities (r.s) in column s of Table 1. The equation omitted from the solution is the longest of the group represented by (5); viz.,

$$P_1 = \sum_r P_r F_{r1}. \quad (12)$$

The other 12 equations in that group are given below in an expanded form in which

$$G_{ss} = F_{ss} - 1,$$

where s is the number of one of the states 2 to 12.

$$P_1 F_{12} + P_2 G_{22} = 0$$

$$P_1 F_{13} + P_3 G_{33} = 0$$

$$P_1 F_{14} + P_4 G_{44} = 0$$

$$P_2 F_{25A} + P_{5A} G_{5A5A} + P_{5B} F_{5B5A} + P_9 F_{95A} + P_{10} F_{105A} + P_{11} F_{115A} = 0$$

$$P_3 F_{35B} + P_{5A} F_{5A5B} + P_{5B} G_{5B5B} + P_{12} F_{125B} = 0$$

$$P_3 F_{36} + P_{5A} F_{5A6} + P_6 G_{66} + P_{12} F_{126} = 0$$

$$P_4 F_{47} + P_7 G_{77} = 0$$

$$P_3 F_{38} + P_{5A} F_{5A8} + P_8 G_{88} + P_{12} F_{128} = 0$$

$$P_6 F_{69} + P_9 G_{99} + P_{10} F_{109} = 0$$

$$P_{5A} F_{5A10} + P_{5B} F_{5B10} + P_{10} G_{1010} = 0$$

$$P_7 F_{711} + P_{11} G_{1111} = 0$$

$$P_8 F_{812} + P_{12} G_{1212} = 0$$

The formulae for F_{rs} are given in Section 5 as the coefficients of P_r in the probabilities (r.s). The formulae for F_{rs} contain the variables E_A , E_B and T_I , which are functions of N , A'' , I'' , B'' , A_b , B_b and T . The numerical values of F_{rs} and G_{ss} are obtained by inserting into the formulae for F_{rs} and G_{ss} the trial value of N , and the experimentally-obtained values of A'' , I'' , B'' , A_b , B_b and T . The simultaneous equations are then solved by matrix algebra to give numerical values of the 13 probabilities, P_1 to P_{12} . (Expanded forms of the formulae for F_{rs} used in

the summations $\sum_s F_{rs} = 1$ are not used in the solution). The solution is verified by inserting the values of the 13 probabilities and the values of the formulae F_{r1} into the right-hand side of equation (12) and observing that it is equal to P_1 . Such a verification is shown in Table 2.

8. FORMULATION OF THE COINCIDENCE COUNT RATE, C'

The coincidence count rate, C' , is defined in Section 3 as the sum of the true and accidental coincidence count rates. A true coincidence is recorded when a pulse in channel A and a pulse in channel B, both generated by the same disintegration, reach the coincidence mixer. The two pulses, in general, will not reach the mixer simultaneously. To ensure that a true coincidence is recorded, there is a period, known as the resolving time, starting when the first pulse arrives at the mixer, for the duration of which the mixer waits for the arrival of the second pulse. The resolving time, T_R , is made large enough to make sure that all true coincidences are recorded. The dead time of the paralysis units, T , is set so that

$$T_R < T/2.$$

Consider a disintegration which is recorded at instant I_1^2 (Figure 44) by channel B but not by channel A. If at instant I_2^{5A} , less than T_R later, another disintegration is recorded by channel A, a coincidence will be recorded which is termed an accidental coincidence. This and other types of accidental coincidence are discussed below.

8.1 Formula for the True Coincidence Rate

For a true coincidence to be recorded, the system must be in state 1 when the disintegration occurs. The number of disintegrations per second which occur when the system is in state 1 is $N P_1$. The number which are also detected by both detectors, i.e. the number of true coincidences per second, is equal to $N P_1 E_A E_B$.

8.2 Formulae for the Accidental Coincidence Rates

The various types of accidental coincidence are classified below according to the state which the system is in when the second disintegration occurs. A formula is derived for the rate of each type.

8.2.1 State 2

Consider disintegrations which occur when the system is in state 2, and which are detected by channel A (as at instant I_2^{5A} in Figure 44). The number of such disintegrations per second is $N P_2 E_A$.

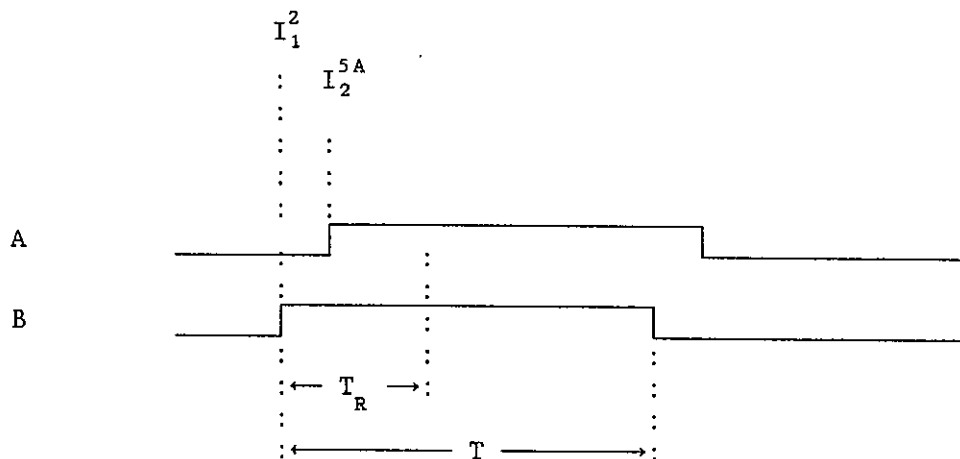


Figure 44 - Accidental coincidence in state 2

It is assumed that all degrees of overlapping of the dead times are equally probable. Therefore the number of disintegrations per second detected by A in the intervals T_R is equal to

$$N P_2 E_A \frac{T_R}{T} .$$

Because these disintegrations occur less than T_R after a disintegration detected by B (at I_1^2), the above formula is the rate for accidental coincidences whose second disintegrations are detected in state 2.

8.2.2 State 3

By similar reasoning, the rate for accidental coincidences in state 3 is equal to

$$N P_3 E_B \frac{T_R}{T} .$$

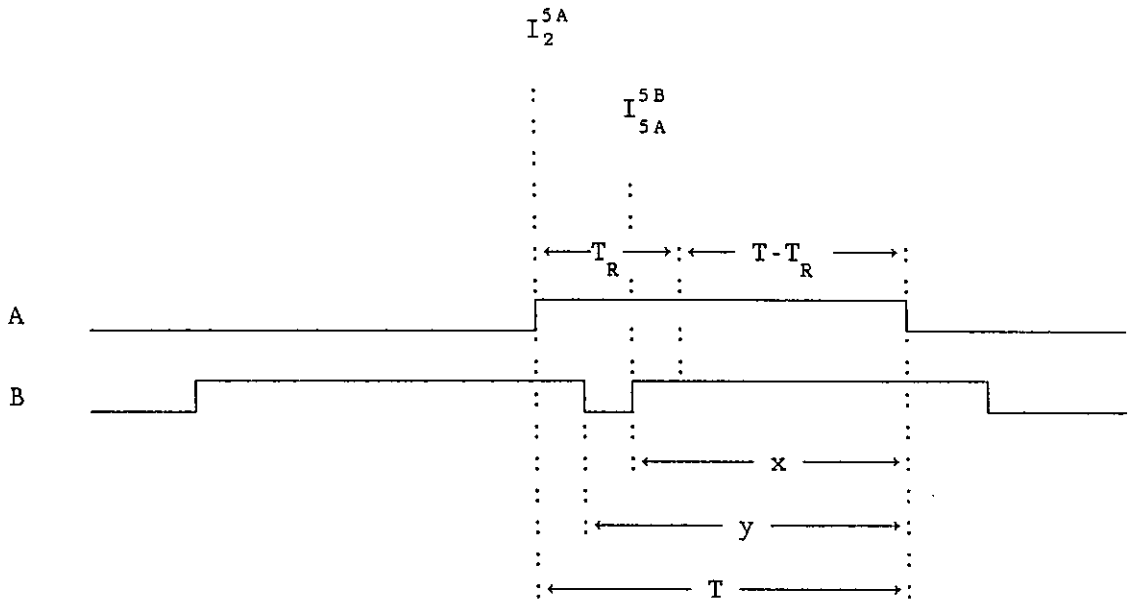


Figure 45 - Accidental coincidence in state 5A

8.2.3 State 5A

An accidental coincidence will be recorded when the disintegration detected at instant I_2^{5A} (Figure 45) is followed by one detected at instant I_{5A}^{5B} in the other channel. For an accidental coincidence to be recorded, the interval x , measured from the finish of the dead time of A, can vary from $T - T_R$ to y , and y can vary from $T - T_R$ to T .

Consider disintegrations which are detected by B at instants between x and $x + \delta x$, and which occur after B has become live at instants between y and $y + \delta y$. The number of such disintegrations, giving rise to accidental coincidences, is

$$N P_{5A} E_B \frac{\delta x}{T} \frac{\delta y}{T}.$$

The rate of accidental coincidences in state 5A

$$\begin{aligned}
 &= N P_{5A} E_B \frac{1}{T^2} \int_{T-T_R}^T \left\{ \int_{T-T_R}^y dx \right\} dy \\
 &= N P_{5A} E_B \frac{1}{T^2} \int_{T-T_R}^T \left\{ y - (T - T_R) \right\} dy
 \end{aligned}$$

$$\begin{aligned}
&= N P_{5A} E_B \frac{1}{T^2} \left\{ \frac{1}{2} \left[T^2 - (T^2 - 2TT_R + T_R^2) \right] - (T - T_R) T_R \right\} \\
&= N P_{5A} E_B \frac{1}{T^2} \left\{ \left[TT_R - \frac{1}{2} T_R^2 \right] - TT_R + T_R^2 \right\} \\
&= N P_{5A} E_B \frac{T_R^2}{2T^2} .
\end{aligned}$$

8.2.4 State 5B

By similar reasoning, it can be shown that the rate for accidental coincidences in state 5B is equal to

$$N P_{5B} E_A \frac{T_R^2}{2T^2} .$$

8.2.5 State 9

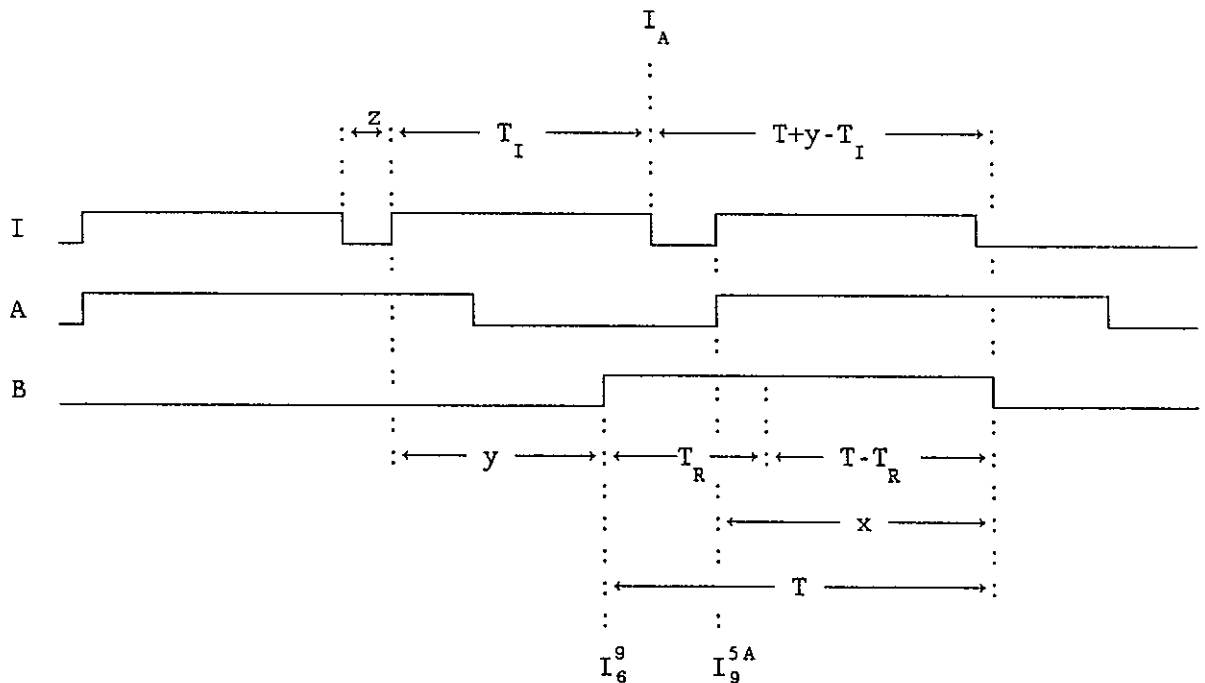


Figure 46 - Accidental coincidence in state 9

The first of the two disintegrations which give rise to an accidental coincidence (Figure 46) occurs at instant I_6^9 , and the system changes from state 6 to state 9. The second disintegration is detected by channel A at instant I_9^{5A} , before the end of the resolving time, T_R . For an accidental coincidence to occur, I_6^9 cannot precede the instant I_A by more than T_R ; i.e. the lower limit of y is $T_I - T_R$. The lower limit of x is $T - T_R$, which marks the end of the resolving time. The second disintegration cannot occur before I_A ; thus, the upper limit of x is $T + y - T_I$.

The element of the accidental-coincidence rate in state 9 is

$$N P_9 E_A \frac{\delta z}{T - T_I} \frac{\delta y}{T_I} \frac{\delta x}{T} .$$

$$\text{Let } N P_9 E_A \frac{1}{(T - T_I) T_I T} = k .$$

The rate in state 9 is equal to

$$\begin{aligned} & k \int_0^{T - T_I} \left\{ \int_{T_I - T_R}^{T_I} \left[\int_{T - T_R}^{T + y - T_I} dx \right] dy \right\} dz \\ &= k \int_0^{T - T_I} \left\{ \int_{T_I - T_R}^{T_I} \left[y - T_I + T_R \right] dy \right\} dz \\ &= k \int_0^{T - T_I} \left\{ \frac{1}{2} \left[T_I^2 - (T_I^2 - 2T_I T_R + T_R^2) \right] + (T_R - T_I) T_R \right\} dz \\ &= k \left\{ T_I T_R - \frac{1}{2} T_R^2 + T_R^2 - T_I T_R \right\} (T - T_I) \\ &= N P_9 E_A \frac{T_R^2}{2 T_I T} . \end{aligned}$$

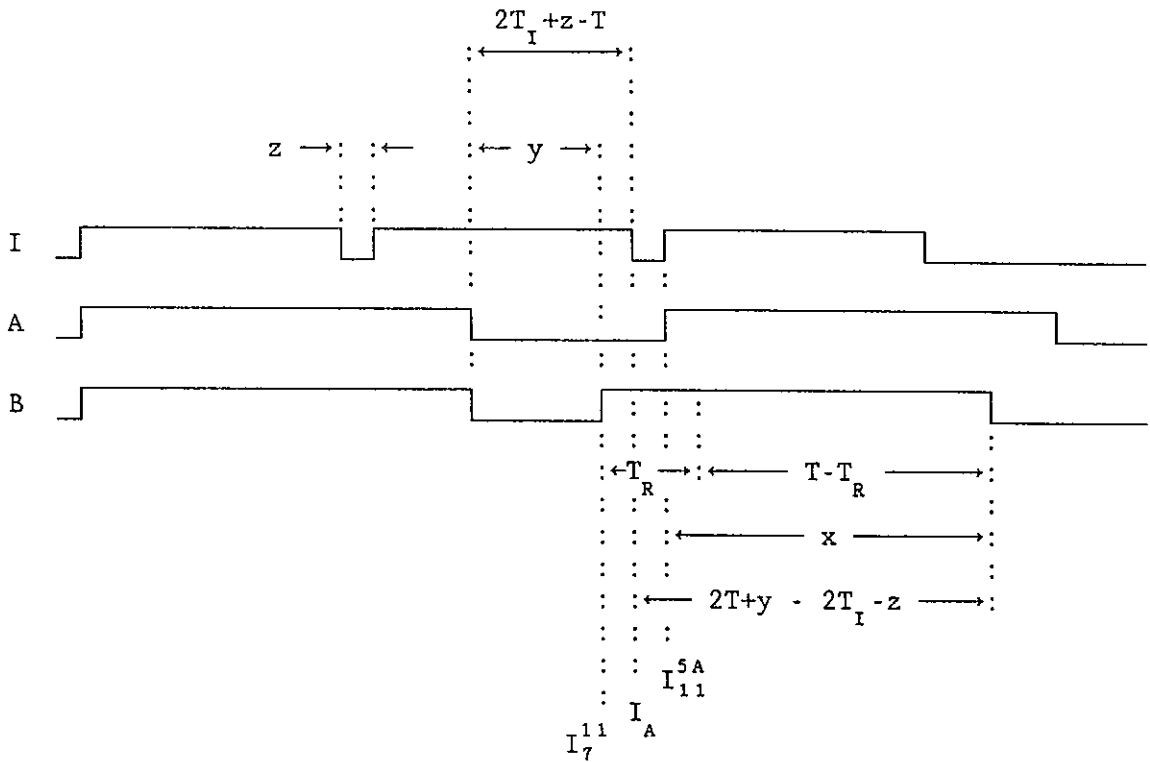


Figure 47 - Accidental coincidence in state 11, category 1

8.2.6 State 11

Accidental coincidences which occur in state 11 are subdivided into 2 categories according to whether T_R is less than, or greater than $2T_I - T$. The accidental-coincidence rate in state 11 is denoted by K_{11} . A formula for K_{11} is derived for each category.

8.2.6.1 Category 1, $T_R < (2T_I - T)$

It is first shown that the lower limit of y , when integrating with respect to y , is $2T_I + z - T - T_R$, and that this is greater than zero. An accidental coincidence is recorded when the disintegration detected at instant I_7^{11} (Figure 47) is followed within T_R by a disintegration detected at I_{11}^{5A} in the other channel. The instant I_7^{11} can never precede the instant I_A by more than an interval T_R . It can be seen that when I_7^{11} precedes I_A by T_R , the value of y is $2T_I + z - T - T_R$. The latter expression is the lower limit of y when integrating with respect to y to obtain K_{11} . The minimum value of the lower limit of y (when $z = 0$) is $2T_I - T - T_R$. Since $(2T_I - T)$ is always positive, and $T_R < (2T_I - T)$, then $2T_I - T - T_R$ is positive, and the lower limit of y , i.e. $2T_I + z - T - T_R$, is greater than zero.

$$\begin{aligned}
\text{The element of } K_{11} &= N P_{\frac{XI}{I}} E_A \frac{\delta z}{T-T_I} \frac{\delta y}{T_I} \frac{\delta x}{T} , \\
&= N P_{11} \frac{2T_I}{3T_I-T} E_A \frac{\delta z}{T-T_I} \frac{\delta y}{T_I} \frac{\delta x}{T} \\
&= k \delta z \delta y \delta x,
\end{aligned}$$

$$\text{where } k = N P_{11} E_A \frac{2}{(3T_I-T)(T-T_I)T} .$$

K_{11} , category 1, is equal to

$$\begin{aligned}
k \int_0^{T-T_I} \left\{ \int_{2T_I+z-T-T_R}^{2T_I+z-T} \left[\int_{T-T_R}^{2T+y-2T_I-z} dx \right] dy \right\} dz \\
= k \int_0^{T-T_I} \left\{ \int_{2T_I+z-T-T_R}^{2T_I+z-T} \left[T - 2T_I - z + T_R + y \right] dy \right\} dz \\
= k \int_0^{T-T_I} \left\{ (T - 2T_I - z + T_R) T_R \right. \\
\left. + \frac{1}{2} \left[(2T_I + z - T)^2 - (2T_I + z - T - T_R)^2 \right] \right\} dz ,
\end{aligned}$$

which, using the identity " $a^2 - b^2 = (a+b)(a-b)$ ",

$$\begin{aligned}
&= k \int_0^{T-T_I} \left\{ (T - 2T_I - z + T_R) T_R \right. \\
&\quad \left. + \frac{1}{2} (4T_I + 2z - 2T - T_R) T_R \right\} dz \\
&= N P_{11} E_A \frac{2}{(3T_I-T)(T-T_I)T} \int_0^{T-T_I} \left(\frac{1}{2} T_R \right) T_R dz \\
&= N P_{11} E_A \frac{T_R^2}{(3T_I-T)T} .
\end{aligned}$$

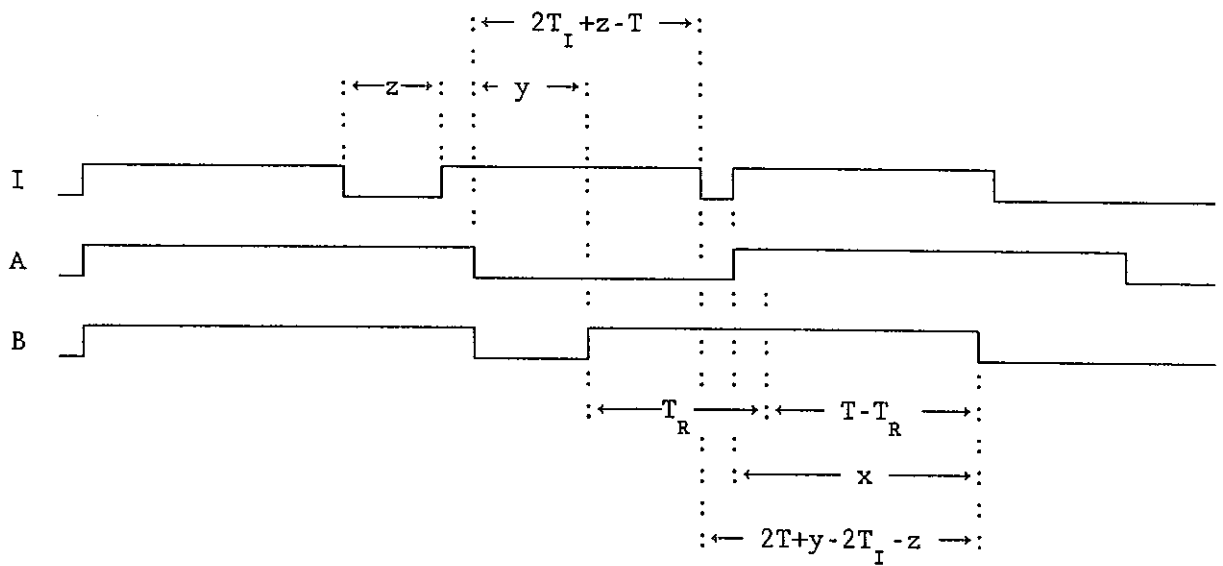


Figure 48 - Accidental coincidence in state 11, category 2, case (a)

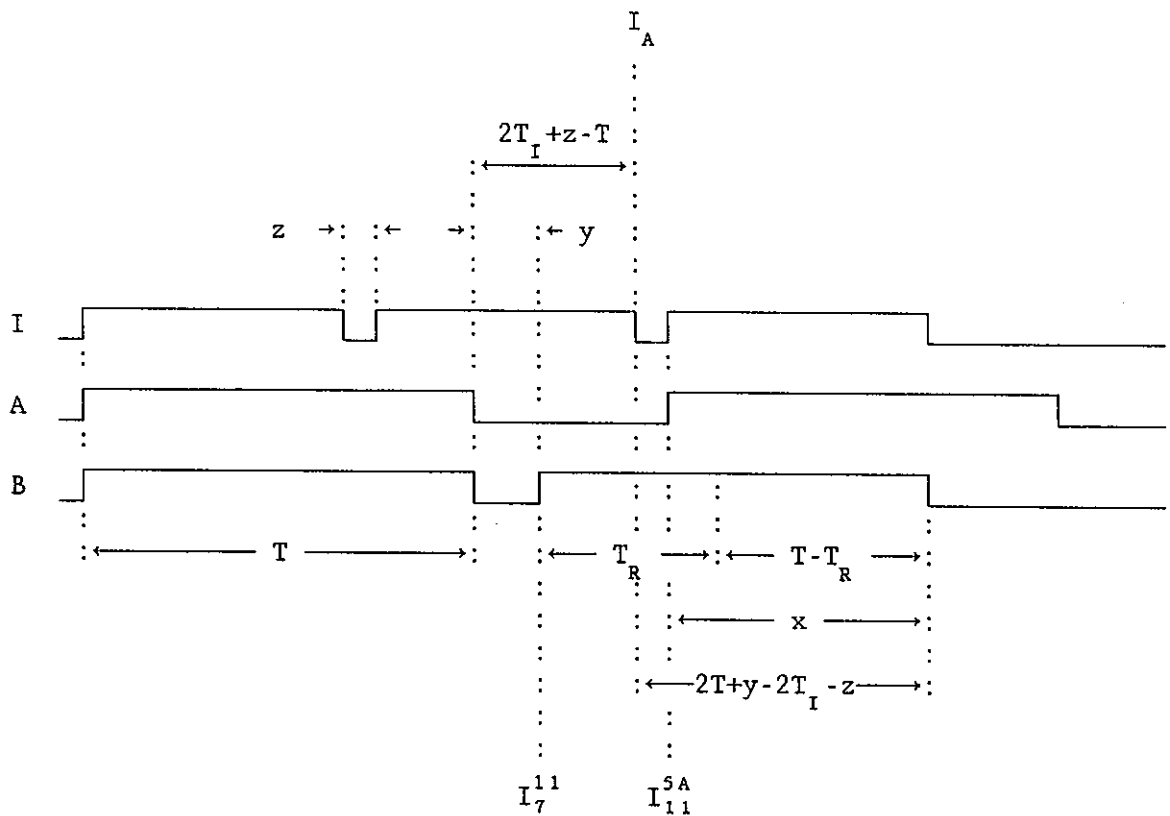


Figure 49 - Accidental coincidence in state 11, category 2, case (b)

8.2.6.2 Category 2, $T_R > (2T_I - T)$

The rate K_{11} , category 2, is the sum of the rates of 2 cases:-

In case (a), $z > (T+T_R - 2T_I)$

" " (b), $z < (T+T_R - 2T_I)$.

It can be seen in Figure 48 that in case (a) the lower limit of y is $2T_I + z - T - T_R$. It will be shown that this is greater than zero. The lower limit of y , when rearranged, is equal to $z - [T_R - (2T_I - T)]$. Since $(2T_I - T)$ is positive, then, according to the rule for category 2, the expression in square brackets is positive. According to the rule for case (a), z is greater than the expression in square brackets. Therefore, $z - [T_R - (2T_I - T)]$ is positive. Hence the lower limit of y is greater than zero.

It is now shown that in case (b) the lower limit of y is zero. It can be seen in Figure 49 that accidental coincidences occur because T_R ends after the instant I_A . According to the rule for category 2, $T+T_R > 2T_I$. Therefore, when z and y are zero, T_R will end after I_A ; thus, when z is zero, the lower limit of y is zero. The lower limit of y is also zero when $z > 0$, provided that $T+T_R > 2T_I + z$. The last inequality, rearranged, is $z < T+T_R - 2T_I$, which is the rule for case (b). Thus, in case (b) the lower limit of y is zero.

Case (a)

K_{11} , category 2, case (a), is equal to

$$k \int_{T+T_R-2T_I}^{T-T_I} \left\{ \int_{2T_I+z-T-T_R}^{2T_I+z-T} \left[\int_{T-T_R}^{2T_I+y-2T_I-z} dx \right] dy \right\} dz ,$$

which, noting that the limits of integration with respect to x and y are the same as in category 1, is equal to

$$\begin{aligned} N P_{11} E_A \frac{T_R^2}{(3T_I - T)(T - T_I) T} \int_{T+T_R-2T_I}^{T-T_I} dz \\ = N P_{11} E_A \frac{T_R^2 (T_I - T_R)}{(3T_I - T)(T - T_I) T} . \end{aligned}$$

Case (b)

K_{11} , category 2, case (b), is equal to

$$\begin{aligned}
 & k \int_0^{T+T_R-2T_I} \left\{ \int_0^{2T_I+z-T} \left[\int_{T-T_R}^{2T+y-2T_I-z} dx \right] dy \right\} dz \\
 &= k \int_0^{T+T_R-2T_I} \left\{ \int_0^{2T_I+z-T} \left[T-2T_I+T_R-z+y \right] dy \right\} dz \\
 &= k \int_0^{T+T_R-2T_I} \left\{ (T-2T_I+T_R-z)(2T_I+z-T) + \frac{1}{2} (2T_I+z-T)^2 \right\} dz \\
 &= k \int_0^{T+T_R-2T_I} \left\{ (T-2T_I+T_R)(2T_I-T) + z(T-2T_I+T_R-2T_I+T) \right. \\
 &\quad \left. - z^2 + \frac{1}{2} (2T_I-T)^2 + (2T_I-T)z + \frac{1}{2} z^2 \right\} dz \\
 &= k \int_0^{T+T_R-2T_I} \left\{ (2T_I-T) \left(\frac{1}{2} T-T_I+T_R \right) + z(T-2T_I+T_R) - \frac{1}{2} z^2 \right\} dz \\
 &= k \left\{ (2T_I-T) \left(\frac{1}{2} T-T_I+T_R \right) (T+T_R-2T_I) + \frac{1}{2} (T+T_R-2T_I)^3 - \frac{1}{6} (T+T_R-2T_I)^3 \right\} \\
 &= N P_{11} E_A \frac{2}{(3T_I-T)(T-T_I)T} (T+T_R-2T_I) \\
 &\quad \left\{ (2T_I-T) \left(\frac{1}{2} T-T_I+T_R \right) + \frac{1}{3} (T+T_R-2T_I)^2 \right\} .
 \end{aligned}$$

Case (a) + Case (b)

K_{11} , category 2, is the sum of K_{11} for cases (a) and (b), which is equal to

$$\begin{aligned}
 & N P_{11} E_A \frac{1}{(3T_I-T)(T-T_I)T} \left\{ T_R^2 (T_I-T_R) + 2(T+T_R-2T_I) \right. \\
 &\quad \left. \left[(2T_I-T) \left(\frac{1}{2} T-T_I+T_R \right) + \frac{1}{3} (T+T_R-2T_I)^2 \right] \right\} .
 \end{aligned}$$

8.2.6.3 K_{11} when $T_R = 2T_I - T$

It is shown that the formulae for K_{11} for both categories become identical when $(2T_I - T)$ is substituted for T_R . Thus,

$$K_{11}, \text{ category 1} = N P_{11} E_A \frac{(2T_I - T)^2}{(3T_I - T) T}$$

$$K_{11}, \text{ category 2} = N P_{11} E_A \frac{1}{(3T_I - T) (T - T_I) T} \left\{ (2T_I - T)^2 (T_I - 2T_I + T) \right\}$$

$$= N P_{11} E_A \frac{(2T_I - T)^2}{(3T_I - T) T}$$

which is the same as for category 1. Hence, in writing the computer programme for calculating N, the rules for the 2 categories are given as:

Category 1, $T_R < (2T_I - T)$;

Category 2, $T_R \geq (2T_I - T)$.

8.2.7 State 12

Accidental coincidences which occur in state 12 are subdivided into 2 categories according to whether T_R is less than, or greater than $T - T_I$. The rate of accidental coincidences in state 12 is denoted by K_{12} . A formula for K_{12} is derived for each category.

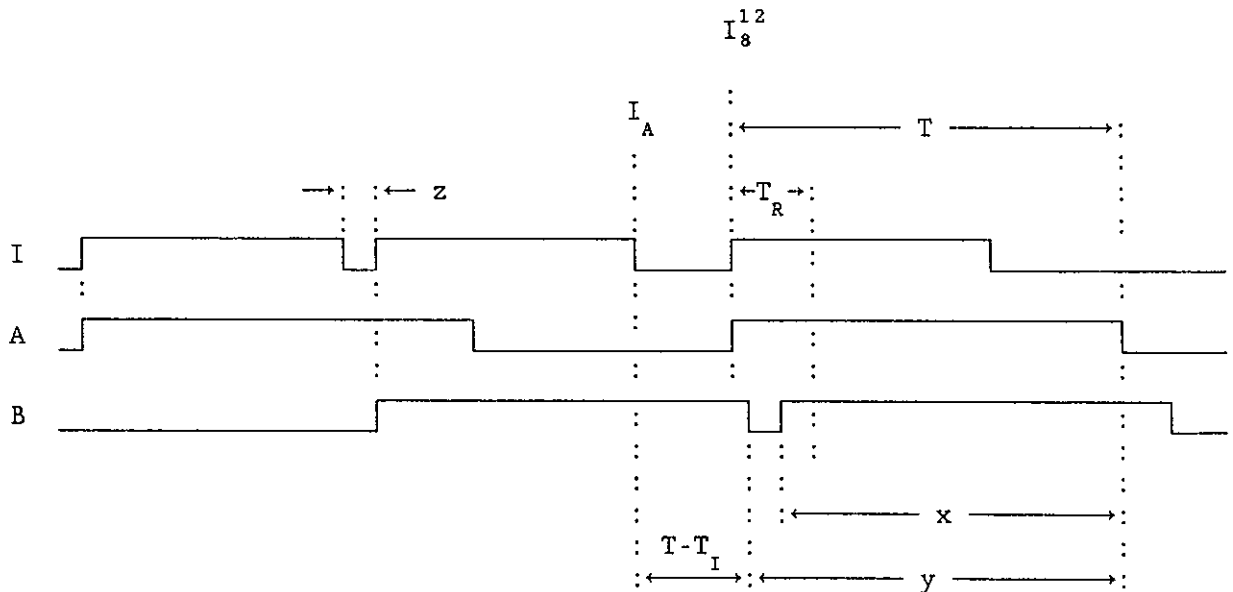


Figure 50 - Accidental coincidence in state 12, category 1

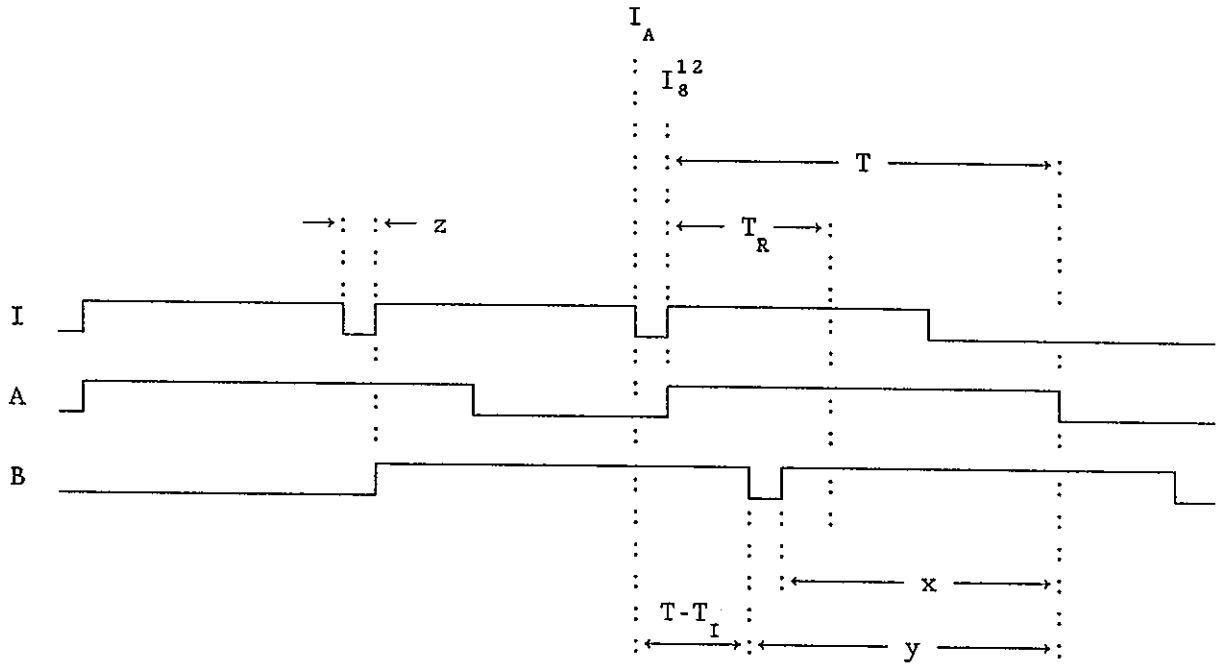


Figure 51 - Accidental coincidence in state 12, category 2

8.2.7.1 Category 1, $T_R < (T - T_I)$

For accidental coincidences to occur, the end of T_R (Figure 50) must be later than the end of the interval labelled " $T - T_I$ ". Thus the minimum value of the interval between the instants I_A and I_8^{12} is $T - T_I - T_R$, and the lower limit of y is $T - T_I$.

The element of K_{12} is

$$N P_{12} E_B \frac{\delta z}{T - T_I} \frac{\delta y}{T - T_I} \frac{\delta x}{T}$$

K_{12} , category 1, is equal to

$$\begin{aligned}
 N P_{12} E_B & \frac{1}{(T - T_I)^2 T} \int_0^{T - T_I} \left\{ \int_{T - T_R}^T \left[\int_{T - T_R}^y dx \right] dy \right\} dz \\
 & = N P_{12} E_B \frac{1}{(T - T_I)^2 T} \int_0^{T - T_I} \left\{ \int_{T - T_R}^T \left[y - (T - T_R) \right] dy \right\} dz \\
 & = N P_{12} E_B \frac{1}{(T - T_I)^2 T} \int_0^{T - T_I} \left\{ \frac{1}{2} \left[T^2 - (T^2 - 2TT_R + T_R^2) \right] - (T - T_R) T_R \right\} dz
 \end{aligned}$$

$$\begin{aligned}
&= N P_{12} E_B \frac{1}{(T-T_I)^2 T} \left\{ T T_R - \frac{1}{2} T_R^2 - T T_R + T_R^2 \right\} (T-T_I) \\
&= N P_{12} E_B \frac{T_R^2}{2 (T-T_I) T} .
\end{aligned}$$

8.2.7.2 Category 2, $T_R > (T-T_I)$

In this category, even when the interval between I_A and I_8^{12} (Figure 51) is zero, the end of T_R is later than the end of the interval labelled " $T-T_I$ ". Thus the lower limit of y is T_I .

K_{12} , category 2, is equal to

$$\begin{aligned}
&N P_{12} E_B \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ \int_{T_I}^T \left[\int_{T-T_R}^y dx \right] dy \right\} dz \\
&= N P_{12} E_B \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ \int_{T_I}^T \left[y - (T-T_R) \right] dy \right\} dz \\
&= N P_{12} E_B \frac{1}{(T-T_I)^2 T} \int_0^{T-T_I} \left\{ (T^2 - T_I^2)/2 - (T-T_R) (T-T_I) \right\} dz \\
&= N P_{12} E_B \frac{1}{(T-T_I) T} \left\{ (T+T_I)/2 - (T-T_R) \right\} (T-T_I) \\
&= N P_{12} E_B \frac{T_R - (T-T_I)/2}{T} .
\end{aligned}$$

8.2.7.3 K_{12} when $T_R = T-T_I$

When $(T-T_I)$ is substituted for T_R in the two formulae derived above for K_{12} , the same expression is obtained:

$$N P_{12} E_B \frac{T-T_I}{2T} .$$

Hence, in writing the computer program for calculating N, the rules for the 2 categories are given as:

Category 1, $T_R < (T - T_I)$;

Category 2, $T_R \geq (T - T_I)$.

8.3 The Equation for the Coincidence Count Rate, C'

The equation for C' is obtained by adding together the formulae derived in Sections 8.1 and 8.2. Thus,

$$C' = N \left[P_1 E_A E_B + (P_2 E_A + P_3 E_B) \frac{T_R}{T} + (P_{5A} E_B + P_{5B} E_A) \frac{T_R^2}{2T^2} + P_9 E_A \frac{T_R^2}{2T_I T} \right] + K_{11} + K_{12},$$

where:

if $T_R < (2T_I - T)$,

$$K_{11} = N P_{11} E_A \frac{T_R^2}{(3T_I - T) T};$$

if $T_R \geq (2T_I - T)$,

$$K_{11} = N P_{11} E_A \frac{1}{(3T_I - T) (T - T_I) T} \left\{ T_R^2 (T_I - T_R) + 2 (T + T_R - 2T_I) \left[(2T_I - T) \left(\frac{1}{2} T - T_I + T_R \right) + \frac{1}{3} (T + T_R - 2T_I)^2 \right] \right\};$$

if $T_R < (T - T_I)$,

$$K_{12} = N P_{12} E_B \frac{T_R^2}{2 (T - T_I) T};$$

if $T_R \geq (T - T_I)$,

$$K_{12} = N P_{12} E_B \frac{T_R - (T - T_I)/2}{T}.$$

9. THE CALCULATION OF THE DISINTEGRATION RATE, N

A trial value of N is calculated by inserting the experimental data into the coincidence-counting equation of Bryant [1963], which is accurate only at low count rates. This first trial value of N is used to solve the simultaneous equations for the 13 probabilities P_1 to P_{12} , as described in Section 7. The numerical values of P_1 , P_2 , P_3 , P_{5A} , P_{5B} , P_9 , P_{11} and P_{12} , the trial value of N , and the experimental values of T , A'' , I'' , B'' , A_b , B_b and T_R are inserted into the equation for C' given in Section 8.3. Unless A'' is low, the value of C' so obtained will not be equal to the experimentally-determined coincidence count rate from which the background has been subtracted. Let the difference be denoted by $\Delta C'$. The trial value of N is progressively reduced until the difference $\Delta C'$ is less than 0.01. The disintegration rate, N , is a trial value of N for which $\Delta C'$ is less than 0.01 counts per second.

After N has been obtained, equation (12) is used to verify the solution of the simultaneous equations. This is done by inserting the values of P_1 to P_{12} , and F_{r1} into the right-hand side of equation (12), which is then found to be equal to P_1 .

10. EXPERIMENTAL VERIFICATION

10.1 The Coincidence-counting Equipment

A block diagram of the $4\pi\beta$ - γ coincidence-counting system used in the Radioisotope Standards Laboratory is shown in Figure 1. The emission of β particles from the radioactive source was detected by the $4\pi\beta$ proportional counter which was filled with a gas mixture consisting of 90% argon and 10% methane. The paralysis units of channels A and B were set at a dead time of 3.88 μs . Because cobalt-60 sources were used in the verification, the upper and lower levels of the γ timing single-channel analyser were set to select the counts in the cobalt-60 photopeaks at 1173 and 1332 keV. The resolving time of the coincidence mixer was set at 0.92 μs .

10.2 Setting the Delay in the γ Channel

Delay measurements showed that when a coincidence occurred, the β pulse was generally generated later than the γ pulse. If the β and γ channels had equal delays after the detectors, the β pulse arrived at the coincidence mixer after the γ pulse. Moreover, the time-lag of the β pulses relative to their coincident γ pulses was variable.

A delay was imposed in the γ channel by the Gate and Delay Generator, so that half the β pulses arrived at the coincidence mixer before their coincident γ pulses, and the other half arrived after [NCRP 1985: 78]. This was done to minimise disconformity with the assumption made in the above derivation that when a coincidence occurs, both paralysis units go dead at exactly the same instant.

10.3 Preparation of Cobalt-60 Sources

The source mounts consisted of plastic films, VYNS or Mylar, which had been coated with gold-palladium alloy. A few micrograms of Catanac spreading agent in an aqueous solution were deposited on the source mounts. Weighed amounts of cobalt-60 solution were deposited on the source mounts from a syringe-pycnometer [Wyllie 1986] and evaporated to dryness. The cobalt-60 residues were then covered with gold-palladium coated VYNS film. Details of the sources and their preparation are given elsewhere [Wyllie 1989a].

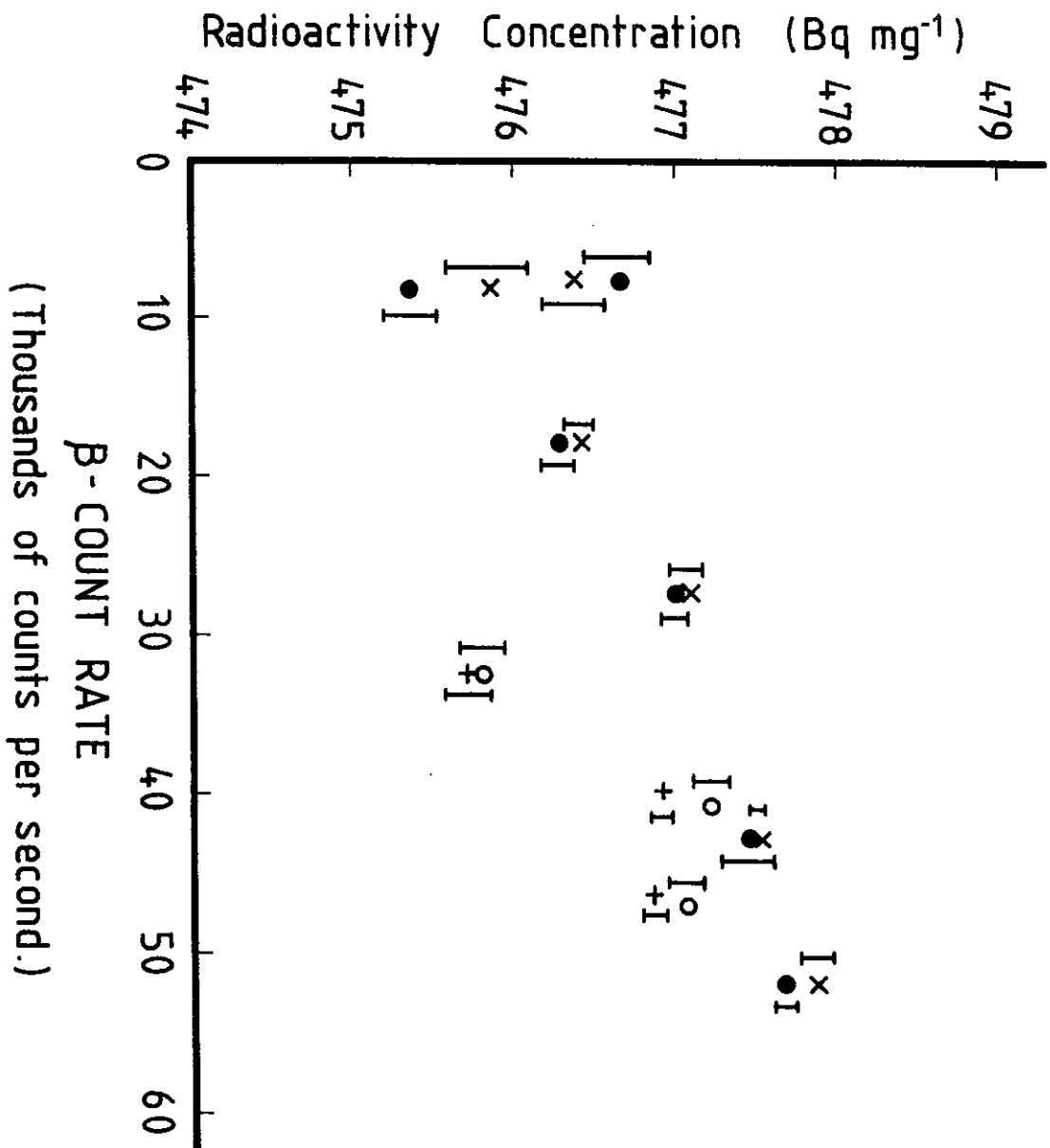
10.4 Varying the Intrinsic Dead Time

As reported earlier [Wyllie 1989b], the intrinsic dead time can be varied by setting different values on the delay control of the timing single-channel analyser in the β channel. When the delay is less than the average width of the β -amplifier output pulses at the discriminator level, the intrinsic dead time is approximately equal to the average pulse width. When the delay is greater than the average pulse width, the intrinsic dead time is approximately equal to the delay.

10.5 Results and Discussion

Two series of radioactivity measurements were made. In the first series the β delay was set at the minimum of $0.2 \mu\text{s}$. The intrinsic dead times were then in the range $2.5 - 2.7 \mu\text{s}$. The dead time set in the paralysis units, $3.88 \mu\text{s}$, satisfied the condition $T/2 < T_I < T$. In the second series the β delay was increased to give intrinsic dead times in the range $3.2 - 3.4 \mu\text{s}$.

The radioactivities of the sources were calculated by the equation of Section 8.3, and then divided by the masses of solution deposited on the source mounts. The resulting values for the radioactivity concentration of the cobalt-60 solution are shown in Figure 52 plotted against the β count rates of the sources.



Source Mount	Intrinsic Dead Time (μs)	
	2.5-2.7	3.2-3.4
VYNS	●	X
Mylar	○	+

Figure 52 - The radioactivity concentration of a cobalt-60 solution, calculated by using the equation of Section 8.3, is plotted against the β count rates of six sources on VYNS and three on Mylar. The set dead time is 3.88 μ s. The intrinsic dead times, which are altered by varying the delay in the β S.C.A., lie within 2 ranges: 2.5 - 2.7 μ s and 3.2 - 3.4 μ s. The bars represent plus and minus one standard deviation.

The radioactivity concentration was calculated also by using the previously published equation [Wyllie 1987a,b], in the derivation of which the effect of intrinsic dead time was not considered.

The values so obtained were higher than those plotted in Figure 52, and thus deviated further from the required constant value. Figure 53 shows the percentage improvement achieved by using the equation of Section 8.3. The graph in Figure 53 was constructed as follows.

The values for the radioactivity concentration in Figure 52 were subtracted from the corresponding values which were obtained by using the earlier equation. The differences in radioactivity concentration, expressed as percentages of the values in Figure 52, were plotted against the β count rates. These differences represent the percentage improvement obtained by considering the effect of intrinsic dead time. Thus, when the β count rate was $40,000 \text{ s}^{-1}$ and the intrinsic dead time was in the range 2.5 to 2.7 μs , there was an improvement of 0.07%.

It appears from the results shown in Figure 53 that the extra correction for the extension of the set dead time by the intrinsic dead time decreases as the intrinsic dead time approaches the set dead time. This is to be expected from the fact that when $T - T_I$ is zero, the set dead time cannot be extended (see Figure 43).

The intrinsic dead time may vary from pulse to pulse. It may be dependent on the count rate and the height of the pulse at the output of the β pre-amplifier. The method for the measurement of the intrinsic dead time described above has the advantage of being carried out concurrently with the coincidence counting. Thus, the value of the intrinsic dead time used in the correction equation is obtained under the same conditions as the other data used in the determination of the disintegration rate.

As can be seen from Figure 52, the calculated radioactivity concentration rises as the β count rate gets higher, instead of remaining constant. This is due, in part, to the fact that the above derivation assumes that when a true coincidence occurs, the paralysis units of both channels go dead simultaneously (Section 1.4).

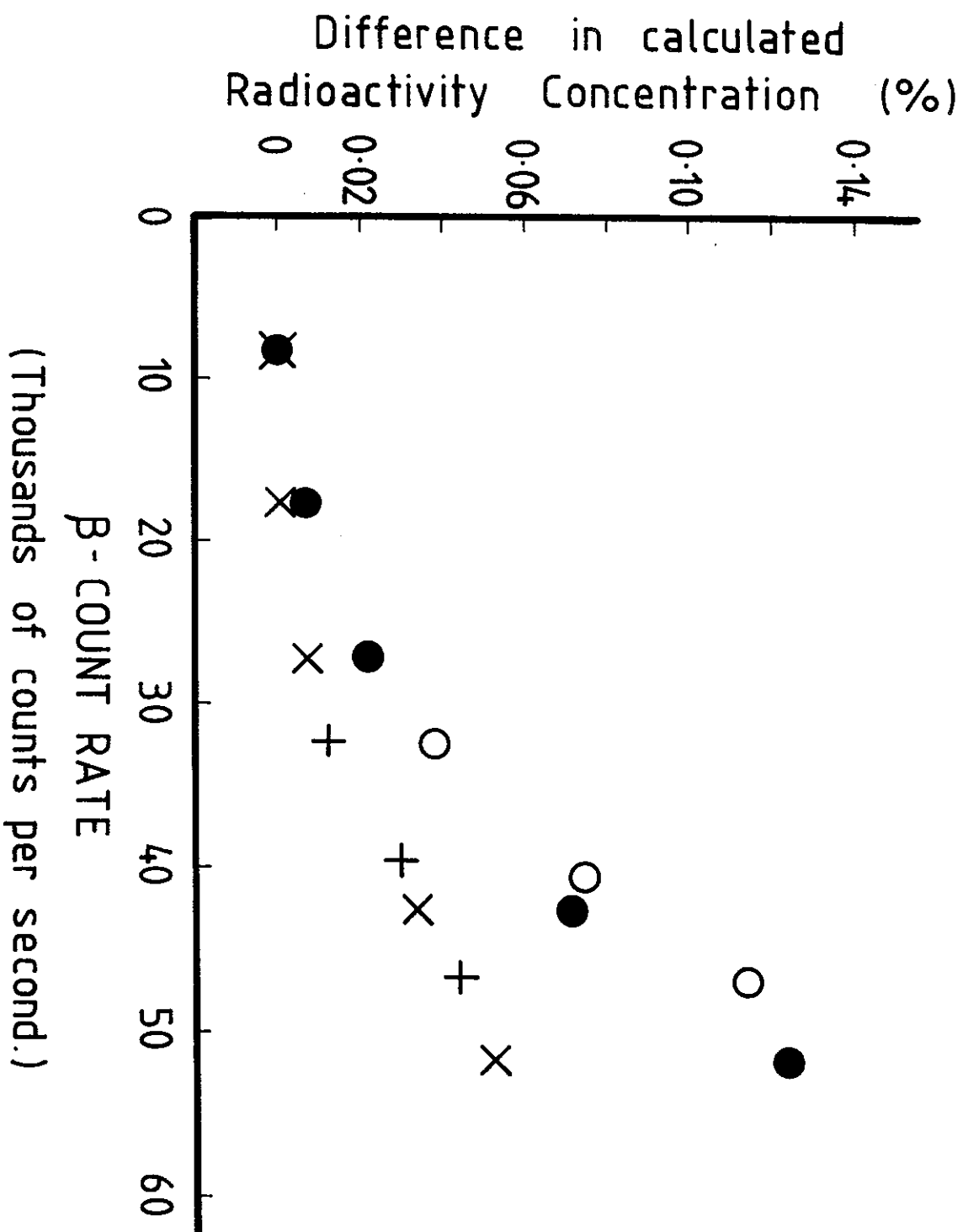


Figure 53 - The difference between the radioactivity concentrations calculated by two equations. The values plotted in Figure 52 are subtracted from the higher values calculated by using the earlier equation [Wyllie 1987a,b]. The differences are plotted as percentages of the values in Figure 52, and represent the percentage improvement obtained by considering the effect of intrinsic dead time.

11. CONCLUSIONS

When determining the disintegration rate of a radioactive source by coincidence counting, the equation of Section 8.3, which takes the intrinsic dead time of the β channel into account, is more effective than an earlier equation [Wyllie 1987a,b] in correcting for dead-time losses and accidental coincidences. Further improvement will require consideration of the variable interval between the arrival of the γ pulses and their coincident β pulses at the coincidence mixer.

12. ACKNOWLEDGEMENT

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