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AUSTRALIAN ATOMIC ENERGY COMMISSION

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July 1974

ISBN 0 642 99643 1

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ABSTRACT

A simple statistical model for the process of forming a compound nucleus is related. It is found that the model predicts extremely small energy spreads for internal single particle states.

National Library of Australia card number and ISBN 0 642 99643 1

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BETA DECAY; BREIT-WIGNER FORMULA; COMPOUND NUCLEI; COMPOUND-NUCLEUS REACTIONS; ENERGY LEVELS; ENERGY SPECTRA; ENERGY TRANSFER; FERMI GAS MODEL; LIFETIME; LINEAR MOMENTUM; NEUTRONS; NEUTRON REACTIONS; NUCLEAR MODELS; RESONANCE; STATISTICAL MODELS; TEMPERATURE DEPENDENCE

CONTENTS

1. INTRODUCTION

In the conventional theory that describes the interaction of low energy neutrons with nuclei, the phenomenon of the appearance of large resonances in the nuclear cross sections can be explained in quantum mechanical terms. The derivation of the single level formula by Breit & Wigner (1936), and the full multilevel theory of Wigner & Eisenbud (1947), attribute the resonance behaviour to the strong reinforcement of the incident neutron's de Broglie waves at particular energies corresponding to possible energy states of the compound system.

On the other hand, such resonance properties as the way in which level densities vary through the periodic table are explained in terms of the behaviour of a Fermi gas (Bethe 1937; Gilbert & Cameron 1965). These two approaches complement each other, and each selects simplified features of nuclear properties to explain the observations. Yet they appear to be somewhat in contrast since the first employs the concept of a potential, hence a field, which produces a strong resonance, while the second attributes the separation in energy of these resonances to the characteristics of a large number of fermions in a gaseous state. This situation raises the question as to whether one can understand the features of an isolated resonance of the compound nucleus in statistical mechanical terms and, likewise, derive valid expressions for the distribution of level densities using ordinary quantum theory. In this report the former question is investigated in terms of a much simplified model and conditions for a Breit-Wigner level to occur are given.

2. DISCUSSION OF THE MODEL

The model proposed here describes the formation of a compound nucleus in three distinct stages:

(i) The target nucleus is pictured as a system of A nucleons moving in an unspecified way in a surrounding meson field that is approximately spherical. It is assumed that the incoming neutron encounters first the meson field of the nucleons which are internally in their ground states. It increases in energy as it penetrates the field and, at the same time, imparts energy to the nuclear system via the field, thus heating up the nucleon system.

(ii) A direct internal collision takes place with a nucleon, then a second and so on, until the neutron has lost enough internal kinetic energy and the other nucleons have gained enough energy to reach a metastable system in approximate thermodynamic equilibrium.

(iii) At the stage where the equilibrium configuration is reached, the possible quantum states of the neutron are fairly well separated, and the changes in its kinetic energy must be considered to be discrete.

3. PENETRATION OF NUCLEAR FIELD

Let the energy of the incident neutron in the overall centre-of-mass system be E and its velocity in the same system be $\underset{\sim}{v}$. In traversing the surface potential field at a particular value of E, let the change in potential energy be Φ and the subsequent internal velocity of the neutron be $\underset{\sim}{v}_1$. If the neutron has mass m, then by conservation of energy

$$E = \tfrac{1}{2} m \underset{\sim}{v}^2 = \tfrac{1}{2} m \underset{\sim}{v}_1^2 + \Phi \; ; \; \Phi < 0 \quad . \qquad (1)$$

Φ is assumed to be somewhat larger than E; it is assumed also to be a fixed quantity. The precise nature of the penetration effect can be understood only in quantum mechanical terms, but to represent its influence we define a penetration number by

$$N\,dv = A\,\delta(v_1 - \sqrt{v^2 + \frac{2}{m}\,|\Phi|})\,dv_1 \quad , \tag{2}$$

and A is a nuclear constant to be determined, by normalising the formation probability to unity over energy. The number of neutrons reaching the nuclear interior in a statistical sample of reactions is, therefore, zero at all energies except E, *i.e.* we are considering a monoenergetic incident beam.

4. COLLISION SEQUENCE

After gaining the large internal velocity v_1, the neutron collides with another nucleon and an exchange of energy and momentum takes place. It is assumed that the initial agitation of the field, though probably small, has the effect of slightly smearing out the discreteness of the energy of the colliding nucleon which occupies a Fermi level. Furthermore, we suppose that before the first collision and before all subsequent collisions, N in number, the nucleons' velocities are normally distributed about a mean value, which would be the value if the nucleon level were of infinitesimal spread, *i.e.*

$$P_j(u_{jk})du_{jk} = \left(\frac{m}{2\pi kt_j}\right)^{1/2} \exp(-m(u_{jk} - \langle u_{jk}\rangle)^2/2kt_j) \times \\ \times du_{jk} \quad , \tag{4}$$

where

where	m	=	nucleon mass,
	k	=	Boltzmann's gas constant,
	t_j	=	level temperature,
	u_{jk}	=	nucleon velocity before jth collision for kth Cartesian component,
and	$P_j(u_{jk})$	=	probability density for values of u_{jk} .

Let P_{jk} be a component of the neutron's momentum, q_{jk} be the component of the incident nucleon's momentum and q'_{jk} be its recoil momentum. By conservation of momentum at each collision

$$P_{jk} + q_{jk} = p_{j+1,k} + q'_{jk} \tag{5}$$

where

$$P_{jk} = m\,v_{jk}\ ,\quad q_{jk} = m\,u_{jk} \quad . \tag{6}$$

The distribution (4) in terms of nucleon momentum is

$$P_j(q_{jk})dq_{jk} = (2\pi mkt_j)^{-1/2} \exp(-[q_{jk} - \langle q_{jk}\rangle]^2/2mkt_j) \times dq_{jk} \tag{7a}$$

$$P_j(q'_{jk})dq'_{jk} = (2\pi mk\tau)^{-1/2} \exp(-[q'_{jk} - \langle q'_{jk}\rangle]^2/2mk\tau_j)\,dq'_{jk} \tag{7b}$$

where τ_j is the level temperature of the state of the recoil nucleon. From Equations (7a), (7b) and (5), we find that the distribution of the neutron momentum transfer per collision is obtained as

$$\int_{-\infty}^{\infty} dq_{jk}\ (2\pi mk)^{-1}(t_j\tau_j)^{-1/2} \exp(-[q_{jk} - \langle q_{jk}\rangle]^2/2mkt\ ) \times \\ \times \exp(-[q'_{jk} - \langle q'_{jk}\rangle]^2/2mk\tau_j\ ) \times dq_{jk} \tag{8}$$

which, after applying Equation (5), yields

$$\bar{P}_j \, dp_{j+1,k} = (2\pi mkT_j)^{-1/2} \exp(-(p_{j+1,k} - p_{j,k} - \Delta_{jk})^2 / 2mkT_j) \times$$
$$\times \, dp_{j+1,k} \quad , \tag{9}$$

where
$$\Delta_{jk} = \langle q'_{jk} \rangle - \langle q_{jk} \rangle \tag{10a}$$

and
$$T_j = t_j + \tau_j \quad . \tag{10b}$$

If we form the joint probability distribution for a sequence of collisions ending in a momentum for the neutron of $p_{N,k}$ for the kth component, and then integrate over all intermediate neutron momenta from $j = 2$ to $N-1$, we obtain the probability density of reaching this state as

$$P_N \, dp_{Nk} = \sqrt{\frac{\beta}{\pi}} \exp(-\beta(p_{1k} - p_{Nk} - \Delta_k)^2) \, dp_{Nk} \quad , \tag{11}$$

where
$$\Delta_k = \sum_{j=1}^{N} \Delta_{jk} = \langle q_{1k} \rangle - \langle q_{Nk} \rangle \quad , \tag{12a}$$

$$\beta = \left(\sum_{j=1}^{N} \frac{1}{\beta_j} \right)^{-1} \quad , \tag{12b}$$

$$\beta_j = (2mkT_j)^{-1} \quad . \tag{12c}$$

The 'temperature' for this distribution is

$$T = \sum_{j=1}^{N} t_j + \sum_{j=1}^{N} \tau_j \quad . \tag{13}$$

One would expect the temperatures t_j to increase as the neutron loses energy and imparts it to the system, although the fact that the neutron loses energy in each collision does not follow from the above conditions alone, but from conservation of energy as well.

5. DISCRETE LEVELS

The total energy of the system remains fixed throughout the neutron's internal collision processes and, in the overall centre-of-mass system where the compound state is at rest, the possible discrete energies that can be assumed by the neutron are well separated. With reference to the whole configuration, it is postulated that, at the Nth collision, the neutron encounters a subsequent energy that coincides with the energy of a single-particle state that has an abnormally low level temperature or, in other words, a state with the minimal of energy spread. At this stage, the neutron's momentum p_N must be much less than its initial momentum p_1. If its energy is ϵ_N and the energy of the state is ϵ_0 then, because of the narrow spread of the state, the probability density with reference to energy can be taken as $\delta(\epsilon_N - \epsilon_0)$. We then define

$$\frac{\underset{\sim}{P}_N^2}{2m} = E_N \; ; \; \frac{\underset{\sim}{p}_N^2}{2m} = \epsilon_N = \text{energy after Nth collision} ,$$

$$\frac{\underset{\sim}{P}_r^2}{2m} = E_r \; ; \; \frac{\underset{\sim}{p}_0^2}{2m} = E_0 = \text{energy of discrete state} ,$$

$$E_N = \epsilon_N + |\Phi| \; ; \; E_r = E_0 + |\Phi| \quad .$$

In terms of the magnitude of P_N, therefore, the probability density for this particular state is given by

$$\mathcal{P}_B(P_r)\,dP_r = \delta(P_N - P_r)\,dP_r \quad , \tag{14}$$

after averaging over angles, and the state has *zero* spread in P_r. The net momentum transfer at this stage is

$$\underset{\sim}{\Delta} \equiv \Delta_i \quad ; \quad i = 1, 2, 3.$$

and the total momentum loss of the neutron must be

$$\underset{\sim}{P}_N = \underset{\sim}{p}_N + \underset{\sim}{\Delta} \quad .$$

The number of neutrons partaking in the three-stage process for each momentum interval dP_1, $d\underset{\sim}{P}_N$, and dP_r is, therefore

$$\mathcal{P}\,dp_1\,d\underset{\sim}{P}_N\,dP_r = A\,\delta(p_1 - \sqrt{p^2 + 2m\,|\Phi|})\,dp_1 \;\text{x}$$

$$\text{x}\;\pi^{-\frac{3}{2}}\,\beta^{\frac{3}{2}}\exp(-\beta\,[\underset{\sim}{p}_1 - \underset{\sim}{P}_N]^2)\,d\underset{\sim}{P}_N \;\text{x}\; \delta(P_N - P_r)\,dP_r \tag{15}$$

where $p = |\underset{\sim}{p}|$ and $\underset{\sim}{p}$ is the incident neutron's momentum. Let us suppose that this momentum is in the direction of the z axis, and that the field deflects the neutron very little before the first collision. Then, integrating Equation (15) over the relative angles between $\underset{\sim}{p}_1$ and $\underset{\sim}{P}_N$, we obtain

$$dp_1\,dP_N\,dP_r = A\,\delta(p_1 - \sqrt{p^2 + 2m\,|\Phi|})\,dp_1 \;\text{x}$$

$$\text{x}\;\pi^{-\frac{1}{2}}\beta^{\frac{1}{2}}\{\exp(-\beta\,[p_1 - P_N]^2) - \exp(-\beta\,[p_1 + P_N]^2)\}\;\text{x}$$

$$\text{x}\;\frac{p_1}{P_N}\,dP_N\;\delta(P_N - P_r)\,dP_r \quad . \tag{16}$$

We can now integrate out the delta functions to obtain a net probability of formation of

$$\mathcal{P}(p,\beta) = \frac{A}{\sqrt{\pi}}\beta^{\frac{1}{2}}\,\frac{p_1}{P_N}\,\{\exp[-\beta(p_1 - P_r)^2] - \exp[-\beta(p_1 + P_r)^2]\} \tag{17}$$

in which we substitute

$$p_1 = \sqrt{p^2 + 2m\,|\Phi|} \quad . \tag{18}$$

Now the parameter β, defined by Equation (12a), will vary from one collision sequence to another, as the degree of nuclear heating depends upon the details of the collision sequence itself. If one views a large number of collision sequences, then it would be expected that, in an infinitesimal interval $d\beta$, the probability of obtaining a specific value of β would be given by $\mathcal{P}_\beta d_\beta$, where $\mathcal{P}_\beta$ is some function to be determined. It must satisfy

$$\int_0^{B_{max}} \mathcal{P}_\beta(\beta)\,d\beta = 1 \tag{19}$$

where $0 \leqslant \beta \leqslant \beta_{max}$.

6. DETERMINATION OF THE TEMPERATURE SUM DISTRIBUTION

If we neglect the energy dependence of the neutron width in the neighbourhood of the compound state level, then the Breit-Wigner (1936) single level formula predicts that the probability of forming a compound nucleus from any channel C is proportional to

$$\mathcal{P}_{CN} = \frac{1}{\pi} \cdot \frac{\Gamma/2}{(E-E_0)^2 + \Gamma^2/4} \quad . \qquad (20)$$

The form (20) was selected to give a normalisation

$$\int_0^\infty \mathcal{P}_{CN}\, dE \approx 1$$

to a close approximation. Γ is the total width of the compound energy state. Referring to Equation (17), it is seen that $|\Phi|$, which is of order of the neutron binding energy, about 8 MeV, is such that

$$p_1 >> 1/\beta_{max} \; ; \quad P_r >> 1/\beta_{max} \qquad (21)$$

because the energy spread in each interval neutron state is supposedly extremely small, and so is the momentum spread, or the sum of level temperatures. Therefore, only the first exponent in Equation (17) contributes significantly to the integral with respect to β. For any entrance channel one should find

$$\frac{\Gamma}{2\pi} \cdot \frac{1}{(E-E_0)^2 + \Gamma^2/4} = \frac{A}{\sqrt{\pi}} \int_0^{\beta_{max}} d\beta \;\beta^{1/2}\, \mathcal{P}_\beta(\beta)\, \frac{p_1}{P_N} \times$$

$$\times \exp(-\beta(p_1 - P_r)^2) \; . \qquad (22)$$

We put $\frac{p_1}{P_r} \approx 1$, owing to the largeness of $|\Phi|$ and substitute the approximate relation

$$p_1 - P_r = p^2 + 2m|\Phi| - 2m(E_0 + |\Phi|)$$

$$\approx \frac{m}{2|\Phi|} \cdot (E-E_0) \; ; \quad |\Phi| >> E, E_0, \qquad (23)$$

which is obtained from the Taylor expansions of the square roots. This yields

$$\frac{\Gamma}{2\pi} \cdot \frac{1}{(E-E_0)^2 + \Gamma^2/4} = \frac{A}{\sqrt{\pi}} \int^{\beta_{max}} d\beta \,.\, \beta^{1/2}\, \mathcal{P}_\beta(\beta) \exp \frac{-\beta m}{2|\Phi|} (E-E_0)^2 \, . \qquad (24)$$

Now we suppose that when $\beta \sim \beta_{max}$, the distribution function $\mathcal{P}_\beta(\beta)$ becomes negligibly small, so we can replace β_{max} by infinity in the limit of the integral. Let

$$y = \frac{m}{2|\Phi|} (E-E_0)^2 \; ; \quad \epsilon = \frac{m\Gamma^2}{8|\Phi|} \quad . \qquad (25)$$

The constant A is found by normalising the right-hand side of Equation (24) to unity after integrating over E. This yields

$$A = \sqrt{\frac{m}{2|\Phi|}} \quad , \qquad (26)$$

and Equation (24) becomes

$$\frac{\Gamma}{2\pi} \cdot \frac{m}{2|\Phi|} \cdot \frac{1}{y+\epsilon} = \sqrt{\frac{m}{2\pi|\Phi|}} \cdot \int_0^\infty d\beta \,.\, \beta^{1/2}\, \mathcal{P}_\beta(\beta)\, e^{-y\beta} \,. \qquad (27)$$

Taking the inverse Laplace transform of Equation (27) we get

$$\mathcal{P}_\beta(\beta)\, d_\beta = \sqrt{\frac{\epsilon}{\pi\beta}} \quad e^{-\epsilon\beta}\, d\beta \,, \qquad (28)$$

as the probability density distribution for the temperature sequence sum. We find the averaged value of β to be

$$<\beta> = 1/2\epsilon = \frac{4|\Phi|}{m\Gamma^2} = (2mK)^{-1} \left\langle \frac{1}{T} \right\rangle^{-1} \,. \qquad (29)$$

Letting Γ be of order 0.1 eV, we find

$$8\,|\Phi|\,k\bar{T} \sim 0.01\ (eV)^2 \,;\ \bar{T} = \left\langle \frac{1}{T} \right\rangle^{-1} \,. \qquad (30)$$

The energy spread of the whole sequence of states is therefore

$$\xi = k\bar{T} \sim 1.6 \times 10^{-10}\ eV \,, \qquad (31)$$

which corresponds to a temperature sum of 1.9×10^{-6} K. Since the temperature of the Fermi gas when the compound state is reached is of order 10^{10} K, corresponding to an energy of order 1 MeV (Preston 1965), each particle state in the gas is, for this model, extremely sharply defined. When there are N collisions, the average temperature is of order ξ/Nk over all collisions. This corresponds to an average lifetime of

$$\tau_{1/2} \sim 4N \times 10^{-5}\ sec \,. \qquad (32)$$

But for the collisions with the neutron, the nucleon states would have a much longer lifetime than the compound state itself, which exists for times of order 10^{-10} sec.

One might wonder how the compound state, once it has reached the metastable configuration, can decay in times much less than 10^{-5} sec. The answer to this, from the above model, would be that the single particle states exist only between nucleon collisions. For collisions, the average momentum transfer has to be much greater than the energy spread of the particle states prior to collision. Therefore, formation and decay can occur much more rapidly than each particle state can decay. It is interesting to speculate that the lifetime (32) is approaching the beta decay lifetimes of nuclides if N is taken to be large, say of order 10^4 The semi-classical treatment given here, however, is over-simplified and a rigorous quantum mechanical equivalent to this theory would provide more secure grounds for such an investigation.

We note that if a wavelength

$$\lambda = h\sqrt{2m\xi} = h\sqrt{\beta}$$

is defined that is characteristic of each possible collision sequence, then, from Equation (28), such wavelengths have a distribution given by

$$\mathcal{P}_\lambda(\lambda)\, d\lambda = \sqrt{\frac{\pi}{2}}\ \lambda_0 \exp(-\lambda^2/2\lambda_0^2)\, d\lambda \ ;\ 0 \leqslant \lambda < \infty \,,$$

which is a Gaussian distribution of variance

$$\lambda_0^2 = h^2/2\epsilon \,.$$

The average wavelength per collision sequence has a similar distribution.

There is one essential difficulty with a semi-classical model of this type. The energy of a nucleon in a single particle state j has a spread of Kt_j, where t_j is the 'level temperature'. If the spread in energy of this state is less than 10^{-10} eV, the momentum spread of the nucleon in the single particle state j has a range of values with spread

$$\Delta p_j = (2mKt_j)^{1/2} \sim 10^{-4} \frac{\text{MeV}}{c} .$$

However, the nucleon is confined to the volume of the target nucleus with dimensions of say $\Delta x = 5$ fermi. By the uncertainty principle, any single particle wave function must then predict a spread in momentum of at least

$$\Delta p_j = \hbar / \Delta x = 40 \frac{\text{MeV}}{c} .$$

The only way this difficulty can be avoided is to argue that the width of single particle states observed in nuclear reactions is some 10^6 times greater than that observed for compound nucleus states, which implies that the average nuclear field during the process must extend well beyond 5 fermi due to resonance excitation; in fact, something like 10^6 fermi. It is for dimensions of this magnitude that such small momentum spreads can occur.

7. REFERENCES

Bethe, H.A. (1937) – Rev. Mod. Phys. 9, 69.

Breit, G. & Wigner, E.P. (1936) – Phys. Rev. 49, 519, 642.

Gilbert, A. & Cameron, A.G.W. (1965) – Can. J. Phys. 43, 1446.

Preston, M.A. (1965) – Physics of the Nucleus, Addison-Wesley, Massachusetts, p. 501.

Wigner, E.P. & Eisenbud, L. (1947) – Phys. Rev. 72, 29.