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LUCAS HEIGHTS

INTEGRALS INVOLVING DOPPLER BROADENED CONTOUR FUNCTIONS

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ABSTRACT

A variety of integrals arising in the study of resonance absorption have been evaluated.

CONTENTS

1. INTRODUCTION

The search for a long reactivity lifetime in a high temperature gas cooled reactor has led to the study of an undermoderated system having an initial material content of commercial plutonium, thorium, and beryllium, all as oxides. Although simple theories of resonance absorption and its temperature dependence are adequate for the assessment of thermal systems, where most neutron absorption into fuel depends only weakly on temperature, improved techniques will be required when dealing with an epithermal system, since the undermoderation inevitably results in the majority of fuel absorptions occurring as resonance reactions. The use of commercial plutonium as the fuel introduces added complications in the exceptionally large cross section of the Pu240 1eV resonance and the presence of the Pu239 resonance at thermal energies. Another problem which will seriously affect lifetime studies is that the isotopic chains set up by neutron absorption in the fertile nuclei, Th232 and Pu240, consist of a complex of heavy isotopes with a multiplicity of resonances with consequent overlapping and a corresponding reduction in the absorption.

To gain a thorough understanding of the process affecting the reactivity of the system, a combined analytical and numerical approach is being adopted. While, in principle, the whole problem could be solved numerically it would involve a prodigious amount of computing and would require facilities far in excess of those available. The main aim of the analytical work will be to remove or improve as many of the usual approximations as possible, and towards this end a detailed study of the Doppler broadening of Breit-Wigner resonance contours has been initiated.

The calculation of the resonance absorption of neutrons invariably leads to integrals involving the Doppler broadened contour functions.

$$\psi(x,t) = \frac{1}{2}\frac{1}{\sqrt{(\pi t)}}\int_{-\infty}^{\infty} \exp\{-(x-y)^2/4t\}\,\frac{dy}{1+y^2}\ , \tag{1.1}$$

$$\phi(x,t) = \frac{1}{2}\frac{1}{\sqrt{(\pi t)}}\int_{-\infty}^{\infty} \exp\{-(x-y)^2/4t\}\,\frac{y\,dy}{1+y^2}\ , \tag{1.2}$$

which were introduced originally by Voigt (1912) in connection with the anomalous dispersion of light. The dimensionless variables x and t in Equations 1.1 and 1.2 are defined by:

$$x = \frac{2}{\Gamma}(E-E_r)\ ,$$

$$t = \frac{4m}{\Gamma^2 M}\,E_r T\ ,$$

where E_r is the resonance energy,

E is the incident neutron energy,

Γ is the total width of the resonance,

M and m are the masses of the absorber atom and the neutron respectively, and

T is the temperature of the absorber atom in energy units.

Bethe and Placzek (1937) and Bethe (1937) were the first to apply the function $\psi(x,t)$ to the analysis of neutron resonances and since that time many authors have studied both the functions $\psi(x,t)$ and $\phi(x,t)$ in this context. For an extensive bibliography of the work, the reader is referred to that compiled by Dresner (1960).

The mathematical treatment of the functions $\psi(x,t)$ and $\phi(x,t)$ can be undertaken by identifying them either as Fourier convolutions, or as confluent hypergeometric functions related to the parabolic cylinder functions. They have also been shown to be a Hilbert transform pair

(Keane and Clancy 1963). The fact that the functions are Fourier convolutions suggests that analysis of the functions and their properties may be simplified by the use of the Fourier transform method, which is the principal analytical tool used in the present work.

In dealing with problems in resonance absorption theory it is likely that approximate methods will always be required but it is highly desirable that they be avoided whenever possible. The primary purpose of this report is to collect and extend a list of those integrals involving Doppler broadened contour functions which can be expressed without approximation in terms of a minimum set of tabulated functions.

2. SOME PROPERTIES OF THE DOPPLER BROADENED CONTOUR FUNCTIONS

2.1 Alternative Expressions for the Functions

Most of the properties of the Doppler broadened contour functions can be obtained in a straightforward manner from alternative integral representations which may be derived by the application of Fourier transforms.

Since the functions $\psi(x,t)$ and $\phi(x,t)$ are Fourier convolutions it is not difficult to show that:

$$\mathcal{F}_c \ \{\psi(x,t)\} = \sqrt{(\pi/2)} \exp(-p-p^2 t) \quad , \qquad (2.1)$$

$$\mathcal{F}_s \ \{\phi(x,t)\} = \sqrt{(\pi/2)} \exp(-p-p^2 t) \quad , \qquad (2.2)$$

where $\mathcal{F}_c$ and $\mathcal{F}_s$ denote the Fourier cosine and sine transforms respectively. Simple inversion of these transforms leads to:

$$\psi(x,t) = \int_0^\infty \exp(-p-p^2 t) \cos px \, dp \quad , \qquad (2.3)$$

$$\phi(x,t) = \int_0^\infty \exp(-p-p^2 t) \sin px \, dp \quad , \qquad (2.4)$$

which are the alternative expressions sought.

From Equations 2.3 and 2.4 we deduce immediately that $\psi(x,t)$ is an even function of x while $\phi(x,t)$ is odd. We can also obtain the central values of the functions, namely:

$$\psi(0,t) = \frac{1}{2} \sqrt{(\pi/t)} \exp(1/4t) \operatorname{erfc}(1/2\sqrt{t}) \quad , \qquad (2.5)$$

$$\phi(0,t) = 0 \quad . \qquad (2.6)$$

2.2 Asymptotic Expansions for Small t

If t is very small, asymptotic expansions may be obtained by expanding the term $\exp(-p^2 t)$ in the integrands of both Equations 2.3 and 2.4.

Thus

$$\psi(x,t) \simeq \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \int_0^\infty p^{2n} e^{-p} \cos px \, dp$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^{2n}}{dx^{2n}} \frac{1}{1+x^2} \quad , \qquad (2.7)$$

and also

$$\phi(x,t) \simeq \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^{2n}}{dx^{2n}} \frac{x}{1+x^2} \quad . \qquad (2.8)$$

2.3 Approximation for Large t

When t is large the term $\exp(-p^2 t)$ dominates the convergence of the integrals in Equations 2.3 and 2.4 and we have:

$$\psi(x,t) \simeq \int_0^\infty \exp(-p^2 t) \cos px \, dp$$

$$= \frac{1}{2} \sqrt{(\pi/t)} \exp(-x^2/4t) \quad , \qquad (2.9)$$

while

$$\phi(x,t) \simeq \int_0^\infty \exp(-p^2 t) \sin px \, dp$$

$$= \frac{x}{2t} \exp(-x^2/4t) \; {}_1F_1\left(\tfrac{1}{2}; \tfrac{3}{2}; \tfrac{x^2}{4t}\right) \quad . \qquad (2.10)$$

2.4 Differential Equations

It follows immediately from Equations 2.3 and 2.4 that:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t} \quad , \qquad (2.11)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t} \quad . \qquad (2.12)$$

After a single differentiation with respect to x in 2.3 and 2.4, integration by parts yields the alternative equations:

$$x\psi = \phi - 2t \frac{\partial \psi}{\partial x} \quad , \qquad (2.13)$$

$$x\phi = 1 - \psi - 2t \frac{\partial \phi}{\partial x} \quad . \qquad (2.14)$$

Combining Equations 2.13 and 2.14 leads to the second order differential equations:

$$4t^2 \frac{\partial^2 \psi}{\partial x^2} + 4xt \frac{\partial \psi}{\partial x} + (1 + x^2 + 2t)\psi = 1 \quad , \qquad (2.15)$$

$$4t^2 \frac{\partial^2 \phi}{\partial x^2} + 4xt \frac{\partial \phi}{\partial x} + (1 + x^2 + 2t)\phi = 0 \quad . \qquad (2.16)$$

3. INTEGRALS DERIVED FROM FOURIER TRANSFORMS

3.1 Transforms of Products of $\psi(x,t)$ and $\phi(x,t)$

It is easy to prove that if

$$\mathcal{F}_c\{f(x)\} = F(p)$$

and

$$\mathcal{F}_s\{f(x)\} = G(p) \quad ,$$

then $$\mathcal{F}_c\ \{f(x)\cos qx\} = \tfrac{1}{2}F(|p+q|) + \tfrac{1}{2}F(|p-q|)\ , \tag{3.1}$$

$$\mathcal{F}_s\ \{f(x)\cos qx\} = \tfrac{1}{2}G(|p+q|) + \tfrac{1}{2}\frac{p-q}{|p-q|}G(|p-q|)\ , \tag{3.2}$$

$$\mathcal{F}_c\ \{f(x)\sin qx\} = \tfrac{1}{2}G(|p+q|) - \tfrac{1}{2}\frac{p-q}{|p-q|}G(|p-q|)\ . \tag{3.3}$$

From Equation 3.1 and the Parseval formula it follows that:

$$\int_0^\infty \psi(x,t_1)\,\psi(x,t_2)\ \cos qx\ dx$$

$$= \frac{\pi}{4}\int_0^\infty \exp(-p-p^2t_1)\ \exp\{-(p+q)-(p+q)^2t_2\}\ dp$$

$$+ \frac{\pi}{4}\int_0^\infty \exp(-p-p^2t_1)\exp\{-|p-q| - (p-q)^2t_2\}\ dp$$

After some manipulation and a change of symbols we find that:

$$\mathcal{F}_c\ \{\psi(x,t_1)\,\psi(x,t_2)\} = T_1 + T_2 + T_3\ , \tag{3.4}$$

where $$T_1 = \frac{\pi}{4}\sqrt{\left(\frac{\tau}{2t_1t_2}\right)}\exp\left\{-p^2\tau-p+\frac{2p\tau}{t_1}+\frac{\tau}{t_1t_2}\right\}\operatorname{erfc}\left\{p\sqrt{\frac{t_2\tau}{t_1}}+\sqrt{\frac{\tau}{t_1t_2}}\right\}.\ ,$$

$$T_2 = \frac{\pi}{4}\sqrt{\left(\frac{\tau}{2t_1t_2}\right)}\exp\left\{-p^2\tau-p+\frac{2p\tau}{t_2}+\frac{\tau}{t_1t_2}\right\}\operatorname{erfc}\left\{p\sqrt{\frac{t_1\tau}{t_2}}+\sqrt{\frac{\tau}{t_1t_2}}\right\}\ ,$$

$$T_3 = \frac{\pi}{4}\sqrt{\left(\frac{\tau}{2t_1t_2}\right)}\exp(-p^2\tau-p)\left\{\operatorname{erf}\left(p\sqrt{\frac{t_2\tau}{t_1}}\right)+\operatorname{erf}\left(p\sqrt{\frac{t_1\tau}{t_2}}\right)\right\}.\ ,$$

and $$\frac{1}{\tau} = \frac{1}{t_1}+\frac{1}{t_2}\ .$$

Similarly using Equations 3.2 and 3.3 we find that:

$$\mathcal{F}_c\ \{\phi(x,t_1)\,\phi(x,t_2)\} = T_1 + T_2 - T_3\ , \tag{3.5}$$

$$\mathcal{F}_s\ \{\psi(x,t_1)\,\phi(x,t_2)\} = T_1 - T_2 + T_3\ . \tag{3.6}$$

3.2 Special Examples

Two particular results which are required in later sections are obtained by setting $t_1 = t_2 = t$ in Equations 3.4 and 3.5, when it follows that:

$$\mathcal{F}_c\ \{\psi^2(x,t)\} = \frac{\pi}{4\sqrt{t}}\ \exp\left\{-\frac{p^2t}{2}-p\right\}\ \text{X}$$

$$[\exp(p+1/2t)\operatorname{erfc}\{p\sqrt{(t/2)}+1/\sqrt{(2t)}\} + \operatorname{erf}(p\sqrt{(t/2)})]\ , \tag{3.7}$$

and

$$\mathcal{F}_c\ \{\phi^2(x,t)\} = \frac{\pi}{4\sqrt{t}}\ \exp\left\{-\frac{p^2 t}{2} - p\right\}\ X$$

$$[\exp(p + 1/2t)\mathrm{erfc}\ \{p\sqrt{(t/2)} + 1/\sqrt{(2t)}\} - \mathrm{erf}\ \{p\sqrt{(t/2)}\}\ ]\ , \tag{3.8}$$

while the same substitution made in Equation 3.6 yields the Fourier sine transform of $\psi(x,t)\ \phi(x,t)$. Setting either t_1 or t_2 equal to zero leads to expressions for the transforms

$$\mathcal{F}_c\left\{\frac{\psi(x,t)}{1+x^2}\right\}\ ,\quad \mathcal{F}_c\left\{\frac{x\,\phi(x,t)}{1+x^2}\right\}\ ,\quad \mathcal{F}_s\left\{\frac{x\,\psi(x,t)}{1+x^2}\right\}\ ,\quad \text{and}$$

$$\mathcal{F}_s\left\{\frac{\phi(x,t)}{1+x^2}\right\}\ .$$

Since $\psi(x,0) = 1/(1+x^2)$, it would be possible to obtain the Fourier cosine transform of $1/(1+x^2)^2$ from Equation 3.4 but it is more simply obtained from the general transform of $1/(1+x^2)^n$. The general expression is well known as:

$$\mathcal{F}_c\left\{\frac{1}{(1+x^2)^n}\right\} = \frac{\left(\frac{1}{2}\right)^{n-1} p^{n-\frac{1}{2}}}{(n-1)!}\ K_{n-\frac{1}{2}}(p)$$

$$= \sqrt{(\pi/2)}(-\tfrac{1}{2})^{n-1}\ \frac{p^{2n-1}}{(n-1)!}\left(\frac{1}{p}\ \frac{d}{dp}\right)^{n-1}\frac{e^{-p}}{p}\ , \tag{3.9}$$

giving in particular:

$$\mathcal{F}_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{(\pi/2)}\exp(-p)\ , \tag{3.10}$$

$$\mathcal{F}_c\left\{\frac{1}{(1+x^2)^2}\right\} = \frac{1}{2}\sqrt{(\pi/2)}\ (1+p)\exp(-p)\ . \tag{3.11}$$

3.3 Tabulation of Results

All the results obtained so far can be generalised by using the theorem that if

$$\mathcal{F}_c\ \{f(x)\} = F(p)$$

then

$$\mathcal{F}_c\ \{f(ax)\} = (1/a)\ F(p/a)\ , \tag{3.12}$$

which for example gives in place of Equation 2.1 the result:

$$\mathcal{F}_c\ \{\psi(ax,t)\} = \frac{1}{a}\sqrt{\frac{\pi}{2}}\ \exp\left(-\frac{p}{a} - \frac{p^2 t}{a^2}\right)\ .$$

Use of Equation 3.12, the Parseval and convolution theorems with the Fourier cosine and sine transforms already derived, together with other standard transforms, has led to the evaluation of the integrals given in Table 1. Many other entries could be made by differentiating these integrals with respect to parameters but such developments are left to the reader.

4. INTEGRALS OF POWERS OF $\psi(x,t)$

Throughout this section we use the notation:

$$\chi_n = \int_0^\infty \psi^n(x,t)\,dx \quad . \tag{4.1}$$

For the particular values n = 1,2 the integrals are well known and have been evaluated (Bethe 1937) in closed form while approximate expressions are available for $n \geqslant 3$ and for some values of t . The integrals have been tabulated by Cook and Elliott (1960) for $t < 1$, while Keane and McKay (1960) have given a tabulation for $t > 1$ derived from an approximate analytical representation of χ_n.

4.1 The Integrals χ_1 and χ_2

Integrating Equation 1.1 with respect to x and inverting the order of integration shows that:

$$\chi_1 = \frac{1}{4\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \frac{dy}{1+y^2} \int_{-\infty}^{\infty} \exp\{-(x-y)^2/4t\}\,dx$$

$$= \frac{1}{2}\pi \;, \tag{4.2}$$

The value of χ_2 can be deduced from previous results but is most easily found by using the Parseval formula with the Fourier cosine transform of $\psi(x,t)$ as given in Equation 2.1. Thus

$$\chi_2 = \frac{1}{2}\pi \int_0^\infty \exp(-2p-2p^2t)\,dp$$

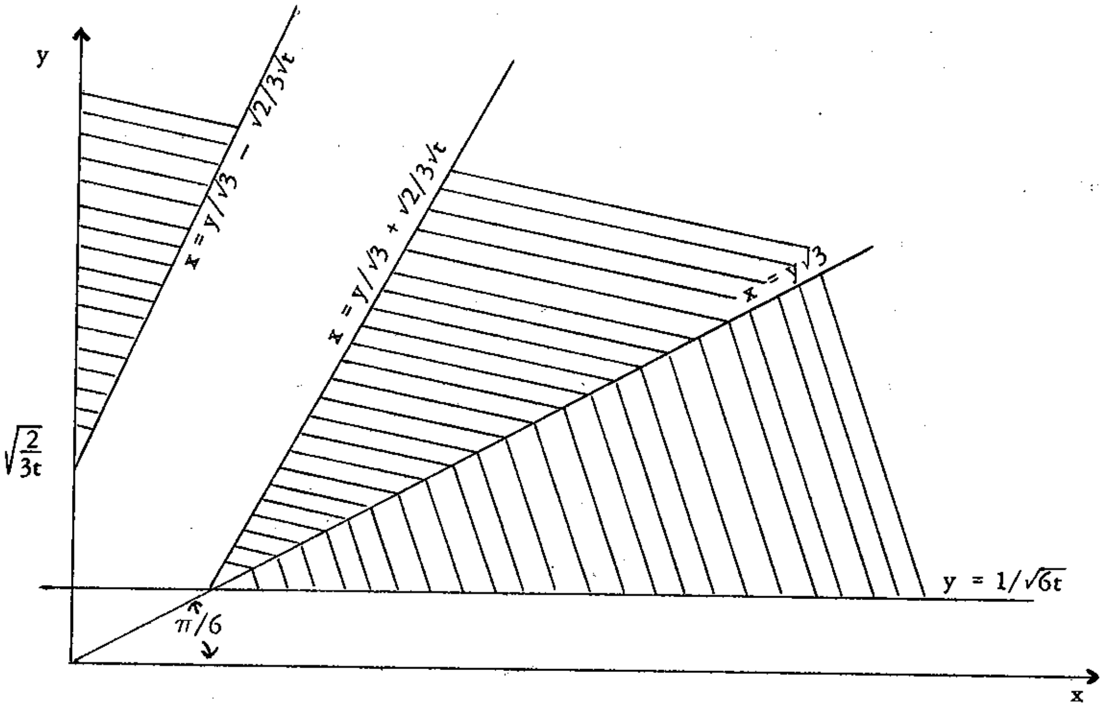
$$= \frac{\pi}{4}\,\psi(0,t/2) \quad . \tag{4.3}$$

4.2 Reduction of χ_3

Using the Parseval formula and the expression for the Fourier cosine transforms of $\psi(x,t)$ and $\psi^2(x,t)$ as given by Equations 2.1 and 3.8, we obtain:

$$\chi_3 = \frac{\pi}{2\sqrt{2t}}\left[e^{\frac{1}{2t}} \int_0^\infty e^{-p-3p^2t/2}\,dp \int_{p\sqrt{\frac{t}{2}}+\frac{1}{\sqrt{2t}}}^{\infty} e^{-x^2}dx + \int_0^\infty e^{-2p-3p^2t/2}\,dp \int_0^{p\sqrt{\frac{t}{2}}} e^{-x^2}dx\right]$$

$$= \frac{\pi}{2\sqrt{3}}\,\frac{1}{t}\,e^{\frac{2}{3}t}\left[\int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2}dy \int_{\frac{y}{\sqrt{3}}+\frac{1}{3}\sqrt{\frac{2}{t}}}^{\infty} e^{-x^2}dx + \int_{\sqrt{\frac{2}{3t}}}^{\infty} e^{-y^2}\,dy \int_0^{\frac{y}{\sqrt{3}}-\frac{1}{3}\sqrt{\frac{2}{t}}} e^{-x^2}\,dx\right] \;.$$



The integrands in the last two double integrals are $\exp(-r^2)$ so that integration over regions similarly situated with regard to the origin will give the same result. The three regions shown in the figure are equivalent for the present purpose so that:

$$\chi_3 = \frac{\pi\sqrt{3}}{2t}\, e^{\frac{2}{3t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2}\, dy \int_{y\sqrt{3}}^{\infty} e^{-x^2} dx$$

$$= \frac{\pi\sqrt{3\pi}}{4t}\, e^{\frac{2}{3t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc} y\sqrt{3}\, dy \quad . \qquad (4.4)$$

Alternatively we could transform to polar coordinates to obtain:

$$\chi_3 = \frac{\pi\sqrt{3}}{2t}\, e^{\frac{2}{3t}} \int_{0}^{\frac{\pi}{6}} d\theta \int_{\frac{\operatorname{cosec}\theta}{\sqrt{6t}}}^{\infty} r\, e^{-r^2}\, dr$$

$$= \frac{\pi\sqrt{3}}{4t}\, e^{\frac{2}{3t}} \int_{0}^{\frac{\pi}{6}} e^{-\frac{1}{6t\sin^2\theta}}\, d\theta$$

$$= \frac{\pi\sqrt{3}}{4t}\, e^{\frac{1}{2t}} \int_{0}^{\frac{\pi}{6}} e^{-\frac{\cot^2\theta}{6t}}\, d\theta$$

$$= \frac{\pi\sqrt{3}}{4t} e^{\frac{1}{2t}} \int_{\sqrt{3}}^{\infty} e^{-\frac{u^2}{6t}} \frac{du}{1+u^2} \tag{4.5}$$

$$= \frac{3\pi\sqrt{2\pi}}{8\sqrt{t}} e^{\frac{1}{2t}} \psi\left(0, \frac{3t}{2}\right) - \frac{\pi\sqrt{3}}{4t} e^{\frac{1}{2t}} \int_{0}^{\sqrt{3}} \frac{e^{-\frac{u^2}{6t}}}{1+u^2} du \quad .$$

The result in Equation 4.5 was originally obtained by McKay and Keane (1960) by an entirely different analysis.

4.3 Reduction of χ_4

It follows from Equation 3.7 and the Parseval formula that:

$$\chi_4 = \frac{\pi^2}{16t} \int_0^{\infty} e^{-2p-p^2t} \left[\operatorname{erf}\left(p\sqrt{\tfrac{t}{2}}\right) + e^{p+\frac{1}{2t}} \operatorname{erfc}\left(p\sqrt{\tfrac{t}{2}} + \tfrac{1}{\sqrt{2t}}\right) \right]^2 dp$$

$$= \frac{\pi}{4t} e^{\frac{1}{t}} \left[\int_{\frac{1}{\sqrt{t}}}^{\infty} e^{-z^2} dz \int_0^{\frac{z}{\sqrt{2}} - \frac{1}{\sqrt{2t}}} e^{-x^2} dx \int_0^{\frac{z}{\sqrt{2}} - \frac{1}{\sqrt{2t}}} e^{-y^2} dy \right.$$

$$+ 2 e^{-\frac{1}{4t}} \int_{\frac{1}{2\sqrt{t}}}^{\infty} e^{-z^2} dz \int_0^{\frac{z}{\sqrt{2}} - \frac{1}{2\sqrt{2t}}} e^{-x^2} dx \int_{\frac{z}{\sqrt{2}} + \frac{1}{2\sqrt{2t}}}^{\infty} e^{-y^2} dy$$

$$\left. + \int_0^{\infty} e^{-z^2} dz \int_{\frac{z}{\sqrt{2}} + \frac{1}{\sqrt{2t}}}^{\infty} e^{-x^2} dx \int_{\frac{z}{\sqrt{2}} + \frac{1}{\sqrt{2t}}}^{\infty} e^{-y^2} dy \right] \quad ,$$

where we have made changes of variables to reduce all the exponents to simple form.

The integrands are now only functions of the square of the distance from the origin so that we may change the regions of integration to any other set similarly situated with respect to the origin. From a study of the geometry of the regions of integration we find that:

$$\int_{\frac{1}{\sqrt{t}}}^{\infty} e^{-z^2} dz \int_0^{\frac{z}{\sqrt{2}} - \frac{1}{\sqrt{2t}}} e^{-y^2} dy \int_0^{\frac{z}{\sqrt{2}} - \frac{1}{\sqrt{2t}}} e^{-x^2} dx = 2 \int_{\frac{1}{\sqrt{3t}}}^{\infty} e^{-z^2} dz \int_{\frac{z}{\sqrt{2}}}^{\infty} e^{-y^2} dy \int_{y\sqrt{3}}^{\infty} e^{-x^2} dx \quad ,$$

$$\int_0^{\infty} e^{-z^2} dz \int_{\frac{z}{\sqrt{2}} + \frac{1}{\sqrt{2t}}}^{\infty} e^{-y^2} dy \int_{\frac{z}{\sqrt{2}} + \frac{1}{\sqrt{2t}}}^{\infty} e^{-x^2} dx = 4 \int_{\frac{1}{\sqrt{3t}}}^{\infty} e^{-z^2} dz \int_{\frac{z}{\sqrt{2}}}^{\infty} e^{-y^2} dy \int_{y\sqrt{3}}^{\infty} e^{-x^2} dx \quad ,$$

$$\int_{\frac{1}{2\sqrt{t}}}^{\infty} e^{-z^2} dz \int_{\frac{z}{\sqrt{2}} + \frac{1}{2\sqrt{2t}}}^{\infty} e^{-y^2} dy \int_0^{\frac{z}{\sqrt{2}} - \frac{1}{2\sqrt{2t}}} e^{-x^2} dx = 3 \int_{\frac{1}{2\sqrt{3t}}}^{\infty} e^{-z^2} \int_{z\sqrt{2}}^{\infty} e^{-y^2} dy \int_{y\sqrt{3}}^{\infty} e^{-x^2} dx \quad .$$

Hence:

$$\chi_4 = \frac{3\pi\sqrt{\pi}}{4t\sqrt{t}}\, e^{\frac{1}{t}} \left[\int_{\frac{1}{\sqrt{3t}}}^{\infty} e^{-z^2} dz \int_{\frac{z}{\sqrt{2}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3})\, dy \right.$$

$$\left. + e^{-\frac{1}{4t}} \int_{\frac{1}{2\sqrt{3t}}}^{\infty} e^{-z^2} dz \int_{z\sqrt{2}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3})\, dy \right] .$$

If we change the order of integration in each of the double integrals:

$$\chi_4 = \frac{3\pi^2}{8t\sqrt{t}}\, e^{\frac{1}{t}} \left[\int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3}) \{\operatorname{erfc}(1/\sqrt{3t}) - \operatorname{erfc}(y\sqrt{2})\}\, dy \right.$$

$$\left. + e^{-\frac{1}{4t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3}) \{\operatorname{erfc}(1/2\sqrt{3t}) - \operatorname{erfc}(y/\sqrt{2})\}\, dy \right]$$

$$= \frac{\sqrt{3\pi}}{2\sqrt{t}}\, e^{\frac{1}{3t}} \left[\operatorname{erfc}(1/\sqrt{3t}) + e^{-\frac{1}{4t}} \operatorname{erfc}(1/2\sqrt{3t})\right] \chi_3$$

$$- \frac{3\pi^2}{8t\sqrt{t}}\, e^{\frac{1}{t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3}) \{\operatorname{erfc}(y\sqrt{2}) + e^{-\frac{1}{4t}} \operatorname{erfc}(y/\sqrt{2})\}\, dy \quad .$$

Therefore

$$\chi_4 = \frac{3}{2}\chi_3 \left[\psi(0, 3t/4) + 2\psi(0, 3t)\right]$$

$$- \frac{3\pi^2}{8t\sqrt{t}}\, e^{\frac{1}{t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3}) \{\operatorname{erfc}(y\sqrt{2}) + e^{-\frac{1}{4t}} \operatorname{erfc}(y/\sqrt{2})\}\, dy \quad . \qquad (4.6)$$

The fivefold integral $\int_0^\infty \phi^4(x,t)\,dx$ can be reduced to a single integral by a similar argument. Noting the difference in sign between Equations 3.7 and 3.8, and repeating the analysis leading to Equation 4.6 yields the result:

$$\int_0^\infty \phi^4(x,t)\,dx = \frac{3}{2}\chi_3 \left[\psi(0, 3t/4) - 2\psi(0,3t)\right]$$

$$- \frac{3\pi^2}{8t\sqrt{t}}\, e^{\frac{1}{t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3}) \{\operatorname{erfc}(y\sqrt{2}) - e^{-\frac{1}{4t}} \operatorname{erfc}(y/\sqrt{2})\}\, dy \quad . \qquad (4.7)$$

4.4 Tabulation of the Functions

Numerical values of X_2 and X_3 are given in Table 2. Instead of tabulating X_4 it was considered more convenient to give values for the components A and B where:

$$A = \frac{3}{2} X_3 \psi(0,3t/4) - \frac{3\pi^2}{8t\sqrt{t}} e^{\frac{1}{t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3}) \operatorname{erfc}(y\sqrt{2}) \, dy \quad ,$$

$$B = 3 X_3 \psi(0,3t) - \frac{3\pi^2}{8t\sqrt{t}} e^{+\frac{3}{4t}} \int_{\frac{1}{\sqrt{6t}}}^{\infty} e^{-y^2} \operatorname{erfc}(y\sqrt{3}) \operatorname{erfc}(y/\sqrt{2}) \, dy \quad .$$

5. INTEGRALS RELATED TO X_n

Provided they exist, integrals of the form $\int_{-\infty}^{\infty} x^m \psi^p \phi^q \, dx$ can be expressed in terms of X_1, X_2, and X_3 if $p + q \leqslant 3$, while if $p + q = 4$ they depend, in addition, on the components A and B of X_4. For greater values of $p + q$ a rapidly increasing number of primitive functions is required to represent the integrals.

5.1 A General Formula

By extending the definition of the Doppler broadened contour functions to the complex domain Keane and Clancy (1963) prove the following formula:

$$\int_{-\infty}^{\infty} x^k (\psi + i\phi)^m \, dx = \begin{cases} \pi i^{m-1} & , \quad m = k+1 \\ 0 & , \quad m > k+1 \end{cases} \quad , \tag{5.1}$$

which reduces on setting $k = 0$ to the special result given by Cook (1958), namely:

$$\int_{-\infty}^{\infty} (\psi + i\phi)^m \, dx = \begin{cases} \pi & , \quad m = 1 \\ 0 & , \quad m > 1 \end{cases} \quad . \tag{5.2}$$

Obviously the integral in Equation 5.1 does not exist when $m < k + 1$.

5.2 General Recurrence Relation

Another useful general formula can be obtained from the differential Equation 2.13 if we multiply by $x^n \phi^p \psi^q$ and integrate from minus infinity to infinity. Thus:

$$\int_{-\infty}^{\infty} x^{n+1} \phi^p \psi^{q+1} \, dx = \int_{-\infty}^{\infty} x^n \phi^{q+1} \psi^q \, dx - 2t \int_{-\infty}^{\infty} x^n \phi^p \psi^q \frac{\partial \psi}{\partial x} \, dx \quad .$$

Since

$$\int_{-\infty}^{\infty} x^n \phi^p \frac{\partial}{\partial x} (\psi^{q+1}) \, dx = -\int_{-\infty}^{\infty} \psi^{q+1} [n x^{n-1} \phi^p + p x^n \phi^{p-1} \frac{\partial \phi}{\partial x}] \, dx \quad ,$$

we can use Equation 2.14 and rearrange the terms to obtain:

$$\left(1+\frac{2p}{q+1}\right)\int_{-\infty}^{\infty} x^{n+1}\,\phi^{p}\psi^{q+1}\,dx = \int_{-\infty}^{\infty} x^{n}\,\phi^{p+1}\,\psi^{q}\,dx + \frac{2nt}{q+1}\int_{-\infty}^{\infty} x^{n-1}\,\phi^{p}\,\psi^{q+1}\,dx$$

$$+\frac{2p}{q+1}\int_{-\infty}^{\infty} x^{n}\,\phi^{p-1}\psi^{q+1}\,dx - \frac{2p}{q+1}\int_{-\infty}^{\infty} x^{n}\,\phi^{p-1}\psi^{q+2}\,dx \quad ,(5.3)$$

provided n, p, and q are such that the integrals converge.

5.3 Tabulation of Results

A systematic application of Equations 5.1, 5.2, and 5.3 yields the values of the integrals given in Table 3. The final numerical evaluation can be achieved with the aid of Table 2.

6. REFERENCES

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TABLE 1

INTEGRALS OBTAINABLE FROM THE TRANSFORMS

$\int_0^\infty \psi(ax,t)\cos px\,dx$	$\frac{\pi}{2a}\exp\left\{-\frac{p}{a}-\frac{p^2}{a^2}t\right\}$	
$\int_0^\infty \phi(ax,t)\sin px\,dx$	$\frac{\pi}{2a}\exp\left\{-\frac{p}{a}-\frac{p^2}{a^2}t\right\}$	
$\int_0^\infty x\,\psi(ax,t)\sin px\,dx$	$\frac{\pi}{2a}\left[-\frac{1}{a}-\frac{2p}{a^2}t\right]\exp\left\{-\frac{p}{a}-\frac{p^2}{a^2}t\right\}$	
$\int_0^\infty \psi(x,t_1)\,\psi(x,t_2)\cos px\,dx$	$T_1+T_2+T_3$	where T_1, T_2, T_3 are defined in Section 3.1
$\int_0^\infty \phi(x,t_1)\,\phi(x,t_2)\cos px\,dx$	$T_1+T_2-T_3$	
$\int_0^\infty \psi(x,t_1)\,\phi(x,t_2)\sin px\,dx$	$T_1-T_2+T_3$	
$\int_0^\infty \frac{\psi(x,t)}{1+x^2}\cos px\,dx$	$\frac{\pi}{4\sqrt{2t}}(E_1+E_2+E_3)$	where $E_1 = e^{p+\frac{1}{t}}\operatorname{erfc}(p\sqrt{t}+1/\sqrt{t})$ $E_2 = e^{-p+\frac{1}{t}}\operatorname{erfc}(1/\sqrt{t})$ $E_3 = e^{-p}\operatorname{erfc}(p\sqrt{t})$
$\int_0^\infty \frac{x\,\phi(x,t)}{1+x^2}\cos px\,dx$	$\frac{\pi}{4\sqrt{2t}}(E_1+E_2-E_3)$	
$\int_0^\infty \frac{x\,\psi(x,t)}{1+x^2}\sin px\,dx$	$\frac{\pi}{4\sqrt{2t}}(-E_1+E_2+E_3)$	
$\int_0^\infty \frac{\phi(x,t)}{1+x^2}\sin px\,dx$	$\frac{\pi}{4\sqrt{2t}}(E_1-E_2+E_3)$	

TABLE 1 (continued)

$\int_0^\infty \psi(x,t_1)\,\psi(x,t_2)\,dx$	$\frac{\pi}{4}\,\psi\left(0,\ \frac{t_1+t_2}{4}\right)$
$\int_0^\infty \phi(x,t_1)\,\phi(x,t_2)\,dx$	$\frac{\pi}{4}\,\psi\left(0,\ \frac{t_1+t_2}{4}\right)$
$\int_0^\infty \frac{\psi(x,t)}{a^2+x^2}\,dx$	$\frac{\pi}{2a(1+a)}\,\psi\left(0, t/(1+a)^2\right)$
$\int_0^\infty \frac{x\,\phi(x,t)}{a^2+x^2}\,dx$	$\frac{\pi}{2a(1+a)}\,\psi\left(0,\ t/(1+a)^2\right)$
$\int_0^\infty \frac{\psi(x,t)}{(a^2+x^2)^2}\,dx$	$\frac{\pi}{4a^3(1+a)}\,\psi\left(0,t/(1+a)^2\right) + \frac{\pi}{8a^2t}\left[1-\psi\left(0,t/(1+a)^2\right)\right]$
$\int_0^\infty \frac{\psi(ax,t)}{1+x^2}\,dx$	$\frac{\pi}{2(1+a)}\,\psi\left(0,t/(1+a)^2\right)$
$\int_0^\infty \frac{\psi(ax,t)}{(1+x^2)^2}\,dx$	$\frac{\pi}{4(1+a)}\,\psi\left(0,t/(1+a)^2\right) + \frac{\pi a}{8t}\left[1-\psi\left(0,t/(1+a)^2\right)\right]$
$\int_0^\infty \frac{\psi^2(x,t)}{1+x^2}\,dx$	$\frac{\pi}{16}\left[2\,\psi^2(0,t/2) + \psi^2(0,t/4)\right]$
$\int_0^\infty \frac{\phi^2(x,t)}{1+x^2}\,dx$	$\frac{\pi}{16}\left[2\,\psi^2(0,t/2) - \psi^2(0,t/4)\right]$
$\int_0^\infty \frac{x\,\psi(x,t)\,\phi(x,t)}{1+x^2}\,dx$	$\frac{\pi}{16}\,\psi^2(0,t/4)$

TABLE 1 (continued)

$\int_0^\infty e^{-\alpha x^2}\psi(ax,t)\,dx$	$\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\;\psi\,(0,t+a^2/4\,\alpha)$	
$^*\int_{-\infty}^{\infty}\frac{\psi(x,t)}{x+a}\,dx$	$\pi\,\phi(a,t)$	where $^*\int$ denotes the principal value
$^*\int_{-\infty}^{\infty}\frac{\phi(x,t)}{x+a}\,dx$	$\pi\,\psi(a,t)$	
$\int_0^\infty \psi(ax,t_1)\,\psi(b\,\overline{u-x},\,t_2)\,dx$	$\frac{\pi}{2}\,\frac{1}{a+b}\,\psi\left(\frac{ab}{a+b}\,u\,,\,\frac{a^2t_2+b^2t_1}{(a+b)^2}\right)$	
$\int_0^\infty \phi(ax,t_1)\,\phi(b\,\overline{u-x},\,t_2)\,dx$	$-\frac{\pi}{2}\,\frac{1}{a+b}\,\psi\left(\frac{ab}{a+b}\,u,\,\frac{a^2t_2+b^2t_1}{(a+b)^2}\right)$	
$\int_0^\infty \frac{\psi(x-u,t)}{1+u^2}\,du$	$\frac{\pi}{4}\,\psi\left(\frac{x}{2}\,,\,\frac{t}{4}\right)$	
$\int_0^\infty \frac{\psi(x-u,t)}{(1+u^2)^2}\,du$	$\frac{\pi}{8}\left(1-\frac{1}{t}\right)\psi\left(\frac{x}{2}\,,\,\frac{t}{4}\right)+\frac{\pi}{8t}\left[1-x\,\phi\left(\frac{x}{2}\,,\,\frac{t}{4}\right)\right]$	
$\int_0^\infty e^{-\alpha(x-u)^2}\psi(ax,t)\,dx$	$\frac{1}{2}\sqrt{(\pi/4)}\,\psi(au\,,\,t+a^2/4\,\alpha)$.	

TABLE 2

χ_2, χ_3, A, AND B

T	CHI 2	CHI 3	A	B	000
1.0	5.14969E-01	2.24084E-01	6.56106E-02	3.93391E-02	000
1.1	5.03261E-01	2.13142E-01	6.07728E-02	3.63225E-02	000
1.2	4.92488E-01	2.03365E-01	5.65626E-02	3.37090E-02	000
1.3	4.82523E-01	1.94563E-01	5.28656E-02	3.14234E-02	000
1.4	4.73261E-01	1.86590E-01	4.95935E-02	2.94079E-02	000
1.5	4.64618E-01	1.79325E-01	4.66774E-02	2.76178E-02	000
1.6	4.56522E-01	1.72674E-01	4.40626E-02	2.60177E-02	000
1.7	4.48914E-01	1.66557E-01	4.17052E-02	2.45791E-02	000
1.8	4.41743E-01	1.60908E-01	3.95692E-02	2.32792E-02	001
1.9	4.34966E-01	1.55674E-01	3.76252E-02	2.20990E-02	001
2.0	4.28546E-01	1.50806E-01	3.58487E-02	2.10230E-02	001
2.2	4.16650E-01	1.42023E-01	3.27198E-02	1.91339E-02	001
2.4	4.05844E-01	1.34304E-01	3.00535E-02	1.75303E-02	001
2.6	3.95960E-01	1.27458E-01	2.77559E-02	1.61531E-02	001
2.8	3.86868E-01	1.21339E-01	2.57566E-02	1.49586E-02	001
3.0	3.78462E-01	1.15832E-01	2.40023E-02	1.39133E-02	001
3.2	3.70654E-01	1.10846E-01	2.24513E-02	1.29916E-02	001
3.4	3.63374E-01	1.06307E-01	2.10709E-02	1.21733E-02	001
3.6	3.56562E-01	1.02155E-01	1.98352E-02	1.14423E-02	002
3.8	3.50168E-01	9.83417E-02	1.87230E-02	1.07858E-02	002
4.0	3.44147E-01	9.48248E-02	1.77172E-02	1.01931E-02	002
4.2	3.38465E-01	9.15700E-02	1.68035E-02	9.65571E-03	002
4.4	3.33088E-01	8.85481E-02	1.59703E-02	9.16642E-03	002
4.6	3.27988E-01	8.57338E-02	1.52076E-02	8.71925E-03	002
4.8	3.23143E-01	8.31059E-02	1.45070E-02	8.30914E-03	002
5.0	3.18530E-01	8.06458E-02	1.38616E-02	7.93180E-03	002
5.5	3.07897E-01	7.51286E-02	1.24512E-02	7.10915E-03	002
6.0	2.98363E-01	7.03626E-02	1.12754E-02	6.42529E-03	002
6.5	2.89744E-01	6.62001E-02	1.02817E-02	5.84881E-03	003
7.0	2.81898E-01	6.25303E-02	9.43200E-03	5.35701E-03	003
7.5	2.74713E-01	5.92683E-02	8.69803E-03	4.93309E-03	003
8.0	2.68096E-01	5.63479E-02	8.05838E-03	4.56435E-03	003
8.5	2.61976E-01	5.37168E-02	7.49655E-03	4.24104E-03	003
9.0	2.56290E-01	5.13330E-02	6.99963E-03	3.95554E-03	003
9.5	2.50988E-01	4.91624E-02	6.55736E-03	3.70180E-03	003
10.0	2.46028E-01	4.71769E-02	6.16151E-03	3.47502E-03	003

(continued)

TABLE 2 (continued)

T	CHI 2	CHI 3	A	B	
0.0	2.46028E-01	4.71769E-02	6.16151E-03	3.47502E-03	0037
1.0	2.36991E-01	4.36720E-02	5.48354E-03	3.08732E-03	0038
2.0	2.28949E-01	4.06735E-02	4.92522E-03	2.76876E-03	0039
3.0	2.21726E-01	3.80768E-02	4.45848E-03	2.50297E-03	0040
4.0	2.15189E-01	3.58049E-02	4.06323E-03	2.27830E-03	0041
5.0	2.09234E-01	3.37992E-02	3.72481E-03	2.08624E-03	0042
6.0	2.03776E-01	3.20148E-02	3.43221E-03	1.92043E-03	0043
7.0	1.98750E-01	3.04162E-02	3.17708E-03	1.77603E-03	0044
8.0	1.94099E-01	2.89754E-02	2.95291E-03	1.64932E-03	0045
9.0	1.89778E-01	2.76698E-02	2.75462E-03	1.53736E-03	0046
0.0	1.85749E-01	2.64808E-02	2.57815E-03	1.43782E-03	0047
2.0	1.78443E-01	2.43944E-02	2.27821E-03	1.26888E-03	0048
.0	1.71976E-01	2.26223E-02	2.03348E-03	1.13127E-03	0049
.0	1.66198E-01	2.10975E-02	1.83056E-03	1.01733E-03	0050
8.0	1.60990E-01	1.97710E-02	1.65998E-03	9.21682E-04	0051
0.0	1.56265E-01	1.86059E-02	1.51487E-03	8.40415E-04	0052
2.0	1.51950E-01	1.75741E-02	1.39016E-03	7.70645E-04	0053
.0	1.47988E-01	1.66537E-02	1.28200E-03	7.10193E-04	0054
.0	1.44333E-01	1.58273E-02	1.18744E-03	6.57391E-04	0055
3.0	1.40947E-01	1.50811E-02	1.10417E-03	6.10935E-04	0056
.0	1.37798E-01	1.44038E-02	1.03038E-03	5.69796E-04	0057
.0	1.34859E-01	1.37862E-02	9.64609E-04	5.33153E-04	0058
.0	1.32107E-01	1.32206E-02	9.05669E-04	5.00339E-04	0059
.0	1.29523E-01	1.27006E-02	8.52602E-04	4.70813E-04	0060
.0	1.27090E-01	1.22210E-02	8.04611E-04	4.44127E-04	0061
.0	1.24794E-01	1.17770E-02	7.61035E-04	4.19910E-04	0062
.0	1.19575E-01	1.07992E-02	6.67990E-04	3.68245E-04	0063
.0	1.14980E-01	9.97426E-03	5.92729E-04	3.26503E-04	0064
.0	1.10891E-01	9.26864E-03	5.30801E-04	2.92190E-04	0065
.0	1.07222E-01	8.65800E-03	4.79093E-04	2.63564E-04	0066
.0	1.03905E-01	8.12420E-03	4.35373E-04	2.39379E-04	0067
.0	1.00885E-01	7.65350E-03	3.98003E-04	2.18720E-04	0068
.0	9.81214E-02	7.23523E-03	3.65752E-04	2.00903E-04	0069
.0	9.55786E-02	6.86104E-03	3.37683E-04	1.85404E-04	0070
.0	9.32287E-02	6.52425E-03	3.13068E-04	1.71820E-04	0071
.0	9.10482E-02	6.21949E-03	2.91338E-04	1.59833E-04	0072

(continued)

TABLE 2 (continued)

T	CHI 2	CHI 3	A	B
100.0	9.10482E-02	6.21949E-03	2.91338E-04	1.59833E-04
110.0	8.71200E-02	5.68923E-03	2.54799E-04	1.39694E-04
120.0	8.36705E-02	5.24344E-03	2.25377E-04	1.23492E-04
130.0	8.06089E-02	4.86331E-03	2.01266E-04	1.10227E-04
140.0	7.78671E-02	4.53525E-03	1.81211E-04	9.92030E-05
150.0	7.53926E-02	4.24918E-03	1.64314E-04	8.99215E-05
160.0	7.31443E-02	3.99748E-03	1.49918E-04	8.20184E-05
170.0	7.10895E-02	3.77428E-03	1.37532E-04	7.52217E-05
180.0	6.92018E-02	3.57496E-03	1.26781E-04	6.93241E-05
190.0	6.74595E-02	3.39587E-03	1.17376E-04	6.41658E-05
200.0	6.58448E-02	3.23406E-03	1.09090E-04	5.96217E-05
220.0	6.29405E-02	2.95311E-03	9.51915E-05	5.19998E-05
240.0	6.03950E-02	2.71751E-03	8.40267E-05	4.58770E-05
260.0	5.81397E-02	2.51705E-03	7.48908E-05	4.08669E-05
280.0	5.61230E-02	2.34439E-03	6.72983E-05	3.67040E-05
300.0	5.43054E-02	2.19410E-03	6.09055E-05	3.32005E-05
320.0	5.26560E-02	2.06207E-03	5.54624E-05	3.02193E-05
340.0	5.11502E-02	1.94517E-03	5.07831E-05	2.76586E-05
360.0	4.97682E-02	1.84091E-03	4.67264E-05	2.54408E-05
380.0	4.84939E-02	1.74734E-03	4.31834E-05	2.35059E-05
400.0	4.73140E-02	1.66290E-03	4.00685E-05	2.18067E-05
420.0	4.62171E-02	1.58630E-03	3.73135E-05	2.03054E-05
440.0	4.51941E-02	1.51650E-03	3.48638E-05	1.89717E-05
460.0	4.42369E-02	1.45262E-03	3.26745E-05	1.77809E-05
480.0	4.33387E-02	1.39395E-03	3.07091E-05	1.67127E-05
500.0	4.24937E-02	1.33987E-03	2.89371E-05	1.57502E-05

TABLE 3

INTEGRALS RELATED TO χ_n

$$\int_0^\infty \psi^2\,dx = \chi_2$$

$$\int_0^\infty \phi^2\,dx = \chi_2$$

$$\int_0^\infty x\,\psi\,\phi\,dx = \frac{1}{2}\chi_1$$

$$\int_0^\infty x^2\,\psi^2\,dx = \frac{1}{2}\chi_1 + t\,\chi_2$$

$$\int_0^\infty \psi^3\,dx = \chi_3$$

$$\int_0^\infty \psi\,\phi^2\,dx = \frac{1}{3}\chi_3$$

$$\int_0^\infty x\,\psi^2\,\phi\,dx = \frac{1}{3}\chi_2 - \frac{1}{9}\chi_3$$

$$\int_0^\infty x\,\phi^3\,dx = \chi_2 - \frac{1}{3}\chi_3$$

$$\int_0^\infty x^2\,\psi^3\,dx = \frac{1}{3}\chi_2 - \frac{1}{9}\chi_3 + \frac{2}{3}\,t\,\chi_3$$

$$\int_0^\infty x^2\,\psi\,\phi^2\,dx = \frac{1}{9}\chi_2 - \frac{1}{27}\chi_3 + \frac{2}{9}\,t\,\chi_3 + \frac{1}{3}\chi_1$$

TABLE 3 (continued)

$$\int_0^\infty x^3\,\psi^2\,\phi\,dx = \frac{5}{12}\chi_1 - \frac{1}{9}\chi_2 + \frac{1}{27}\chi_3 + \frac{5}{6}t\,\chi_2 - \frac{1}{3}t\,\chi_3$$

$$\int_0^\infty x^4\,\psi^3\,dx = \frac{5}{12}\chi_1 - \frac{1}{9}\chi_2 + \frac{1}{27}\chi_3 + \frac{7}{6}t\,\chi_2 - \frac{4}{9}t\,\chi_3 + \frac{2}{3}t^2\,\chi_3$$

$$\int_0^\infty \psi^4\,dx = \chi_4 = A + B$$

$$\int_0^\infty \psi^2\,\phi^2\,dx = \frac{1}{3}\,A$$

$$\int_0^\infty \phi^4\,dx = A - B$$

$$\int_0^\infty x\,\psi^3\,\phi\,dx = \frac{1}{4}\chi_3 - \frac{1}{4}B - \frac{1}{2}t\,\chi_4$$

$$\int_0^\infty x\,\psi\,\phi^3\,dx = \frac{1}{4}\chi_3 - \frac{1}{4}B - \frac{1}{2}t\,\chi_4$$

$$\int_0^\infty x^2\,\psi^4\,dx = \frac{1}{4}\chi_3 - \frac{1}{4}B$$

$$\int_0^\infty x^2\,\psi^2\,\phi^2\,dx = \frac{2}{9}\chi_2 - \frac{17}{108}\chi_3 + \frac{1}{6}t\,\chi_4 + \frac{1}{12}B + \frac{1}{9}t\,A$$

$$\int_0^\infty x^2\,\phi^4\,dx = \frac{4}{3}\chi_2 - \frac{43}{36}\chi_3 + t\,\chi_4 + \frac{3}{4}B + \frac{2}{3}t\,A$$

TABLE 3 (continued)

$$\int_0^\infty x^3\,\psi^3\,\phi\,dx = \frac{4}{15}\chi_2 - \frac{4}{15}\chi_3 - \frac{1}{15}t\,\chi_3 + \frac{1}{10}t\,\chi_4 - \frac{2}{5}t^2\,\chi_4 + \frac{3}{20}B - \frac{1}{15}t\,A - \frac{1}{5}t\,B$$

$$\int_0^\infty x^3\,\psi\,\phi^3\,dx = \frac{4}{15}\chi_2 - \frac{4}{15}\chi_3 - \frac{1}{15}t\,\chi_3 + \frac{1}{10}t\,\chi_4 - \frac{2}{5}t^2\,\chi_4 + \frac{3}{20}B - \frac{1}{15}t\,A - \frac{1}{5}t\,B$$

$$\int_0^\infty x^4\,\psi^4\,dx = \frac{4}{15}\chi_2 - \frac{4}{15}\chi_3 + \frac{37}{120}t\,\chi_3 + \frac{1}{10}t\,\chi_4 - \frac{2}{3}t^2\,\chi_4 + \frac{3}{20}B + \frac{1}{15}t\,A - \frac{23}{40}t\,B$$

$$\int_0^\infty x^4\,\psi^2\,\phi^2\,dx = \frac{5}{18}\chi_1 - \frac{22}{135}\chi_2 + \frac{46}{405}\chi_3 + \frac{7}{9}t\,\chi_2 - \frac{193}{540}t\,\chi_3 - \frac{1}{30}t\,\chi_4 + \frac{3}{10}t^2\,\chi_4$$
$$- \frac{1}{20}B + \frac{1}{45}t\,A + \frac{3}{20}t\,B + \frac{1}{9}t^2\,A$$

$$\int_0^\infty x^5\,\psi^3\,\phi\,dx = \frac{1}{3}\chi_1 - \frac{56}{225}\chi_2 + \frac{128}{675}\chi_3 + \frac{34}{25}t\,\chi_2 - \frac{212}{225}t\,\chi_3 - \frac{3}{50}t\,\chi_4 - \frac{4}{75}t^2\,\chi_3 + \frac{91}{150}t^2\,\chi_4$$
$$- \frac{16}{25}t^3\,\chi_4 - \frac{9}{100}B - \frac{1}{75}t\,A + \frac{14}{25}t\,B - \frac{1}{25}t^2\,A - \frac{8}{25}t^2\,B$$

$$\int_0^\infty x^6\,\psi^4\,dx = \frac{1}{3}\chi_1 - \frac{56}{225}\chi_2 + \frac{128}{675}\chi_3 + \frac{152}{75}t\,\chi_2 - \frac{362}{225}t\,\chi_3 - \frac{3}{50}t\,\chi_4 - \frac{287}{400}t^2\,\chi_3 + \frac{257}{300}t^2\,\chi_4$$
$$- \frac{173}{75}t^3\,\chi_4 - \frac{9}{100}B - \frac{1}{75}t\,A + \frac{187}{200}t\,B + \frac{19}{150}t^2\,A - \frac{703}{400}t^2\,B$$