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RESEARCH ESTABLISHMENT

LUCAS HEIGHTS

TEMPERATURES IN A SPHERE WITH DISTRIBUTED SOURCE AND VARIABLE SURFACE HEAT TRANSFER

by

J. J. THOMPSON *

* Attached, from The University of New South Wales

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ABSTRACT

An iterative perturbation theory solution is developed for a general, spherical, steady state heat conduction problem, to provide a basis for the estimation of hot spot factors for the fuel elements in a pebble bed reactor.

CONTENTS

1. INTRODUCTION

In the thermal analysis of a pebble bed reactor with randomly packed spheres which are recirculated through the core, the use of mean values must be corrected by the use of "hot spot" factors. The processes to which a single ball is subjected are random and the significance of the results of any nuclear or thermal analysis, and indeed the accuracy to which such computations are carried out, are determined by the nature of the relevant probability distributions. Thus dispersions about mean values are parameters of equal importance to means.

The investigation reported here arose in connection with the problem of the statistics of pebble bed processes. One effect of some importance is the variation of heat transfer coefficient round the surface of a ball. This variation will depend on local packing and flow conditions, and a single ball throughout its transit through the core will have the heat transfer coefficient at the surface at a particular point slowly varying in a random fashion. Temperature at a point in a ball will therefore also be a random variable. Peak values of temperatures and associated thermal stresses are significant design parameters. In addition nuclear calculations use temperature-dependent cross sections and it is of interest to estimate the "errors" in nuclear calculations due to the uncontrollable temperature fluctuations. A final point is that a mechanism exists for thermal fatigue due to this effect.

Measurements have been made of heat transfer coefficients around packed spheres and further work is to be done at Lucas Heights. The theory presented here should give some guidance as to how the data processing of experimental results should be carried out for useful application.

2. THEORY

A uniform sphere, radius R, thermal conductivity k, has a distributed heat source S. Measurements of surface heat flux over the sphere as a member of a random packing define a surface variable heat transfer coefficient with reference to surface temperature T_S and reference temperature T_O. With all temperatures measured relative to T_O the governing equations are:

$$k \nabla^2 T + S = 0 \quad , \tag{1a}$$

$$k \frac{\partial T}{\partial r} + h T = 0 \quad \text{at} \quad r = R \, . \tag{1b}$$

Using spherical coordinates with $r = \rho R$, note that

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{\rho^2} \frac{\partial}{\partial \theta} + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \, . \tag{2}$$

It is convenient to separate out mean values of S and h and introduce the perturbation parameters γ and λ to rewrite the basic equations as follows:

$$\frac{R^2 S}{k} = Q \left(1 + \gamma \, q(\rho, \theta, \phi)\right) \quad , \tag{3a}$$

$$\frac{hR}{k} = \alpha \left(1 + \lambda H(\theta, \phi)\right) \quad , \tag{3b}$$

$$\nabla^2 T + Q(1 + \gamma q) = 0 \quad , \tag{4a}$$

$$\frac{\partial T}{\partial \rho} + \alpha(1 + \lambda H) T = 0 \quad , \text{ for } \rho = 1 \, . \tag{4b}$$

Also

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi} q \, \rho^2 \sin \theta \, dr \, d\theta \, d\phi = \int_0^{2\pi} \int_0^{\pi} H \sin \theta \, d\theta \, d\phi = 0 \, . \tag{5}$$

Boundary condition (4b) is non-linear so solutions to this equation cannot be obtained by superposition. On the other hand it is convenient to recognize three components in the solution, to facilitate discussion of perturbations.

Thus, let $T = T_Q + T_q + T_h$,

where the components satisfy the conditions:

$$\nabla^2 T_Q + Q = 0 , \tag{6a}$$

$$\frac{\partial T_Q}{\partial \rho} + \alpha T_Q = 0 , \quad (\rho = 1) . \tag{6b}$$

$$\nabla^2 T_q + \gamma q Q = 0 , \tag{7a}$$

$$\frac{\partial T_q}{\partial \rho} + \alpha T_q = 0 , \quad (\rho = 1). \tag{7b}$$

$$\nabla^2 T_h = 0 , \tag{8a}$$

$$\frac{\partial T_h}{\partial \rho} + \alpha (1 + \lambda H) T_h + \alpha \lambda H (T_Q + T_q) = 0 , \quad (\rho = 1) . \tag{8b}$$

(a) T_Q solution.

This solution is trivial.

$$T_Q = \frac{Q}{6}\left(1 + \frac{2}{\alpha} - \rho^2\right) . \tag{9}$$

(b) T_q solution.

To solve (7a) and (7b), eigenfunction expansion is most convenient. The relevant functions satisfy:

$$(\nabla^2 + K^2)\phi = 0 ,$$

with the boundary condition $\frac{\partial \phi}{\partial \rho} + \alpha \phi = 0$, at $\rho = 1$.

Thus

$$\phi_{n,m,s} = j_n (K_{ns} \rho) P_n^m (\cos \theta) [\cos (m\phi) \text{ or } \sin (m\phi)] , \tag{10}$$

with the characteristic equation for eigenvalues:

$$(\alpha - n - 1) j_n (K_{ns}) + K_{ns} j_{n-1} (K_{ns}) = 0 , \begin{cases} s = 1,2 .. \\ n = 1,2 .. \end{cases} . \tag{11}$$

In the following $\phi_{n,m}$ will be written as a product of the spherical Bessel function j_n and the spherical harmonic:

$$Y_{nm} = P_n^m (\cos \theta) \cos m\phi ,$$

it being understood that in practice the odd function:

$$P_n^m (\cos \theta) \sin m \phi ,$$

must also be included.

It is also convenient to normalize the functions to give the orthonormal set $\phi^*_{n,m,s}$. Thus

$$\phi^*_{n,m,s} = j_n^* (K_{ns}\rho)\ Y^*_{nm}\ (\theta,\phi)\ , \tag{12}$$

$$j_n^*(K_{ns}\,\rho) = j_n(K_{ns}\rho)\left[\frac{1}{2}\, j_n^{\prime 2}\,(K_{ns})\left\{1 + \frac{(\alpha+n)(\alpha-n-1)}{K^2_{ns}}\right\}\right]^{-\frac{1}{2}}\ , \tag{13}$$

$$Y^*_{nm}\ (\theta,\phi) = Y_{nm}\ (\theta,\ \phi)\left[\frac{4\pi\,(n+m)\,!}{\epsilon_m(n-m)!\,(2n+1)}\right]^{-\frac{1}{2}}\ , \tag{14}$$

with $\epsilon_o = 1\ ,\ \epsilon_m = 2\ ,(\ m > 0\ )$.

Expansion of both T_q and $q(\rho,\ \theta,\ \phi)$ as series of eigenfunctions leads to the solution for T_q:

$$T_q\ (\rho,\ \theta,\ \phi) = \gamma\ Q \sum_{n=o}^{\infty}\ \sum_{m=o}^{n}\ \sum_{s}\ \frac{\phi^*_{n,m,s}\ (\rho,\theta,\phi)}{K^2_{ns}} \int_o^1\int_o^{2\pi}\int_o^{\pi} q\ \phi^*_{n,m,s}\ \rho^2 \sin\theta\ d\theta\ d\phi\ d\rho\ . \tag{15}$$

The boundary conditions on T_h involve $T_Q + T_q$ at $\rho = 1$.

From Equations 9 and 15, using 13:

$$(T_Q + T_q)_{\rho=1} = Q\left\{\frac{1}{3\alpha} + \gamma \sum_n \sum_m q'_{nm}\ Y_{nm}\ (\theta,\phi)\right\}\ , \tag{16}$$

$$q'_{nm} = \sum_s \frac{1}{K_{ns}}\ \left(\frac{2}{K^2_{ns} + (\alpha+n)(\alpha-n-1)}\right)^{\frac{1}{2}} \int_o^1\int_o^{2\pi}\int_o^{\pi} q\,\phi^*_{n,m,s}\ \rho^2 \sin\theta\ d\theta\ d\phi\ d\rho\ . \tag{17}$$

(c) <u>T_h solution</u>.

A closed solution for T_h is not possible, but an integral equation for T_h on the surface can be established. An iterative solution then leads to a series in the perturbation parameter λ.

Equation (8a) is satisfied by:

$$T_h = \sum_n \sum_m\ A_{nm}\ \rho^n\ Y^*_{nm}\ (\theta,\phi)\ , \tag{18}$$

$$\therefore\quad T_h(\rho = 1) = \sum_n \sum_m\ A_{nm}\ Y^*_{nm}\,(\theta,\phi)\ , \tag{19}$$

$$\therefore\quad A_{nm} = \oint_s T_h\ Y^*_{nm}(\theta,\phi)\ ds\ , \tag{20}$$

when s denotes the surface and $ds = \sin\theta\ d\theta\ d\phi$. A knowledge of T_h on the surface therefore leads to the complete solution, from Equation 18.

From Green's Theorem,

$$\int_v \rho^n\ Y^*_{nm}\ \nabla^2 T_h\ dv = \int_v T_h\ \nabla^2(\rho^n Y^*_{nm})\ dv + \oint_s (Y^*_{nm}\ \frac{\partial T_h}{\partial\rho} - n\ T_h\ Y^*_{nm}\ )\ ds\ ,$$

$$\therefore\quad \oint_s Y^*_{nm}\ \frac{\partial T_h}{\partial \rho}\ ds = \oint n\ T_h\ Y^*_{nm}\ ds\ . \tag{21}$$

Hence using the boundary condition of Equation 8b ,

$$(n+\alpha)\oint_s T_h Y^*_{nm}\, ds + \lambda\alpha \oint_s H\, T_h\, Y^*_{nm}\, ds + \lambda\alpha \oint_s H\,(T_Q + T_q)\, Y^*_{nm}\, ds \quad . \qquad (22)$$

The series of (19) gives the required integral equation:

$$T_h = -\lambda\alpha \sum_n \sum_m \left(\frac{1}{n+\alpha}\right) Y^*_{nm} \left\{ \oint_s H\, T_h\, Y^*_{nm}\, ds + \oint_s H\,(T_Q + T_q)\, Y^*_{nm}\, ds \right\} . \qquad (23)$$

As the values T_Q and T_q are now known, this equation can be written:

$$T_h = -\lambda\alpha \sum_n \sum_m \left(\frac{1}{n+\alpha}\right) Y^*_{nm} \left\{ \oint_s H\, T_h\, Y^*_{nm}\, ds + Q\,\Omega_{nm} \right\} \quad . \qquad (24)$$

The first and second iterations produce the following:

$$T_h^{(1)} = -\lambda\alpha\, Q \sum_n \sum_m Y^*_{nm} \left(\frac{\Omega_{nm}}{n+\alpha}\right) \quad ,$$

$$T_h^{(2)} = T_h^{(1)} + \lambda^2 \alpha^2 Q \sum_n \sum_m \sum_p \sum_q Y^*_{nm} \frac{\Omega_{pq}}{(n+\alpha)(p+\alpha)} \int_s Y^*_{nm}\, H\, Y^*_{pq}\, ds \ .$$

Introducing the significant integrals of H which occur in these series as:

$$H^{pq}_{nm} = \int_0^{2\pi}\!\!\int_0^{\pi} Y^*_{nm}(\theta,\phi)\, H(\theta,\phi)\, Y^*_{pq}(\theta,\phi) \sin\theta\, d\theta\, d\phi \quad , \qquad (25)$$

the series may be written more concisely as:

$$T_h(\theta,\phi) = -\lambda\alpha \sum_n \sum_m \left(\frac{1}{n+\alpha}\right) Y^*_{nm}(\theta,\phi) \left\{ \Omega_{nm} - \lambda\alpha \sum_p \sum_q \left(\frac{H^{pq}_{nm}}{p+\alpha}\right) \Omega_{pq} \right.$$

$$\left. + \lambda^2 \alpha^2 \sum_p \sum_q \left(\frac{H^{pq}_{nm}}{p+\alpha}\right) \sum_r \sum_s \left(\frac{H^{rs}_{pq}}{r+\alpha}\right) \Omega_{rs} + \ldots\ldots \right\} \quad . \qquad (26)$$

3. THE SOLUTION FOR UNIFORM HEAT SOURCE AND AXIALLY SYMMETRIC HEAT TRANSFER DISTRIBUTION

Writing $m = 0$, $Y^*_{no} = \sqrt{\frac{2n+1}{4\pi}}\, P_n(\mu)$, with $\mu = \cos\theta$, the simplified integral equation becomes:

$$T_h(\mu) = -\lambda\alpha \sum_n P_n(\mu) \frac{2n+1}{2(n+\alpha)} \left\{ \int_{-1}^{+1} H(\mu)\, T_h(\mu)\, P_n(\mu)\, d\mu + \frac{Q}{4\alpha} \int_{-1}^{+1} H(\mu)\, P_n(\mu)\, d\mu \right\}, \qquad (27)$$

$$\therefore\ T_h(\mu) = -\lambda\alpha \sum_n \left(\frac{(2n+1)P_n(\mu)}{2(n+\alpha)} \int_{-1}^{+1} H(\mu)\, T_h(\mu)\, P_n(\mu)\, d\mu \right) - \frac{\lambda Q}{4} \sum_n \frac{2n+1}{2(n+\alpha)} H_n\, P_n(\mu) \ . \qquad (28)$$

As a particular example of interest, let $H(\mu) = P_m(\mu)$. Note however that solutions for various values of m cannot be summed to give the complete solution when H is written as a sum of Legendre polynomials.

With

$$H_n = \frac{2}{2m+1} \qquad \delta_{nm} \quad ,$$

$$T_h(\mu) = -\lambda\alpha \sum_n \frac{2n+1}{2(n+\alpha)} H_{mn} P_n(\mu) - \frac{\lambda Q}{4(m+\alpha)} P_m(\mu) \quad , \tag{29}$$

with $$H_{mn} = \int_{-1}^{+1} P_m(\mu)\, T_h(\mu)\, P_n(\mu)\, d\mu \quad . \tag{30}$$

In terms of the integrals

$$X^m_{p,q} = \int_{-1}^{+1} P_m(\mu)\, P_p(\mu)\, P_q(\mu)\, d\mu \quad , \tag{31}$$

the first terms of the series are:

$$T_h = -\frac{\lambda Q}{4(m+\alpha)} \Bigg[P_m(\mu) - \lambda\alpha \sum_n \frac{2n+1}{2(n+\alpha)} P_n(\mu)\, X^m_{mn}$$

$$+ \lambda^2\alpha^2 \sum_n \frac{2n+1}{2(n+\alpha)} P_n(\mu) \sum_p \frac{2p+1}{2(p+\alpha)} X^m_{mp} X^m_{pn} - \ldots\ldots \Bigg] . \tag{32}$$

The significance of a heat transfer coefficient given by a single Legendre Polynomial lies in the comparative simplicity of Equation 32 which is amenable to hand computation. Thus Equation 32 is a convenient starting point for numerical work to study convergence.

4. CONCLUSION

With variable heat transfer, the surface temperature satisfies an integral equation, written most concisely as:

$$0 = T_h(\mu,\phi) + \lambda\alpha \iint_S [H(\mu',\phi')\, T_h(\mu',\phi') + (T_Q + T_q)(\mu',\phi')\, H(\mu',\phi')]\, G[\mu,\phi|\mu',\phi']\, d\mu'\, d\phi' , \tag{33}$$

where $$G[\mu,\phi|\mu',\phi'] = \sum_n \sum_m \frac{1}{n+\alpha} Y^*_{nm}(\mu,\phi)\, Y^*_{nm}(\mu,\phi) \quad . \tag{34}$$

Equation 33 is most naturally solved in terms of a series in λ, which is essentially an expansion of T_h in spherical harmonics over the surface of the sphere.

Alternatively Equation 33 may be solved numerically as an integral equation, using the precomputed $G(\mu,\phi|\mu',\phi')$ from Equation 34. Once T_h over the surface is available numerically the distribution of T_h is given by:

$$T_h(\rho,\theta,\phi) = \sum_n \sum_{m=0}^{n} \rho^n\, Y^*_{nm}(\theta,\phi) \iint_S T_h(\theta,\phi)\, Y^*_{nm}(\theta,\phi)\, d\mu\, d\phi \quad . \tag{35}$$

If the original equations define heat transfer to a medium whose temperature is not T_o, a constant, but $T_o + T^*_o(\theta,\phi)$, the only change involved is in the integral equation for the temperature T_h, now measured relative to the mean value T_o.

The new equation is:

$$T_h + \lambda\alpha \int_s (H\,T_h + (T_Q + T_q)\,H)\,G\,ds' = \alpha \int_s (1 + \lambda H)\,T_o^*\,G\,ds' \quad . \qquad (36)$$

Equation 26 is no more difficult than Equation 33.

The equations which have been obtained give temperatures in terms of perturbation parameters λ, for heat transfer and γ, for heat source. Also the solutions occur naturally in terms of series of mutually orthogonal eigenfunctions. They are therefore in a convenient form for variance analysis and a study of departures from the mean, when the ball history is described as a random process.

5. ACKNOWLEDGMENT

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6. NOTATION

h	Heat transfer coefficient
H	Departure of non-dimensional heat transfer coefficient about the mean value α, measured relative to α
$j_n(K_{ns}\rho)$	Spherical Bessel function of order n
k	Thermal conductivity
$P_n^m(\mu\,\phi)$	Associated Legendre polynomial
Q	Mean value of heat source having dimensions of temperature
$q(\rho,\theta,\phi)$	Departure of heat source about the mean Q and measured relative to Q
r,θ,ϕ	Spherical co-ordinates
R	Outer radius of sphere
S	True heat source
s	Denotes surface of sphere
$T(\rho,\theta,\phi)$	Temperature, departure from T_o
T_o	Mean coolant temperature
$T_o^*(\theta,\phi)$	Departure of coolant temperature from T_o
T_Q, T_q, T_h	Components of temperature field due to Q,q,H
$Y_n^m(\theta,\phi)$	Spherical harmonic $P_n^m \cos\theta \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}$
v	Denotes volume of sphere
α	Average heat transfer coefficient, non-dimensional
γ	Perturbation parameter, relevant to q
ϵ_m	Neumann factor

K_{ns}	s'th eigenvalue, n'th spherical Bessel function
λ	Perturbation parameter, relevant to H
μ	$\cos\theta$
ρ	Non-dimensional co-ordinate (r/R)
*	Denotes orthonormal eigenfunction