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PART V - ADAPTATION OF MODEL TO PACKINGS IN CYLINDRICAL VESSELS

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ABSTRACT

Following the exploratory study reported in Part I, detailed experimental determinations have been made of the properties of a range of unbiased random packings with cylinder-to-sphere diameter ratios of 11.4 and 8.71. Properties at the dense end of the range depend on the diameter ratio. Properties at the loose end of the range are independent of the diameter ratio, and agree closely with those computed from the model developed in Part IV for the semi-infinite case.

The model is adapted to the cylindrical case by means of quadratic equations whose coefficients depend on the cylinder-to-sphere diameter ratio. Although the adapted model is based on experiments with comparatively small diameter ratios, it is shown to hold for larger ratios, without the need to call further on experimentally determined properties.

It is shown that commonly used preparation methods, such as placing or pouring spheres on the top surface, can result in packings with radial bias. Two methods of preparing unbiased random packings are described.

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CONTAINERS; CYLINDRICAL CONFIGURATION; CYLINDERS; FUEL ASSEMBLIES; GEOMETRY; MOTION; PEBBLE BED REACTORS; RANDOMNESS; REACTOR CORES; REACTOR FUELLING; SHAPE; SPHERES

PREFACE

This report is a part of a series on "Some Geometrical Properties of Packings of Equal Spheres in Cylindrical Vessels" as follows:

Part I Exploratory Study of Random Packings in Small Vessels.
G. A. Tingate, AAEC/E208.

Part II The Cylindrically Ordered Packing.
F. A. Rocke, AAEC/E240.

Part III Basic Model away from the Influence of Wall Effects.
N. W. Ridgway, G. A. Tingate, AAEC/E202.

Part IV Extension of Model to Outer Region of Semi-infinite Vessel with Plane Wall.
G. A. Tingate, AAEC/E223.

Part V Adaptation of Model to Packings in Cylindrical Vessels.
G. A. Tingate, this report, AAEC/E234.

Part VI Discussion and Conclusions.
N. W. Ridgway, F. A. Rocke, and G. A. Tingate, AAEC report in preparation.

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(continued)

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1. INTRODUCTION

In 1966 the Australian Atomic Energy Commission completed a feasibility study of a high-temperature gas-cooled reactor system based on a pebble bed concept, using spheres of fuelled beryllium oxide. The work reported in this series is a study of the structure of random packings of equal spheres made in support of this project.

Part I is an account of exploratory experimental determinations of some of the properties of random packings of spheres of 1 in. nominal diameter in a 5.75 in. diameter cylindrical vessel. It shows that errors were incurred between observed and computed values, partly because of the small cylinder-to-sphere diameter ratio (D_v/D_p) used, and partly because of end effects. An accuracy of about ± 1 per cent should be attainable with a value of D_v/D_p of 8 or greater, and a packing height to sphere diameter ratio (H/D_p) of 12, provided end effects are eliminated and packing methods without deliberate radial bias are used. The same accuracy should also be attainable by the wax freezing, machining and weighing method commonly used by other workers, provided certain additional observations are made, and better accuracy is unlikely by any other method.

The present study includes an experimental programme using an improved procedure to resolve the remaining discrepancies. In addition, the model for the semi-infinite case, developed in Parts III and IV, has been adapted to the cylindrical case. A key feature of this adaptation is the use of the properties of the appropriate cylindrically ordered packing reported by Rocke (1971) at the dense end of the range, in place of those of the rhombohedral array.

During the analysis it was found desirable to amend the equation for OF_W, the occupancy factor of the spheres touching the wall. The original Equation 1 in Part I has a factor F in the denominator to account for the effect of the curvature of the wall, that is, OF_W is expressed in terms of the densest possible packing for that particular D_v/D_p ratio. Here it is expressed in terms of the rhombohedral array by omitting F to give the amended equation

$$OF_W = \frac{n\, D_p^2}{3.628\, H(D_v - D_p)} \quad , \qquad (1)$$

where n is the number of spheres touching the wall.

2. DIMENSIONS OF SPHERES AND VESSELS USED IN EXPERIMENTS

The nominal diameter of the spheres was 1 in. Two types were used, machined aluminium and bonded zircon sand. Measurements with a micrometer, with a minimum of 3 readings on each of 20 spheres of each type gave diameters of the aluminium spheres ranging ± 0.2 per cent about a mean of 0.9902 in., and those of the zircon spheres ± 2.0 per cent about a mean of 1.003 in.

Two cylindrical vessels with flat bases were made from Perspex, the axial length of each being 21.5 in., and the mean diameters (measured by micrometer) 11.393 in. and 8.711 in. As discussed in Section 9 of Part I, the smaller vessel was expected to permit determinations of the mean properties of random packings to be made to an accuracy of ± 1 per cent. The intention was therefore to use both vessels initially, and then discontinue the use of the larger one unless irreconcilable differences were found between the two. For this accuracy to be achieved, errors in individual observations should not exceed about ± 0.3 per cent.

3. EXPERIMENTAL PROCEDURE

Fine horizontal circumferential lines were scratched on the cylindrical walls of the vessels at axial distances 1 in., 7 in., 13 in. and 19 in. above the upper surface of the base. This allowed the base to be covered with a dense layer of spheres, giving a test section 18 in. high with a dead section 2.5 in. high above it. In practice, determinations were not made in the bottom 6 in. of the test section, to avoid residual axial bias due to the base. The test section used for the experiments therefore consisted of two halves each of 6 in. axial height. This eliminated the effect of the base whether or not it was first covered with a dense layer of spheres.

A removable crown piece, made of the same tubing as the vessel, was placed on top during the preparation of the packing. Its axial length was about 3 in. and the packing was always prepared initially to the top of the crown piece, ensuring that the vessel was completely filled by the preparation method under investigation. It also enabled packings to be consolidated by vibration or jolting while maintaining reasonably uniform submerged conditions over the top of the vessel.

Before removing the crown piece a mark was made on the outside of the cylindrical wall opposite each sphere judged to be in the test section and touching the wall. A count was made for each half of the test section with a view to rejecting any packing with axial bias in excess of ±1 per cent about the mean. About one packing in every three was rejected on this account. Furthermore, as the dense end of the range was approached, it was found impossible to meet this standard because the bottom half could not be vibrated down to the same density as the top half. In such cases observations were limited to the top half; each experiment was duplicated, and the results were combined to represent the equivalent complete packing.

Having prepared a given packing of the desired standard, the crown piece was eased off to allow some of the surplus spheres to roll off without disturbing the remainder. The surplus spheres were carefully removed by hand until no part of any sphere could be seen projecting above the line defining the top of the test section.

The top half of the packing was then unpicked carefully by hand until no part of any sphere could be seen projecting above the line of demarcation between the two halves of the test section. During unpicking it was usually found that one or more of the spheres previously judged to be touching the wall were in fact just clear of it, and the record was adjusted accordingly. The following properties were determined:

Total number of spheres in packing	N
Number of spheres touching cylindrical wall	n
Volume of intruding sphere material	V_I*
Volumes of intruding sphere material in inner and outer parts of outer region	V_{II}*, V_{IO}*
Occupancy factor of spheres in central region	OF_C

The same procedure was then applied to the bottom half of the packing.

4. EXPERIMENTS

4.1 Experiments with 1 in. Spheres in 11.4 in. Vessel

The first experiments were carried out in the 11.4 in. diameter vessel, using aluminium spheres because of their better uniformity and precision. In addition to the observations stated under Section 3, a record was made of the number and radial location of all spheres between D_p and 1.25 D_p from the wall. This provided confirmation of the existence of a cusp in the void fraction curve at a distance D_p from the wall, and the correctness of the interpolated curve shown in Figure 3 of Part I.

* For consistency with Part IV, values are presented in this part in the standardised forms V_I/V_W, V_{II}/V_W and V_{IO}/V_W, where V_W is the volume of the spheres touching the cylindrical wall.

During the experiments with the 11.4 in. cylinder, several practical difficulties were encountered which would have made adherence to the original standard of axial bias too time consuming. The chief difficulty was that packings with better than ± 1 per cent axial bias at the wall usually fell short of this standard in the central region, while the intruding spheres fell even further short. In addition, the weight of the packing made densifying by vibration or jolting* unwieldy and difficult with the equipment available. The use of the 11.4 in. cylinder was therefore limited to the experiments listed in Table 1. From the results tabulated, the extent to which individual packings fell short of the desired standard of axial bias can be seen. However, the overall axial bias, as indicated by the totals, was well within the standard, except for the intruding spheres which were only slightly outside it. It is therefore concluded that the precautions taken to avoid axial bias were successful, and that the axial bias in a given packing was due to unavoidable experimental scatter. Because of the experimental difficulties with the 11.4 in. vessel, and because the differences between it and the 8.71 in. vessel due to curvature are small compared with experimental scatter, the smaller vessel was considered adequate for further experimental determinations.

4.2 Experiments with 1 in. Spheres in 8.71 in. Vessel

Results of experiments with both the aluminium and the zircon spheres are given in Tables 2 and 3. The axial bias of individual packings, and the overall axial bias indicated by the totals, agree closely with those for the 11.4 in. diameter cylinder.

5. ADAPTATION OF MODEL

5.1 Basic Approach

Relationships are required for the various properties of interest over the practical range of random packings in terms of the D_v/D_p ratio. Maximum use is to be made of the model and minimum use of experimental results. It is only necessary to adapt the model in the case of the $OF_C - OF_W$ and $V_I/V_W - OF_W$ relationships, since the other properties of interest can be derived from them.

The model can be applied directly at the loose end of the range of packings in cylindrical vessels. In Section 7 of Part I, experimentally determined values are given of the mean void fractions in the central region of packings in cylindrical vessels with various D_v/D_p ratios. The values for dense and intermediate random packings vary with the D_v/D_p ratio, but those for the loosest random packings are independent of this ratio. In Section 2 of Part IV, experimentally determined values are given for the loosest random packings prepared in a prismatic vessel, simulating as far as possible a semi-infinite vessel with a plane wall. Within the limits of experimental accuracy, the values are the same as those obtained with any cylindrical vessel, all lying in the range 0.420 to 0.423.

Thus the value of OF_C of the loosest possible random packing is not influenced by the walls of the vessel. It follows that the $OF_C - OF_W$ relationship can be represented by a family of curves running together at the loose end of the range.

The experimentally determined properties of the intruding spheres given in Part IV and in Section 4 of this part indicate that the $V_I/V_W - OF_W$ relationship is partly amenable to this treatment. As shown in Part IV, the values of V_I calculated from the model correspond to the observed volumes V_{II} in the inner part of the outer region. A significant fraction of V_I, V_{IO}, is found in the outer part of the outer region over the range of random packings normally obtained in practice, but this cannot be derived from the model. It is therefore necessary to consider the terms V_{II} and V_{IO} separately. The model is then used to determine the $V_{II}/V_W - OF_W$ relationship, which can also be represented by a family of curves running together at the loose end of the range. The $V_{IO}/V_W - OF_W$ relationship can only be derived from experimental values.

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* A ¼ hp mechanical jolting machine was used to produce the densest packings. It delivered vertical impulses at a rate of about two per second, but was clearly overloaded and only partly effective with the 11.4 in. diameter packings.

TABLE 1

EXPERIMENTAL RESULTS, 1 in. ALUMINIUM SPHERES IN 11.4 in. CYLINDER

Preparation Method	Top Half of Packing				Bottom Half of Packing				Complete Packing				Remarks
	n	N	V_I/V_W	OF_C	n	N	V_I/V_W	OF_C	OF_W	OF_C	V_{II}/V_W	V_{IO}/V_W	
10.5 in. load tube	178	704	0.1710	0.813	179	689	0.1795	0.789	0.773	0.801	0.1619	0.0134	
Heavy thumping	214	758	0.0799	0.856	214	740	0.1105	0.837	0.927	0.847	0.0836	0.0116	
Dropped from 18 in. a hand-ful at a time	193	725	0.1536	0.842	189	727	0.1379	0.827	0.827	0.834	0.1421	0.0037	
Hand packed to give loose structure at wall	159	696	0.3196	0.800	164	706	0.2574	0.822	0.700	0.811	0.2543	0.0342	Radially biased
1¼ in. load tube biased to wall	203	730	0.1021	0.833	205	735	0.1091	0.835	0.884	0.834	0.1051	0.0005	Radially biased
Dense regular shell at wall, loose packing in centre	224	736	0.0425	0.827	223	709	0.0365	0.787	0.969	0.807	0.0395	0.0000	Radially biased
Dense regular shell at wall, loose packing in centre	231	753	0.0322	0.847	231	732	0.0285	0.813	1.000	0.830	0.0304	0.0000	Radially biased
Dense regular shell at wall, loose packing in centre	231	726	0.0281	0.804	231	730	0.0279	0.810	1.000	0.807	0.0280	0.0000	Radially biased
Dense regular shell at wall, dense random packing in centre	231	762	0.0551	0.853	231	756	0.0524	0.844	1.000	0.848	0.0538	0.0000	Radially biased
TOTALS (indicating overall axial bias)	1864	6590	0.9841	7.475	1867	6524	0.9397	7.364					

TABLE 2

EXPERIMENTAL RESULTS, 1 in. ALUMINIUM SPHERES IN 8.71 in. CYLINDER

Preparation Method	Top Half of Packing				Bottom Half of Packing				Complete Packing				Remarks
	n	N	V_I/V_W	OF_C	n	N	V_I/V_W	OF_C	OF_W	OF_C	V_{II}/V_W	V_{IO}/V_W	
7.25 in. load tube	125	399	0.1768	0.805	127	404	0.1721	0.816	0.732	0.811	0.1546	0.0198	
7.25 in. load tube	118	387	0.2432	0.770	118	394	0.2295	0.797	0.686	0.783	0.2059	0.0304	
7.25 in. load tube	121	393	0.1969	0.795	120	393	0.2338	0.785	0.700	0.790	0.1872	0.0281	
Medium vibration	137	420	0.1311	0.849	135	408	0.1248	0.820	0.790	0.835	0.1146	0.0134	
Heavy vibration	147	422	0.0925	0.837	146	417	0.0917	0.825	0.851	0.831	0.0786	0.0135	
TOTALS (indicating overall axial bias)	648	2021	0.8405	4.056	646	2016	0.8519	4.043					

TABLE 3

EXPERIMENTAL RESULTS, 1 in. ZIRCON SPHERES IN 8.71 in. CYLINDER

Preparation Method	Top Half of Packing				Bottom Half of Packing				Complete Packing				Remarks
	n	N	V_I/V_W	OF_C	n	N	V_I/V_W	OF_C	OF_W	OF_C	V_{II}/V_W	V_{IO}/V_W	
7.25 in. load tube	110	372	0.2330	0.792	119	380	0.1818	0.802	0.683	0.797	0.1767	0.0307	
7.25 in. load tube	110	372	0.2317	0.793	111	379	0.2406	0.809	0.660	0.801	0.2015	0.0347	
Light vibration	119	389	0.2038	0.824	119	375	0.1923	0.781	0.710	0.802	0.1684	0.0297	
Medium vibration	138	403	0.0962	0.844	135	403	0.1253	0.841	0.815	0.842	0.0949	0.0159	
Medium thumping	138	406	0.1370	0.835	136	399	0.1294	0.822	0.818	0.829	0.1096	0.0236	
Dense regular shell at wall, loose packing in centre	168	397	0.0106	0.761	168	401	0.0096	0.776	1.000	0.769	0.0106	0.0000	Radially biased
Heavy thumping *	142	412	0.1011	0.857	141	394	0.0829	0.809	0.845	0.833	0.0759	0.0161	
Heavy thumping *	144	405	0.0812	0.835	140	411	0.1165	0.853	0.848	0.844	0.0779	0.0210	
Heavy thumping *	143	407	0.1134	0.814	142	401	0.0787	0.847	0.851	0.831	0.0848	0.0113	
TOTALS (indicating overall axial bias)	783	2339	0.9123	4.849	788	2337	0.8790	4.831					

* Two separate packings in each case; results not included in totals

As shown in Part IV, the model also has the limitation that it does not take into account floating spheres. These are only found at the loose end of the range, and do not affect the normal range of random packings. Corrections, based on experimental determinations, are made on this account for values of OF_W less than 0.7.

Returning to the $OF_C - OF_W$ and $V_{II}/V_W - OF_W$ relationships, if a given curve is approximated by a quadratic expression, two of the three coefficients can be calculated from the model in terms of the third. This is because all the curves of a family are taken as having a common point and a common gradient at the loose end of the range. The third coefficient cannot be obtained from the model, but is calculated from the known value at the dense end of the range for that particular D_v/D_p ratio. Only if this approach failed to give a satisfactory correlation would it be necessary to make use of experimentally determined intermediate values to develop a higher order polynomial.

5.2 Derivation of Equations

5.2.1 $OF_C - OF_W$ relationship

The quadratic relationship is of the form

$$OF_C = A_C + B_C\,OF_W + C_C\,OF_W^2 \quad , \qquad (2)$$

where A_C, B_C and C_C are coefficients to be determined.

From the model in Part IV, the value of OF_W for the loosest possible random packing is 0.615, while the values of OF_C and the gradient of the $OF_C - OF_W$ curve are 0.7420 and 0.4926 respectively (see Appendix 1 for derivation). These values enable B_C and C_C to be expressed in terms of A_C, giving

$$OF_C = A_C(1 - 3.252\,OF_W + 2.644\,OF_W^2) + (1.920\,OF_W - 1.161\,OF_W^2) \qquad (3)$$

For an infinite D_v/D_p ratio, A_C is calculated directly from the properties of the cylindrically ordered packing. The value of OF_W is 1.0 and, as shown by Rocke (1970) the mean void fraction of the central region is 0.2885. This corresponds to the value of OF_C of 0.9609, so that $A_C = 0.5137$ and

$$OF_C = 0.5137 + 0.2497\,OF_W + 0.1975\,OF_W^2 \qquad (4)$$

The application of the method is not so straightforward for packings with finite D_v/D_p ratios. This is because the properties of cylindrically ordered packings vary with D_v/D_p, affecting the values of both OF_W and OF_C by several per cent for small values of D_v/D_p. Corrections for this must therefore be made, and for any other effect greater than about 0.3 per cent, to meet the required standard of accuracy.

OCCUPANCY FACTOR OF SPHERES TOUCHING WALL

Three corrections were made. The first, a reduction in $\overline{\Delta h}$, the mean axial spacing of the rings, is due to the curvature of the wall. It is calculated using the factor F2 derived in Appendix 1 of Part I. The second correction, an increase in $\overline{\Delta h}$, is due to the rings not being true horizontal circles, but having a wavelike structure. The spheres have both vertical and radial wave displacements in the inner rings, but only vertical displacements in the outer ring, radial displacements being prevented there by the constraint of the wall. The structure in the outer ring is such that a given sphere maintains a central position with respect to the two directly beneath it, as shown in Figure 1. It is thus supported by one of them only, resulting in an axial gap δh which varies cyclically around the ring. This second correction is the greater of the two, so there is a net increase in $\overline{\Delta h}$, whose value can be computed from the following expression derived in Appendix 2.

$$\frac{\overline{\Delta h}}{D_p} = \frac{0.8660}{F2} + \frac{0.2453}{\sqrt{\left(\frac{D_v}{D_p} - 1\right)^2 + 0.5938}} \quad . \tag{5}$$

The third correction is due to discontinuities in the numbers of spheres in the outer rings at particular transitional values of D_v/D_p . At values between the transitional values significant differences can arise between the theoretical and the actual numbers of spheres. For example, when $D_v/D_p = 7.37$, the theoretical number is 19.9, about 5 per cent greater than the 19 obtained in practice. It was not known at this point whether the theoretical or the whole number should be used in the adapted model, so computations were carried out for both. When the theoretical number is replaced by the whole number, the reduction in OF_w is partly offset by a reduction in $\overline{\Delta h}$. Analysis showed that the fractional reduction in $\overline{\Delta h}$ is about one third of the fractional increase in the circumferential spacing of the spheres. This ratio was incorporated in the adapted model when values of $\overline{\Delta h}$ so obtained were found to agree with analytically determined values within 1 part in 10,000. Values of interest are given in Table 4. The observed number is in general the whole number part of the theoretical number, with some inconsistencies when the theoretical number is close to an integer. When $D_v/D_p = 30.63$, only 92 spheres are accommodated where the theoretical number is 93.08, whereas when $D_v/D_p = 14.99$, 44 spheres are accommodated where the theoretical number is only 43.93. In the absence of a precise rule for determining the transition point, the whole number part of the theoretical number has been used in the adapted model. On the average the decimal part of the theoretical number is 0.5.

The terminal value of OF_w at the dense end of the range, based on the theoretical number of spheres, is given by

$$OF_{w_{max}} = \frac{\sqrt{3}}{2} \cdot \frac{F1}{(\overline{\Delta h}/D_p)} = \frac{0.8660\ F1}{(\overline{\Delta h}/D_p)} \quad , \tag{6}$$

where F1 is as derived in Appendix 1 of Part I.

The value based on the whole number of spheres is given by

$$OF_{w_{max}} = \frac{0.8660\ F1(1 - F3)}{(\overline{\Delta h}/D_p)\ (1 - F3/3)} \quad , \tag{7}$$

where $$F3 = 1 - \frac{\text{whole number}}{\text{theoretical number}} \quad .$$

Values so calculated are given in Table 4.

Two other effects pertinent to small values of D_v/D_p are discussed and evaluated in Appendix 2. It is not practicable to incorporate them in the adapted model, which in any case is subsequently shown to match the observed properties of random packings within the desired accuracy. However, the two effects would probably need to be taken into account when computing the properties of specific cylindrically ordered packings. Clarification of this point would involve further detailed experiments which are beyond the scope of this study.

Values derived from Equation 6 lie on the upper curve in Figure 2. Various values derived from Equation 7 are plotted as discrete points, since they do not lie on a common curve. In any general model it is desirable to incorporate a continuous function, so a third equation was introduced, based on the mean difference between the theoretical and the whole number of spheres:

TABLE 4

PROPERTIES OF SELECTED CYLINDRICALLY ORDERED PACKINGS

$\frac{D_v}{D_p}$	$\frac{D_p}{D_v}$	Number of Spheres in Outer Ring		F1	$OF_{W_{max}}$	
		Theoretical	Actual		Theoretical	Actual
40.37	0.0248	123.67	123	0.9999	0.9928	0.9892
30.63	0.0326	93.06	92	0.9998	0.9904	0.9828
25.28	0.0396	76.25	76	0.9997	0.9883	0.9861
19.01	0.0526	56.55	56	0.9995	0.9842	0.9777
14.99	0.0667	43.92	44	0.9991	0.9796	0.9807
11.52	0.0868	33.00	33	0.9985	0.9727	0.9727
11.39*	0.0878	32.59	32	0.9985	0.9724	0.9606
9.51	0.1052	26.66	26	0.9977	0.9662	0.9501
8.71*	0.1148	24.15	24	0.9972	0.9627	0.9586
7.52	0.1330	20.40	20	0.9961	0.9558	0.9432
5.77*	0.1733	14.87	14	0.9926	0.9393	0.9017

* Values used in present study

$$OF_{W_{max}} = \frac{0.8660\ F1\ (1-F4)}{(\overline{\Delta h}/D_p)\ (1-F4/3)} , \qquad (8)$$

where $$F4 = 1 - \frac{\text{theoretical number} - 0.5}{\text{theoretical number}} .$$

Values so derived are plotted as the lower curve in Figure 2.

OCCUPANCY FACTOR OF SPHERES IN CENTRAL REGION

The same considerations apply to the numbers of spheres in the rings in the central region. The treatment is simplified by the observed fact that $\overline{\Delta h}/D_p$ is constant for all rings, so Equation 5 applies also to the central region. The uncertainty regarding the theoretical or whole number of spheres in a given ring still holds, so the effects of both were evaluated.

A further major correction is necessary because of discontinuities in the number of rings in the central region at particular transitional values of D_v/D_p. Rocke (1971) invariably found that just below the transitional value the central ring contains 4 spheres. Just above the transitional value a new 'ring' appears as a single central sphere, while the second ring acquires a fifth sphere. This results in a significant discontinuity in the value of OF_C ; for example, at the transitional value of D_v/D_p = 7.798, the number of rings jumps from 4 to 5, corresponding to computed values of OF_C of 0.860 and 0.947 respectively. The effect diminishes rapidly as D_v/D_p increases, but even when D_v/D_p = 36.688, where the number of rings jumps from 21 to 22, OF_C increases by about 1 per cent. A very small change in D_v/D_p can therefore give rise to a substantial change in the value of OF_C, whether it is calculated on the basis of the theoretical or the whole numbers of spheres in the rings.

It is necessary to ascertain whether the smaller or the larger values of OF_C are appropriate. To this end, optimised* quadratic equations were derived directly from the experimentally determined properties of random packings in the 11.4 in., 8.71 in. and 5.75 in. diameter cylinders. The terminal values of OF_C at the dense end of the range were then calculated.

The results for the largest cylinder were unsatisfactory, chiefly because it had not been possible to obtain dense unbiased random packings with the available equipment. The results for the 8.71 in. and 5.75 in. cylinders provided good coverage of the range and resulted in the following equations:

$$OF_C = 0.3330 + 0.8375\, OF_W - 0.2804\, OF_W^2 \qquad (9)$$

and

$$OF_C = 0.2759 + 1.0232\, OF_W - 0.4314\, OF_W^2 \qquad (10)$$

Values of A_C, terminal values of OF_W calculated from Equations 6 and 7, and terminal values of OF_C calculated from Equations 4, 9 and 10 are given in Table 5.

TABLE 5

TERMINAL VALUES OF OF_W AND OF_C

$\frac{D_v}{D_p}$	$\frac{D_p}{D_v}$	A_C	F1	$OF_{W_{max}}$ theoretical	$OF_{C_{max}}$ theoretical	$OF_{W_{max}}$ actual	$OF_{C_{max}}$ actual
Infinity	0.0	0.5137	1.0	1.0	0.9609	1.0	0.9609
8.71	0.1148	0.3330	0.9972	0.9627	0.8794	0.9586	0.8754
5.77	0.1733	0.2759	0.9926	0.9393	0.8563	0.9021	0.8517

The values of A_C are plotted against D_p/D_v in Figure 3, where the points are seen to lie approximately in a straight line. The line of best fit is given by

$$A_C = 0.5137 - 1.434\, D_p/D_v \qquad (11)$$

and is also plotted in Figure 3.

Substituting in Equation 3, the following general relationship is obtained

$$OF_C = (0.5137 - 1.434\, D_p/D_v) + OF_W\,(0.2497 + 4.662\, D_p/D_v)$$
$$+ OF_W^2\,(0.1975 - 3.791\, D_p/D_v) \quad . \qquad (12)$$

Various values of $OF_{C_{max}}$ were next computed. Those derived from Equations 6 and 12 lie on the upper broken curve in Figure 4 and those derived from Equations 8 and 12 lie on the unbroken curve. The experimentally based values of $OF_{C_{max}}$ given in Table 5 are also plotted in Figure 4. From this consideration there is little to choose between the two alternatives.

Values of $OF_{C_{max}}$ were then computed for the appropriate cylindrically ordered packings by the method described in Appendix 3. Four values were determined for each transitional D_v/D_p ratio, since both the theoretical and the whole numbers of spheres in each ring had to be considered. The points plotted in Figure 4 include every value in the range $D_p/D_v = 0.0273$ to 0.1460 ($D_v/D_p = 36.688$ to 6.098) inclusive.

* All optimised equations in this Part were derived by the method of non-linear least squares, using the BMDX85 computer programme made available by the University of California, Los Angeles.

The disposition of the points shows that the values of $OF_{C_{max}}$ based on the theoretical numbers of spheres with the larger numbers of rings are comparatively independent of D_v/D_p. The best fit of the optimised curves is provided by the values based on the whole numbers of spheres with the smaller numbers of rings. As would be expected, the overall differences between the theoretical and whole numbers correspond approximately to half the volume of a sphere per ring. The lower optimised curve is consistent with this and the overall agreement is within 0.1 per cent for values of D_v/D_p greater than about 14 (D_p/D_v less than 0.7), while the maximum individual discrepancy is 0.4 per cent. Values of $OF_{C_{max}}$ were also computed for the cylindrically ordered packings on the basis of the actual number of spheres in each ring being 0.5 less than the theoretical number. These are plotted as the lower broken curve in Figure 4. This is consistent with the lower optimised curve, and the two are in agreement within 0.2 per cent for values of D_v/D_p greater than about 11 (D_p/D_v less than 0.09).

The computed properties of the cylindrically ordered packings diverge considerably from the optimised curves as D_p/D_v increases beyond 0.7. The discrepancies are presumably due partly to the curvature of the wall and partly to the uneconomical use of space near the central 'ring' of 4 spheres.

It should be noted further that the two D_v/D_p ratios, 8.71 and 5.77, used to establish the optimised linear $A_C - D_p/D_v$ relationship, are away from the nearest transitional D_v/D_p ratios. Both theory and experiment show that their central 'rings' contain 2 and 3 spheres respectively. This indicates that intermediate values of $OF_{C_{max}}$ lie on or close to a continuous curve regardless of the number of spheres in the central ring.

It is therefore concluded that the linear $A_C - D_p/D_v$ relationship applies to practical random packings with small D_v/D_p ratios such as those in this study and for all larger ratios including the infinite case. Equations 8 and 12, based on the mean difference between the theoretical and the whole numbers of spheres, were therefore incorporated in the adapted model*.

The resulting $OF_C - OF_W$ relationships for D_v/D_p ratios of infinity and 8.71 are plotted in Figure 5 together with the relationship derived in Part IV for the semi-infinite vessel.

CORRECTION FOR FLOATING SPHERES

A final correction is necessary on account of floating spheres, which were observed in loose packings in a prismatic vessel with a plane wall as reported in Part IV. In the loosest packings so prepared, 4.1 per cent of the spheres touching or capable of touching the wall were floating spheres. In the central region 4.7 per cent were estimated to be floating, giving a mean of 4.4 per cent, which is the best available basis for any correction on this account. Floating spheres were not observed in the range normally obtained in practice, that is, where OF_W is greater than about 0.7.

The existence of floating spheres was not known at the time of the experimental determinations with the cylindrical vessels. However, the values of OF_C given by the three curves in Figure 5 are similar for values of OF_W less than 0.7, so it is likely that the percentages of floating spheres in the central region of cylindrical packings are also similar to those of semi-infinite packings.

A final correction was therefore made to the $OF_C - OF_W$ relationship for values of OF_W less than 0.7 on the basis that 4.4 per cent of the spheres in the central region are floating when $OF_W = 0.615$. A quadratic expression was assumed, giving the following factor by which the uncorrected value of OF_C is multiplied:

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* The theoretical number of spheres was adopted for the equations developed in Part I, even though the final constraint of an integral number of spheres must prevail at the dense end of the range. It was assumed that this constraint would not be felt over the range of random packings normally obtained in practice, but this assumption is now not sustained.

$$F_C = 1 + 6.37(0.7 - OF_W)^2 \quad . \qquad (13)$$

Values of OF_C so corrected are represented by the dotted curve in Figure 5.

The experimentally determined values of OF_W and OF_C given in Tables 2 and 3 are also plotted in Figure 5.

5.2.2 $V_{II}/V_W - OF_W$ relationship

Both the $V_{II}/V_W - OF_W$ and the $V_{IIU} - OF_W$ quadratic approximations were investigated. Since the latter was found more suitable for incorporation into the adapted model, the following derivation is given.

The quadratic relationship is of the form:

$$V_{IIU} = A_{IIU} + B_{IIU}\,OF_W + C_{IIU}\,OF_W^2 \quad , \qquad (14)$$

where A_{IIU}, B_{IIU} and C_{IIU} are coefficients to be determined.

From Part IV, the value of OF_W for the loosest possible random packing is 0.615, while the values of V_{IIU} and the gradient of the $V_{IIU} - OF_W$ curve are 0.09144 and –0.1691 respectively (see Appendix 4 for derivation). These values enable B_{IIU} and C_{IIU} to be expressed in terms of A_{IIU}, giving:

$$V_{IIU} = A_{IIU}\,(1 - 3.252\,OF_W + 2.644\,OF_W^2) + (0.4665\,OF_W - 0.5167\,OF_W^2) \, . \qquad (15)$$

The value of A_{IIU} was derived from the properties of the appropriate cylindrically ordered packing. The mean volume of the intruding spherical segment was first calculated in accordance with Appendix 5. Values of interest are given in Table 6.

In an infinite cylinder the number of intruding spheres can be treated as being the same as the number of spheres touching the wall, so that $V_{IIU_{max}} = 0.03216/0.8660 = 0.03714$. Substituting in Equation 15 we have $A_{IIU} = 0.2230$ and

$$V_{IIU} = 0.2230 - 0.2587\,OF_W + 0.07288\,OF_W^2 \qquad (16)$$

TABLE 6

MEAN VOLUMES OF INTRUDING SPHERICAL SEGMENTS OF SPHERES OF UNIT DIAMETER

Vessel	Packing	Mean Volume of Segment
Semi-infinite with Plane Wall	Rhombohedral	0.04642
Infinite Cylinder	Cylindrically ordered	0.03216
Finite Cylinder $D_v/D_p = 8.71$	Cylindrically ordered, 1 sphere in central ring	0.02633
	Cylindrically ordered, 4 spheres in central ring	0.01346

In a finite cylinder the number of intruding spheres is less than the number touching the wall. For a D_v/D_p ratio of 8.71 the theoretical numbers are 18.54 and 24.15. For consistency with the rest of the adapted model the actual numbers are taken as 0.5 less than these. The value of $V_{IIU_{max}}$ is therefore

$$(18.04/23.65)\ (0.01346/0.866)\, OF_W \;=\; 0.01185 \times 0.9493 = 0.01125.$$

Substituting this value in Equation 15 gives $A_{IIU} = 0.1154$ and

$$V_{IIU} = 0.1154 + 0.09127\ OF_W - 0.2117\ OF_W^2 \quad . \tag{17}$$

Values of V_{II}/V_W are then calculated from the relationship:

$$V_{II}/V_W = 1.654\ V_{IIU}/OF_W \quad . \tag{18}$$

A correction was also made on the basis of 4.4 per cent of the spheres throughout the packing being floating spheres. The value of F_C given by Equation 13 cannot be used directly, since the volumes of the intruding spherical segments are not all the same. The uncorrected value of V_{IIU} is multiplied by the factor F_{IIU} given by:

$$F_{IIU} = 1 + 2.61\,(F_C - 1) \quad . \tag{19}$$

The derivation is given in Appendix 6.

Observed and computed values of V_{II}/V_W are plotted in Figure 6.

As stated at the beginning of this section, the $V_{II}/V_W - OF_W$ quadratic relationship was derived directly along similar lines. The equation for the infinite cylinder is:

$$V_{II}/V_W = 1.140 - 2.054\ OF_W + 0.9752\ OF_W^2 \tag{20}$$

and for a D_v/D_p ratio of 8.71 is:

$$V_{II}/V_W = 0.9725 - 1.508\ OF_W + 0.5315\ OF_W^2 \quad . \tag{21}$$

Values derived from Equations 17 and 21 are virtually the same because of the flatness of the curve, but values derived from Equations 16 and 20 for the infinite cylinder are significantly different, as can be seen from Figure 6, where both are plotted. The disposition of the curves shows that the $V_{II} - OF_W$ quadratic approximation is more suitable for incorporation in the adapted model. If the $V_{II}/V_W - OF_W$ quadratic approximation had been adopted, it would almost certainly have proved inadequate for large D_v/D_p ratios. The $V_{IIU} - OF_W$ quadratic approximation is unlikely to require further adjustment.

5.2.3 $V_{IO}/V_W - OF_W$ relationship

The model in Part IV for the semi-infinite vessel offered little guidance, so it was necessary to derive an optimised relationship directly from the experimental results. These show considerable scatter, as can be seen from Figure 7. Linear and quadratic expressions were investigated, the data being segregated according to D_v/D_p and also all grouped together. For consistency with the model, spheres were treated as not having broken contact with the wall when OF_W was greater than 0.9045.

The scatter does not permit any better resolution than is given by a linear relationship with the data all grouped together, including that for the semi-infinite case. The relationship is given by:

$$V_{IO}/V_W = 0.146\ (0.9045 - OF_W) \quad . \tag{22}$$

A correction was made for floating spheres by multiplying the uncorrected value of V_{IOU} by the factor F_{IOU} given by:

$$F_{IOU} = 1 + 7.37\,(F_C - 1) \quad . \qquad (23)$$

The derivation is given in Appendix 7.

The resulting relationship is plotted as the unbroken curve in Figure 7. For comparison the relationship derived from the data for $D_v/D_p = 8.71$ is plotted as the broken curve.

5.2.4 Derivation of other properties

Other properties of packings can be derived from the relationships given in Appendix 3 of Part IV.

The property of chief interest here is N-n, the number of spheres not touching the wall, which is derived from n_C, the number of spheres in the central region, and n_I, the number of intruding spheres, where

$$n_C = \frac{\pi}{2\sqrt{2}} \cdot \frac{(D_v - 2D_p)^2\, H}{D_p^3} \cdot OF_C \qquad (24)$$

$$n_I = \frac{6\, V_I}{\pi\, D_p^3} \qquad (25)$$

and

$$N\text{-}n = n_C + n_I \quad . \qquad (26)$$

6. DISCUSSION

6.1 $OF_C - OF_W$ Relationship

$D_v/D_p = 8.71$

From figure 5 it can be seen that the greatest discrepancies between the observed and computed values of OF_C occur for values of OF_W less than 0.7. The discrepancies are up to 3.6 per cent, and there appears to be a significant difference between the two types of sphere used. The match at this end of the range cannot be improved in the absence of experimental determinations of the numbers of floating spheres, which is beyond the scope of this study.

This unknown is of little or no significance for values of OF_W greater than 0.7, where the average discrepancy* is 1.3 per cent with a maximum of 1.9 per cent. This falls short of the expected ± 1 per cent, but time did not permit the considerable repetition of the experiments needed to establish whether the agreement is, in general, better than indicated.

Other D_v/D_p ratios

Similar checks were made for $D_v/D_p = 5.77$ and 11.4, but the discrepancies were about double those for $D_v/D_p = 8.71$.

A check was also made of the results of Benenati and Brosilow (1962) given in Table 2 of Part I. This gives some of the properties of random packings with various D_v/D_p ratios prepared by a common method.

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* See column 6 of Table 9 for details

The values of OF_C were derived from these results and are given in Column 2 of Table 8. Unfortunately it was not possible to derive the corresponding values of OF_W from the published results, so an experimental investigation was made of the dependence of OF_W on D_v/D_p for a given packing method. Although the preparation method adopted by Benenati and Brosilow is not stated, it is assumed that the spheres were added to the top surface by pouring or placing without impact and without deliberately disturbing the part of the packing already formed.

The method chosen for the investigation was to place the spheres as uniformly as possible over the top surface of the packing a handful at a time. Glass spheres of measured mean diameter 0.9823 in. were used in a prismatic vessel with a plane transparent wall and in cylindrical vessels with D_v/D_p ratios of 11.60, 7.69 and 5.85. The test area of the plane wall was 12 in. x 16 in. and the axial heights of the cylindrical test sections were 12, 12 and 13 in. respectively. The experiments were carried out by two independent observers, the number of experiments being such as to give essentially the same total test area for each vessel.

The results are given in Table 7, where the mean values of OF_W are seen to be independent of D_v/D_p.

TABLE 7

VALUES OF OF_W GIVEN BY A COMMON PREPARATION METHOD

$\frac{D_v}{D_p}$	Observed Values of OF_W	Average OF_W
Infinite	0.792, 0.770, 0.801, 0.810, 0.779, 0.792	0.791
11.60	0.793, 0.791, 0.793	0.792
7.69	0.789, 0.789, 0.800, 0.803	0.795
5.85	0.786, 0.774, 0.799, 0.808, 0.786	0.791

Returning to the results of Benenati and Brosilow, the observed value of OF_C in the infinite case is 0.831. This packing was prepared against a flat base plate, and so should correspond to the semi-infinite case. The value of OF_W obtained from the model is 0.764. If this value were to hold for the cylinders, the values of OF_C, derived from the adapted model, would be as in Column 3 of Table 8.

If the infinite case were treated as a semi-infinite cylinder, the value of OF_W obtained from the adapted model would be 0.784. If this value were to hold for the cylinders, the derived values of OF_C would be as in Column 5.

TABLE 8

TEST OF ADAPTED MODEL AGAINST RESULTS OF BENENATI AND BROSILOW

$\frac{D_v}{D_p}$	Observed OF_C	Computed			
		On basis of semi-infinite vessel		On basis of infinite cylinder	
		OF_C	Discrep. per cent	OF_C	Discrep. per cent
(1)	(2)	(3)	(4)	(5)	(6)
Infinity	0.831	0.831	0.0	0.831	0.0
20.3	0.828	0.816	+ 1.5	0.825	+0.4
14.1	0.809	0.814	−0.6	0.823	−1.7
5.6	0.808	0.804	+0.5	0.811	+0.4

The discrepancies are much the same in each case, leaving the correct choice unresolved.

However, this is of more theoretical than practical interest. Rocke had no greater difficulty in preparing cylindrically ordered packings with a D_v/D_p ratio of 40.4 than with smaller ratios. The adapted model should therefore be applicable up to this value, and presumably up to any limiting value beyond which cylindrically ordered packings cannot be prepared, if such a practical limit exists.

The values in Table 8 show that the adapted model accounts for the overall observed trend for OF_C to vary with D_v/D_p, other things being equal. The discrepancy of 2.1* per cent between the observed values of OF_C for D_v/D_p = 20.3 and 14.1 occurred even though duplicate runs were carried out and the results averaged. This indicates that the discrepancies of up to 2 per cent obtained with a D_v/D_p ratio of 8.71 are unavoidable in individual packings, at least for D_v/D_p ratios up to about 14. It is therefore doubtful whether experiments with the 11.4 in. cylinder would have given more accurate or consistent results, even if it had been practicable to cover the same range of random packings with the available equipment. However, the work of Benenati and Brosilow offers some hope that averaging the results of duplicated experiments would significantly reduce the greatest discrepancies between observed and computed values.

6.2 $V_{II}/V_W - OF_W$ Relationship

D_v/D_p = 8.71

The discrepancies are greater than for the $OF_C - OF_W$ relationship, the average being about 5 per cent with a maximum of 11 per cent. From Figure 6 there appears to be a significant difference between the two types of sphere used for values of OF_W less than 0.7.

The discrepancies are, however, comparatively minor in terms of the total volume of the spheres in the outer region. The average discrepancy on this basis is about 0.6 per cent with a maximum of 1.2 per cent. In terms of the total volume of the spheres not touching the wall the average discrepancy is about 0.4 per cent with a maximum of 0.7 per cent.

The overall accuracy of Equation 16 was checked against the optimised quadratic derived directly from the experimental results. The optimised equation is:

$$V_{IIU} = 0.1201 + 0.07576\ OF_W - 0.1991\ OF_W^2 \tag{27}$$

and values computed from it agree closely with values computed from Equation 16. For example, when OF_W = 0.85, the discrepancy is 1.7 per cent, but this is only 0.1 per cent of the volume of the spheres in the outer region. The discrepancies decrease rapidly as OF_W decreases further.

Other D_v/D_p ratios

Similar checks were made for D_v/D_p = 5.77 and 11.4, and the discrepancies are about double those for D_v/D_p = 8.71.

6.3 $V_{IO}/V_W - OF_W$ Relationship

D_v/D_p = 8.71

Despite the severe experimental scatter in Figure 7, the net effects of the discrepancies are comparatively minor. In terms of the total volume of the spheres in the outer region, the average discrepancy is about 0.4 per cent with a maximum of 1.2 per cent. In terms of the total volume of the spheres not touching the wall the average discrepancy is about 0.2 per cent with a maximum of 0.7 per cent.

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* that is, about 1 per cent discrepancy of the individual values about the mean.

Other D_v/D_p ratios

The discrepancies for D_v/D_p = 5.77 and 11.4 are about 50 per cent higher than those for D_v/D_p = 8.71.

6.4 V_I/V_W – OF_W Relationship

D_v/D_p = 8.71

Observed values of V_I being the sum of V_{II} and V_{IO}, the discrepancies have a strong tendency to be additive. In terms of the total volume of spheies in the outer region, the average discrepancy is about 0.8 per cent with a maximum of 2.0 per cent. In terms of the total volume of the spheres not touching the wall the average discrepancy is about 0.4 per cent with a maximum of 1.2 per cent.

6.5 (N–n) – OF_W Relationship

D_v/D_p = 8.71

In comparing the observed and computed values of N–n, it is of particular interest to ascertain whether the discrepancies in n_C tend to compensate for those in n_I. Observed values of n_C were obtained by subtracting experimentally determined values of n_I from experimentally determined values of N–n. Errors in the observed values of n_I will therefore not contribute to discrepancies between observed and computed values of N–n. Values of interest are given in Table 9.

The errors partly compensate, the worst errors being reduced by about one-third. For values of OF_W greater than 0.7, however, the improvement is marginal. The average discrepancy is 1.2 per cent with a maximum of 1.7 per cent.

D_v/D_p = 11.5

As stated in Section 4.1, experiments with the 11.4 in. vessel were discontinued because of practical difficulties, including difficulties in forming the denser packings. Referring to Table 1, only three of the packings are free of deliberate radial bias. Observed and computed values of N–n are given in Table 10.

All that can be inferred from this small number of observations is that the discrepancies are much the same as for D_v/D_p = 8.71.

D_v/D_p = 5.77

Observed and computed values of N–n for packings of 0.997 in. zircon sand spheres in a 5.75 in. cylinder are given in Table 11. The identification numbers are taken from Figure 5 of Part I. Only those packings without deliberate radial bias are included.

As would be expected with small packings, the discrepancies of up to 5 per cent are about double those for D_v/D_p = 8.71 and 11.5. Even when the results for similar packings are averaged, the discrepancies are up to 2.5 per cent. Even so, the totals agree within about 1 per cent as for the two larger D_v/D_p ratios.

TABLE 9

OBSERVED AND COMPUTED VALUES OF n_C AND N–n

(0.9902 in. aluminium and 1.003 in. zircon spheres in 8.71 in. cylinder)

Preparation Method	D_p	OF_w	n_C Observed	n_C Computed	% Discrep.	N–n Observed	N–n Computed	% Discrep.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
7.25 in. load tube	0.9902	0.686	489.2	485.1	– 0.8	545	537.5	– 1.4
		0.700	493.1	488.6	– 0.9	545	538.8	– 1.1
		0.732	506.4	497.5	– 1.8	551	543.0	– 1.5
	1.003	0.660	477.8	460.4	– 3.6	530	516.5	– 2.6
		0.683	475.5	463.1	– 2.6	523	514.5	– 1.6
Light vibration	1.003	0.710	478.9	469.6	– 1.9	526	517.0	– 1.7
Medium vibration	0.9902	0.790	521.2	512.8	– 1.6	556	548.9	– 1.3
	1.003	0.815	502.8	495.9	– 1.4	533	526.8	– 1.2
Medium jolting	1.003	0.818	494.5	496.6	+ 0.4	531	526.9	– 0.8
Heavy vibration	0.9902	0.851	519.0	528.0	+ 1.7	546	552.9	+ 1.3
Heavy jolting	1.003	0.845	497.0	502.8	+ 1.2	523	528.3	+ 1.0
		0.848	503.9	503.5	– 0.1	532	528.4	– 0.7
		0.851	495.6	504.2	+ 1.7	523	528.5	+ 1.1
TOTALS			6,454.9	6,408.1	– 1.0	6,964	6,908.0	– 1.0

TABLE 10

OBSERVED AND COMPUTED VALUES OF N–n

(0.9902 in. aluminium spheres in 11.4 in. cylinder)

Preparation Method	OF_w	N–n		% Discrep.
		Observed	Computed	
10.5 in. load tube	0.773	1,036	1,047.0	+ 1.1
Dropped from 18 inches	0.827	1,070	1,064.6	– 0.5
Heavy jolting	0.927	1,070	1,093.5	+ 2.2
TOTALS		3,176	3,205.1	+ 0.9

$D_v/D_p = 24.2$

Experiments carried out by Price (1967) in connection with pressure losses and flow distributions of fluids through stationary packings of equal spheres included several determinations of N and n. All were carried out with 0.504 in. diameter wooden spheres in a 12.19 in. diameter cylinder 7.75 in. high. Three of the determinations were repeated several times. As can be seen from Table 12, the values of N–n of individual packings prepared by a given method have a scatter of about ± 0.5 per cent about their mean value.

The observed and computed average values agree within ± 0.9 per cent, except for the packings prepared by pouring from a beaker. For these the average discrepancy is 1.2 per cent; however, as discussed in Section 6.8, packings produced by such methods are susceptible to radial bias.

The discrepancies generally may be partly attributable to the end effect. The spheres in contact with the base of the vessel were included in the counts. The end effect was avoided at the tops of the packings, but an error of 1/32 in. in the measurement of the packing height would give rise to an error of 0.4 per cent.

TABLE 11

OBSERVED AND COMPUTED VALUES OF N–n

(0.997 in. zircon spheres in 5.75 in. cylinder)

Ident. No.	Preparation Method	OF_w	N–n		% Discrep.
			Observed	Computed	
1	4.75 in. load tube	0.721	260	262.5	+1.0
2		0.717	259	262.7	+1.4
3	Placed singly in centre	0.750	271	260.8	–3.8
4		0.741	259	261.5	+0.6
5		0.741	272	261.5	–3.9
6		0.750	265	260.8	–1.6
7	Placed singly near wall	0.724	272	262.3	–3.6
8	Placed on top a handful at a time	0.731	259	262.0	+1.2
9		0.718	267	262.6	–1.6
10	Dropped singly from top of vessel	0.770	267	259.4	–2.9
11	Dropped from top of vessel a handful at a time	0.805	258	256.0	–0.8
12		0.786	256	258.0	+0.8
13		0.786	261	258.0	–1.1
14	Bulk poured in 15 sec.	0.776	269	258.8	–3.8
15	Bulk poured in 5 sec.	0.792	249	257.3	+3.3
16		0.741	258	261.5	–1.6
18	Dropped from 6 ft.	0.786	248	258.0	+4.0
19	Vibrated	0.870	256	247.2	–3.4
22	Loose packing shaken down bodily in stages	0.689	271	263.8	–2.7
23		0.744	269	261.3	–2.9
24		0.786	271	258.0	–4.8
25		0.838	251	252.0	+0.4
26		0.906	254	241.1	–5.1
TOTALS			6,022	5,947.0	–1.1

TABLE 12

OBSERVED AND COMPUTED VALUES OF N-n

(0.504 in. wooden spheres in 12.19 in. cylinder)

Preparation Method	OF_w	N-n		% Discrep.
		Observed	Computed	
Central load tube	0.729 0.735 0.740	6,905 6,870 6,866	6,903 6,927 6,944	0.0 +0.8 +1.1
AVERAGES	0.735	6,881	6,925	+0.6
Poured on top a beakerful at a time	0.715 0.731 0.719	6,963 6,951 6,963	6,851 6,912 6,866	-1.6 -0.6 -1.4
AVERAGES	0.722	6,959	6,876	-1.2
Loaded while vibrating	0.805 0.779 0.785 0.802	7,183 7,165 7,165 7,202	7,188 7,168 7,115 7,177	-0.1 0.0 -0.7 -0.4
AVERAGES	0.798	7,179	7,162	-0.2
Vibrated 2 hours Vibrated 5 hours	0.847 0.837	7,264 7,268	7,346 7,311	+1.1 +0.6
AVERAGES*	0.842	7,266	7,329	+0.9
TOTALS		84,765	84,708	-0.07

* Although the vibration times are so different, the resulting properties are close enough together to be averaged.

$D_v/D_p = 5.81$

These experiments were carried out early in the study to compare the effect of various preparation methods on spheres of different materials in the same vessel. They have not been reported previously (Table 13).

TABLE 13

OBSERVED AND COMPUTED VALUES OF N–n

(0.9902 in. aluminium spheres in 5.75 in. cylinder)

Preparation Method	OF_w	N–n		% Discrep.
		Observed	Computed	
4.75 in. load tube	0.724	180	182.0	+ 1.1
Dropped from top of vessel a handful at a time	0.823	184	176.4	– 4.2
Dropped from 6 ft.	0.866	176	172.3	–2.1
TOTALS		540	530.7	–1.7

The results may be compared with the values given in Table 11 for the same preparation methods.

$D_v/D_p = 8.46$

These experiments were also carried out early in the study to investigate the effect of various preparation methods, and have not been reported previously (Table 14).

TABLE 14

OBSERVED AND COMPUTED VALUES OF N–n

(1.008 in. wooden spheres in 8.53 in. cylinder)

Preparation Method	OF_w	N–n		% Discrep.
		Observed	Computed	
6.4 in. load tube, fast withdrawal	0.714	705	709.0	+0.6
	0.769	717	716.2	–0.1
6.4 in. load tube, slow withdrawal	0.796	719	718.8	0.0
Placed on top a handful at a time	0.737	757	712.4	–5.9
Vibrated for 20 min.	0.909	724	722.9	–0.2
TOTALS		3,622	3,579.3	–1.2

The effects of various preparation methods are discussed further in Section 6.8.

6.6 $V_I - V_{VI}$ Relationship

In Part I, the volume of the intruding sphere material was found experimentally to decrease linearly with the volume of the spheres touching the wall, that is, V_I increases linearly with V_{VI}, the mean void volume available for intruding spheres in the outer region.

Values were computed from the model and the adapted model to check the linearity of the relationship. They are plotted in Figure 8 in the standardised forms V_{IU} and V_{VIU}. Values of V_{IIU} are also plotted for comparison. The relationships are close enough to linear to account for the indication of linearity in the exploratory experiments reported in Part I.

6.7 Effect of Axial Bias on Radial Bias

This effect was determined from the adapted model. Computations were based on a given packing being divided into two cylinders of equal volume. One was given a value of OF_W one per cent greater than the average, the other a value one per cent less. The two values of N–n were computed and the sum was compared with the computed value of N–n for the whole packing.

Over the range of values of D_v/D_p and OF_W of interest to this study the discrepancy was generally about ± 0.01 per cent with a maximum of ± 0.025 per cent, which is negligible for practical purposes. These values are for packings without end effects. Radial bias due to end effects is not necessarily negligible.

6.8 Influence of Preparation Methods on Structure of Random Packings

The two chief considerations when evaluating a given preparation method are the degree of axial and radial bias and the degree of repeatability. The observed and computed values of N–n for the various preparation methods were therefore examined and although the individual values show too much scatter for any conclusive evaluation, some guidance can be obtained from the total values given in Table 15. All the packings have been equally weighted on the basis of the observed values of N–n all being 260, this being the value for the first packing examined. The various preparation methods have been grouped into three categories, generally in accordance with Table 4 of Part I.

Only two of the methods, 1 and 7, have resulted in freedom from radial bias. No conclusion can be drawn for Method 6 since only two such packings were prepared. However, for reasons given in Section 6.9, packings prepared by this method would be expected to have much the same radial bias as those produced by Method 7.

The reason for the comparatively large discrepancy with Method 2 was found unexpectedly during the experimental determinations of the values of OF_W given in Table 7. Although the two observers had obtained consistent results for their packings in the two smallest cylinders without difficulty, it was found impossible to reconcile their results for the largest cylinder. Closer inspection showed that this was caused by one observer inadvertently building up a shallow cone in the central region. The top of the packing was observed to slump intermittently as further spheres were added. Packings prepared in this way had an average value of OF_W of about 1 per cent less than for those prepared by uniform placement. Packings known to have been affected in this way have been excluded from Table 7.

It was later found that preferential placement near the wall resulted in values of OF_W as much as 3.5 per cent lower than those prepared by uniform placement. Thus any departure from uniform placement will result in a reduced value of OF_W. The value of OF_C will not necessarily be reduced by the same amount, so any method susceptible to this effect must be assumed to give rise to radial bias unless it has been shown otherwise by experiment. This applies to Methods 2 and 3, both of which give a negative discrepancy, indicating that the outer region is more affected than the central region. Note that the packings prepared by Method 3, with deliberate bias, have the greater discrepancy. The effects of bias towards the wall and towards the central region do not tend to counteract one another. Their effects would therefore be additive if combined in a single

TABLE 15

VALUES OF N–n GIVEN BY VARIOUS PREPARATION METHODS

General Category	Ident. No.	Preparation Method	N–n Observed	N–n Computed	% Discrep.
Loose	1	Load tube	4160	4149.5	–0.3
	2	Placed on top (no deliberate bias)	1560	1533.8	–1.7
	3	Placed on top (deliberate bias)	1300	1269.3	–2.4
Medium	4	Dropping or pouring from about 18 in.*	2080	2061.2	–0.9
	5	Medium vibration or jolting	2340	2314.5	–1.1
Dense	6	Dropping from 6 ft.	520	525.0	+1.0
	7	Heavy vibration, jolting or shaking	2600	2603.6	+0.1

* In most cases dropping or pouring took place from the top of the vessel, giving a variable drop height. They are therefore axially biased, but this has not resulted in a significant effect on the radial bias compared with Method 5.

packing. As can be seen from Table 12, this also applies to the method of pouring spheres on top of the packing; the tabulated values suggest that the radial bias has given rise to a discrepancy of about –0.6 per cent in this case.

The medium packings also show a significant negative discrepancy. Presumably any departure from uniform placement gives rise to radial bias as with Method 2, while the impact of the falling spheres tends to counteract it by partly consolidating the portion of the packing already formed. It should be noted, however, that light or medium vibration does not necessarily give rise to radial bias. Table 9 shows that the discrepancies for such packings are much the same as for the loose packings from which they were prepared. Furthermore, the packings loaded while vibrating in Table 12 show no significant radial bias.

With the dense packings, on this basis, the degree of consolidation would be great enough to nullify any initial radial bias. Further support for this proposition is given in Section 6.9.

The repeatability of the results with the smaller vessels leaves much to be desired. Closer control of the preparation methods would probably improve this. Close mechanical control was not exercised over any of the methods. For example, the diameters of the load tubes were not optimised, but were merely selected from the most readily available tubing of about 80 per cent of the vessel diameter. They were not flanged to remain central and the withdrawal rate was left to the discretion of the experimenter. At least two, and sometimes four observers prepared packings by a given method. This was done to avoid personal bias in the averaged results, but at the expense of greater scatter of individual results.

Despite the scope for improvement, it is considered that a repetition of the experiments with small D_v/D_p ratios is not warranted. The time would be better spent on experiments with a D_v/D_p ratio of 14 or more. This would avoid errors arising from the unexplained differences between the lower optimised curve and the lower broken curve in Figure 4.

The preparation methods would have to be carefully controlled and checked for repeatability, and the experimental procedure described in Section 3 would have to be followed to eliminate end effects and minimise axial bias.

Experimental determinations of the numbers and radial positions of floating spheres would be essential for values of OF_w less than 0.7.

6.9 Recommended Methods for Preparing Unbiased Random Packings

At the commencement of the study it was expected that some of the methods commonly adopted by other investigators* when preparing random packings would result in a common structure, free of significant radial bias. The results of this study do not rule this out altogether, but they show that packings so prepared are very susceptible to radial bias. It would be essential to exercise closer control than has generally been used in the past. These methods are not therefore recommended at this stage. In any case, packings prepared by such methods are limited to a comparatively narrow portion of the range of random packings.

The recommended method is to prepare a loose packing by withdrawing a suitable load tube, and vibrate it down to the desired degree of consolidation. Jolting is less favoured because packings so prepared are probably susceptible to axial bias.

Where vibration is inconvenient or impracticable, such as in the preparation of large dense packings, an alternative method is to drop the spheres uniformly over the top surface. This method was investigated by Bonhote et al. (1972) for the preparation of dense packings of irregular granules of activated charcoal. Three separate sizes were used; 6–8 mesh, 8–12 mesh and 12–16 mesh.

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* that is, methods such as placing or pouring spheres uniformly over the top surface without impact, vibration, or other deliberate disturbance of the part of the packing already formed.

Granules were poured through a mesh so that they fell uniformly onto the top surface from a height sufficient to give maximum consolidation. This method gave a repeatability of mean density to about ± 0.2 per cent, provided the feed rate was kept below a critical value. Above this value less consolidation was achieved.

The same method was used to prepare dense random packings of various types of sphere, including 3 mm glass, 1/8 in. steel and a mixture of 1/8 in. and 7/64 in. steel. All resulted in packings of densities close to the maxima attainable by vibration of the same spheres.

It is possible that unbiased packings of medium density can be prepared by this method by reducing the drop height. The feed would have to be controlled, and the mesh would have to be raised mechanically to maintain a constant drop height. The loosest packing obtainable by this method would have a void fraction of about 0.395, corresponding to zero drop height.

Both recommended methods would require testing by the user, and perhaps further development, to ascertain whether the desired degree of consolidation and freedom from axial and radial bias had been achieved.

7. SUMMARY AND CONCLUSIONS

1. Detailed experimental studies of random packings of equal spheres without deliberate radial bias have been carried out in cylindrical vessels with D_v/D_p ratios of 11.4 and 8.71.

2. With the larger vessel, a record was made of the number and radial location of all spheres coming within a distance of 1.25 times the sphere diameter of the wall. The results confirmed the existence of a cusp in the void fraction curve at a distance of one sphere diameter from the wall, and the correctness of the interpolated curve in Figure 3 of Part I.

3. Practical difficulties precluded the use of the larger vessel for random packings at the dense end of the range of interest. However, the experimental results obtained with smaller vessels are adequate for adaptation of the model developed in Part IV for the semi-infinite case.

4. The model has been adapted to give equations in terms of D_v/D_p and OF_W from which OF_C, V_I and related properties can be calculated.

5. The $OF_C - OF_W$ relationship is a quadratic whose coefficients are derived from the properties of the loosest random packing, which is common to all vessels, together with the values of OF_C and OF_W for the densest packing for the particular vessel. The former are obtained directly from the model and the latter are obtained from the experimentally determined properties of random packings by optimisation and extrapolation.

6. For values of D_v/D_p greater than 14, the same relationship can be derived from the properties of the loosest random packing and the appropriate cylindrically ordered packing, without the need to call on the properties of intermediate random packings.

7. For consistency with Part IV, V_I is treated in the standardised form V_I/V_W. Since the model does not cover the volume of the intruding sphere material in the outer part of the outer region, it is necessary to consider V_I/V_W in two parts, V_{II}/V_W and V_{IO}/V_W.

8. The $V_{II}/V_W - OF_W$ relationship is obtained indirectly by means of a $V_{IIU} - OF_W$ quadratic whose coefficients are derived from the properties of the loosest random packing and the appropriate cylindrically ordered packing.

9. The $V_{IO}/V_W - OF_W$ relationship is independent of the D_v/D_p ratio, and is obtained from experimentally determined values by optimisation.

10. An allowance is made for floating spheres at the loose end of the range (values of OF_W less than 0.7). This does not include the range of random packings normally obtained in practice.

11. Observed and computed values of OF_C and N–n are compared for a D_v/D_p ratio of 8.71. For values of OF_W greater than 0.7, the discrepancies are about 1.3 per cent with a maximum of 1.9 per cent. Experimental results of other workers indicate that the anticipated accuracy of ± 1 per cent would be approached if the experiments were repeated and the results averaged.

12. For values of OF_W less than 0.7 the discrepancies are about 2 per cent with a maximum of about 3 per cent. The reason for the larger discrepancies cannot be given in the absence of experimental determinations of the number of floating spheres.

13. Possible future experiments to check the adapted model with greater accuracy are suggested. A vessel with a D_v/D_p ratio of 14 or more should be used.

14. The influence of various preparation methods on the structure of random packings is discussed. It is shown that packings prepared by some of the methods commonly used, such as placing or pouring of spheres, are susceptible to radial bias.

15. It is recommended that unbiased random packings be prepared by using a load tube and vibrating the resulting loose packing. Where vibration is inconvenient, uniform dropping of spheres may be used. Both methods require testing, and perhaps further development, by the user.

8. ACKNOWLEDGEMENTS

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9. NOTATION

D_v	diameter of cylindrical vessel
D_p	diameter of sphere
H	packing height
N	total number of spheres in packing
n	number of spheres touching wall
OF_W	occupancy factor of spheres touching wall
$OF_{W_{max}}$	terminal value of OF_W at dense end of range
OF_I	occupancy factor of intruding spheres
OF_C	occupancy factor of spheres in central region
$OF_{C_{max}}$	terminal value of OF_C at dense end of range
V_W	volume of spheres touching wall
V_{WU}	mean volume of spheres touching wall in unit volume of outer region
V_I	volume of intruding sphere material
V_{IU}	mean volume of intruding spheres in unit volume of outer region
V_{II}	volume of intruding spheres in inner part of outer region

V_{IIU}	the part of V_{IU} in inner part of outer region
V_{IO}	volume of intruding spheres in outer part of outer region
V_{IOU}	the part of V_{IU} in outer part of outer region
A_C, B_C, C_C	coefficients of quadratic $OF_C - OF_W$ relationship
F1,F2,F	factors to allow for curvature of wall
$\overline{\Delta h}$	mean axial spacing of rings in cylindrically ordered packing
α	angular displacement of adjacent spheres in same outer ring of cylindrically ordered packing
δh	axial gap between outer rings corresponding to angle α
F3,F4	factors to allow for differences between theoretical and actual numbers of spheres in ring in cylindrically ordered packing
F_C	factor to allow for floating spheres in central region
$A_{IIU}, B_{IIU}, C_{IIU}$	coefficients of quadratic $V_{IIU} - OF_W$ relationship
F_{IIU}, F_{IOU}	factors to allow for floating intruding spheres
n_C	number of spheres in central region
n_I	number of intruding spheres (= V_I /volume of one sphere)

10. REFERENCES

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References relevant to the overall study are given in Parts I, II and III.

APPENDIX 1

DERIVATION OF PROPERTIES OF LOOSEST POSSIBLE RANDOM PACKING RELEVANT TO THE $OF_C - OF_W$ RELATIONSHIP

$OF_W = 0.615$ (experimentally determined)

From Appendix 3 of Part IV,

$$OF_W = \frac{2}{m}$$

and

$$OF_C = \sqrt{\frac{2}{3}} \cdot \frac{2}{m} \cdot \frac{1}{\sqrt{1-\frac{m}{6}}} \quad .$$

Gradient of $OF_C - OF_W$ curve

$$= \frac{d(OF_C)}{dm} \Bigg/ \frac{d(OF_W)}{dm}$$

$$= \left[\sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{1-\frac{m}{6}}}\right] \left[1 - \frac{m}{12\left(1-\frac{m}{6}\right)}\right] \quad .$$

When $OF_W = 0.615$, the values calculated from these equations are

$$m = 3.252$$

$$OF_C = 0.7420$$

and gradient of $OF_C - OF_W$ curve $= 0.4926$.

APPENDIX 2

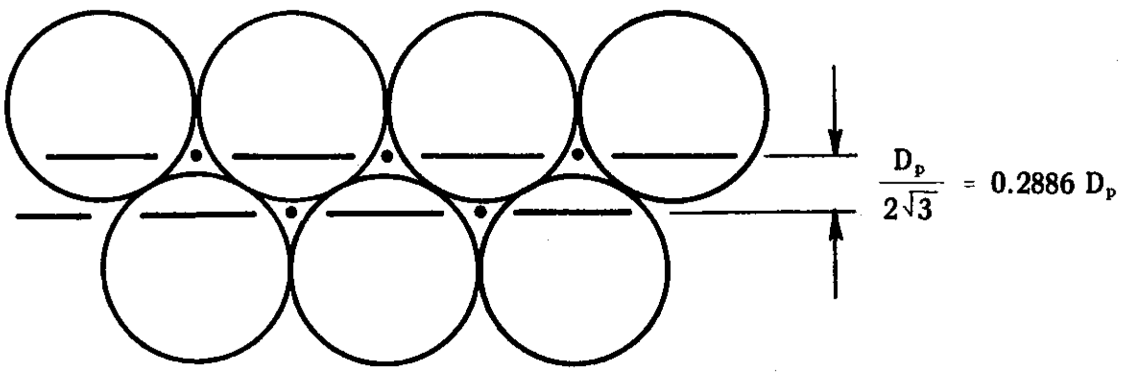
DERIVATION OF VARIATIONS IN AXIAL SPACING OF OUTER RINGS WITH D_v/D_p

Rocke (1971) has shown that the theoretical number of waves in a ring is $2\pi\overline{\Delta R}/D_p$, where $\overline{\Delta R}$, the mean radial ring spacing for an infinite D_v/D_p ratio, is given by

$$\frac{\overline{\Delta R}}{D_p} = \frac{1}{\sqrt{6}} + \frac{3\sqrt{3}}{4} \arcsin\left(\frac{1}{3}\right) = 0.8497....$$

The theoretical number of waves is $2\pi\overline{\Delta R}/D_p = 5.339$, but in practice a whole number, usually 5 or 6, is observed. The theoretical number is appropriate to any general treatment, so the theoretical wavelength is $(\pi/5.339)(D_v - D_p) = 0.5884\ (D_v - D_p)$.

The amplitude of the waves is calculated from the relevant property of the rhombohedral array, the distance between the rows of "cusps" shown in the following diagram.



This would be twice the amplitude of the waves in a ring in the outer shell of a cylindrically ordered packing if the rings in the next shell were truly horizontal. In practice they are not, so it is assumed that the displacements are taken up equally by the two shells, corresponding to an amplitude of $0.0722\ D_p$.

If the waves are treated as being sinusoidal, then the maximum value of the angle α in Figure 1 is given by

$$\tan\alpha = 2\pi\left(\frac{0.0722\ D_p}{0.5884(D_v - D_p)}\right) = \frac{0.7706}{\frac{D_v}{D_p} - 1} .$$

The corresponding increase in axial spacing is given by

$$\delta h_{max} = \frac{D_p}{2} \sin\alpha_{max}$$

$$= \frac{0.3853\ D_p}{\sqrt{\left(\frac{D_v}{D_p} - 1\right) + 0.7706^2}} = \frac{0.3853\ D_p}{\sqrt{\left(\frac{D_v}{D_p} - 1\right) + 0.5938}}$$

In practical packing the waves in the rings of a given shell are in step, for example the maxima occur at the same radial positions for all rings. Since the theoretical value of δh falls to zero at intermediate positions it might be expected that the effect of δh at other positions would be cumulative, giving rise to gross distortions after a few layers, but this is not so. The spacing is the same at all radial positions, the rings adopting an intermediate displacement, which for a sinusoidal wave is $(2/\pi)\ \delta h_{max}$.

Hence the axial spacing of the outer rings is given by

$$\frac{\overline{\Delta h}}{D_p} = \frac{\sqrt{3}}{2F2} + \frac{2}{\pi} \cdot \frac{0.3853}{\sqrt{(\frac{D_v}{D_p} - 1)^2 + 0.5938}}$$

$$= \frac{0.8660}{F2} + \frac{0.2453}{\sqrt{(\frac{D_v}{D_p} - 1)^2 + 0.5938}}$$

Rocke's observed values and values computed by this equation are presented in the following table.

$\frac{D_v}{D_p}$	Observed	Computed, Based on Theoretical Number of Spheres		Computed, Based on Whole Number of Spheres	
	$\frac{\overline{\Delta h}}{D_p}$	$\frac{\overline{\Delta h}}{D_p}$	Error %	$\frac{\overline{\Delta h}}{D_p}$	Error %
40.37	0.87_4	0.8722	–0.21	0.8706	–0.39
30.63	0.87_4	0.8743	+0.03	0.8709	–0.35
25.28	0.87_7	0.8761	–0.10	0.8751	–0.22
19.01	0.88_0	0.8795	–0.06	0.8766	–0.39
14.99	0.88_2	0.8833	+0.15	0.8839	+0.22
11.52	0.88_1	0.8889	+0.90	0.8889	+0.90
9.51	0.89_1	0.8942	+0.36	0.8869	–0.46
7.52	0.88_6	0.9025	+1.86	0.8966	+1.20

The errors are close to the required limit of ± 0.3 per cent for values of D_v/D_p greater than about 14. The source of the larger errors associated with the smaller values of D_v/D_p is not known; this would involve a detailed experimental investigation beyond the scope of this study. However, two effects can be postulated, which in the appropriate combinations could account for most of the discrepancies.

(i) An integral number of waves instead of the theoretical 5.339, a smaller number resulting in an increase in δh_{max} , a larger number resulting in a decrease.

(ii) A departure from a sinusoidal wave configuration as D_v/D_p decreases. Thus for 6 waves, when the number of spheres in an outer ring is less than 36, a limited number of intermediate values of δh is possible. The mean displacement should therefore tend towards 0.5 δh_{max} , and reach this value when the number of spheres is 24.

The errors in the computed values of $\overline{\Delta h}/D_p$ caused by neglecting these effects are as in the following table.

APPENDIX 2 (continued)

$\frac{D_v}{D_p}$	Percentage Error in Computed Value of $\overline{\Delta h}/D_p$ Caused by Neglecting		
	Effect (i)		Effect (ii)
	5 Waves	6 Waves	
40.37	–0.05	+ 0.08	+ 0.15
30.63	–0.06	+ 0.10	+ 0.20
25.28	–0.08	+ 0.13	+ 0.25
19.01	–0.11	+ 0.17	+ 0.33
14.99	–0.13	+ 0.22	+ 0.43
11.52	–0.08	+ 0.29	+ 0.57
9.51	–0.22	+ 0.35	+ 0.69
7.52	–0.28	+ 0.46	+ 0.90

APPENDIX 3

DERIVATION OF PROPERTIES OF CENTRAL REGION OF CYLINDRICALLY ORDERED PACKINGS WITH TRANSITIONAL VALUES OF D_v/D_p

(a) Calculation of Transitional Value of D_v/D_p

Rocke (1971) has shown that the mean radial ring spacing for an infinite D_v/D_p ratio is given by

$$\frac{\overline{\Delta R}}{D_p} = \frac{1}{\sqrt{6}} + \frac{3\sqrt{3}}{4} \arcsin\left(\frac{1}{3}\right) = 0.8497....$$

If this value is assumed to hold for finite cylinders, then for the smallest vessel accommodating ν rings, including a central sphere which is counted as a ring,

$$\frac{D_v}{D_p} = 2\left(\frac{\overline{\Delta R}}{D_p}(\nu-1)+0.5\right) = 1.699\ \nu - 0.699\ .$$

Rocke has compared values computed by this equation with his experimentally determined values. At first, about 200 approximately uniformly distributed values of D_v/D_p from 7.8 to 24.8 were used. These experiments showed that the transitional values of D_v/D_p depended on whether light or heavy vibration was used in the preparation of the packings. Then the ranges associated with values of ν of 6, 10, 13 and 17 were investigated with higher resolution.

The values in the following table resulted from the lightest and heaviest vibration for which ordered packings would form. The computed values fall well within the observed range in each case.

Number of Rings (ν)	Computed Transitional D_v/D_p Ratio	Observed D_v/D_p Ratio			
		Maximum with 4 Spheres in Centre		Minimum with 1 Sphere in Centre	
		Light Vibration	Heavy Vibration	Light Vibration	Heavy Vibration
6	9.497	9.399	9.635	9.420	9.670
10	16.295	16.180	16.436	16.216	16.442
13	21.393	21.251	21.447	21.291	21.473
17	28.191	27.992	28.292	28.037	28.325

(b) Calculation of Mean Radial Ring Spacing with 4 Spheres in Central Ring

Since the packing contains $\nu - 1$ rings, and the diameter of the central ring is $\sqrt{2}\ D_p$, the radial ring spacing is given by

$$\frac{\overline{\Delta R_v}}{D_p} = \frac{\frac{1}{2}\left(\frac{D_v}{D_p}\right) - \frac{1}{2} - \frac{\sqrt{2}}{2}}{(\nu-2)} = \frac{\frac{D_v}{D_p} - 2.414}{2(\nu-2)}\ .$$

APPENDIX 3 (continued)

The values of $\overline{\Delta R_v}/D_p$ in the following table were computed by this equation for the four D_v/D_p ratios,

Computed Transitional D_v/D_p Ratio	$\frac{\overline{\Delta R_v}}{D_p}$ with 4 Spheres in Central Ring
9.497	0.8854
16.295	0.8675
21.393	0.8627
28.191	0.8592

(c) Calculation of $OF_{C_{max}}$

The theoretical number of spheres in the peripheral ring is given by:

$$n_{R_p} = \frac{\pi}{\arcsin\left(\frac{D_p}{D_v - D_p}\right)}$$

(See Appendix 1 of Part I for derivation).

The theoretical number of spheres in the i-th ring (counted from the centre) is obtained similarly by treating it as the peripheral ring in a vessel with an effective diameter ratio of $D_v/D_p - 2(\nu - i)\overline{\Delta R_v}/D_p$, giving,

$$n_{R_i} = \frac{\pi}{\arcsin\left(\frac{D_p}{D_v - 2(\nu - i)\overline{\Delta R_v} - D_p}\right)}$$

The actual number of spheres in the i-th ring is then taken as the whole number part of n_{R_i}, except for the two central rings where the observed values are used.

Values are given in the following table for the transitional D_v/D_p ratio of 9.497.

4 Spheres in Central Ring				1 Sphere in Central Ring			
Ring No.	Effective Diameter Ratio	Number of Spheres		Ring No.	Effective Diameter Ratio	Number of Spheres	
		Theoretical	Actual			Theoretical	Actual
1	2.414	4.000	4	1	1.000	1.000	1
2	4.185	9.837	10	2	2.699	4.994	5
3	5.956	15.462	15	3	4.399	10.520	10
4	7.726	21.053	21	4	6.098	15.913	15
5	9.497	26.633	26	5	7.798	21.278	21
				6	9.497	26.633	26
Totals		76.984	76	Totals		80.337	78

APPENDIX 3 (continued)

The value of $OF_{C_{max}}$ is calculated from Equation 3 of Part I, that is

$$OF_C = \frac{(N-n)D_p^3 - 1.9099V_I}{1.1107\,H\,(D_v - 2D_p)^2} .$$

The following example is given for four spheres in the central ring, on the basis of the actual number of spheres in each ring

$D_v/D_p = 9.497$

$N = 76$

$n = 26$

Mean volume of intruding spherical segment

$= 0.01517\ D_p^3$ (see Appendix 5 for derivation)

$H = \overline{\Delta h} = 0.8872\ D_p$ (from Equations 5 and 7)

Number of spheres in second ring from wall = 21

$V_I = 21 \times 0.01517\ D_p^3 = 0.3186\ D_p^3$.

Hence $$OF_{C_{max}} = \frac{50\ D_p^3 - 1.9099(0.3186\ D_p^3)}{1.1107(0.8872\ D_p)(7.497\ D_p)^2}$$

$= 0.8918$.

APPENDIX 4

DERIVATION OF PROPERTIES OF LOOSEST POSSIBLE RANDOM PACKING RELEVANT TO THE $V_{IIU} - OF_w$ RELATIONSHIP

From Appendix 3 of Part IV,

$$OF_w = \frac{2}{m}$$

and

$$V_{IIU} = \frac{\frac{\pi}{3}\left(1-\sqrt{1-\frac{m}{6}}\right)^2\left(0.5+\sqrt{1-\frac{m}{6}}\right)}{\frac{\sqrt{3}}{4}\,m}$$

Gradient of $V_{IIU} - OF_w$ curve

$$= \frac{d(V_{IIU})}{dm} \Big/ \frac{d(OF_w)}{dm}$$

$$= \frac{2\pi}{3\sqrt{3}}\left[\left(1-\sqrt{1-\frac{m}{6}}\right)^2\left(0.5+\sqrt{1-\frac{m}{6}}\right)-0.25\,m\left(1-\sqrt{1-\frac{m}{6}}\right)\right] .$$

When $OF_w = 0.615$, the values calculated from these equations are

$$m = 3.252$$

$$V_{IIU} = 0.09144$$

and gradient of V_{IIU}/OF_w curve $= -0.1691$.

APPENDIX 5

CALCULATION OF MEAN VOLUMES OF INTRUDING SPHERICAL SEGMENTS OF SPHERES OF UNIT DIAMETER

The volume of a segment of a sphere of radius r is given by

$$V_{SSEG} = \frac{\pi}{3} h^2 (3r - h)$$

where h is the height of the segment.

In the cases to be considered here, r = 0.5 and h is the intrusion distance d_I, so that $V_{SSEG} = \frac{\pi}{3} \cdot d_I^2 (1.5 - d_I)$.

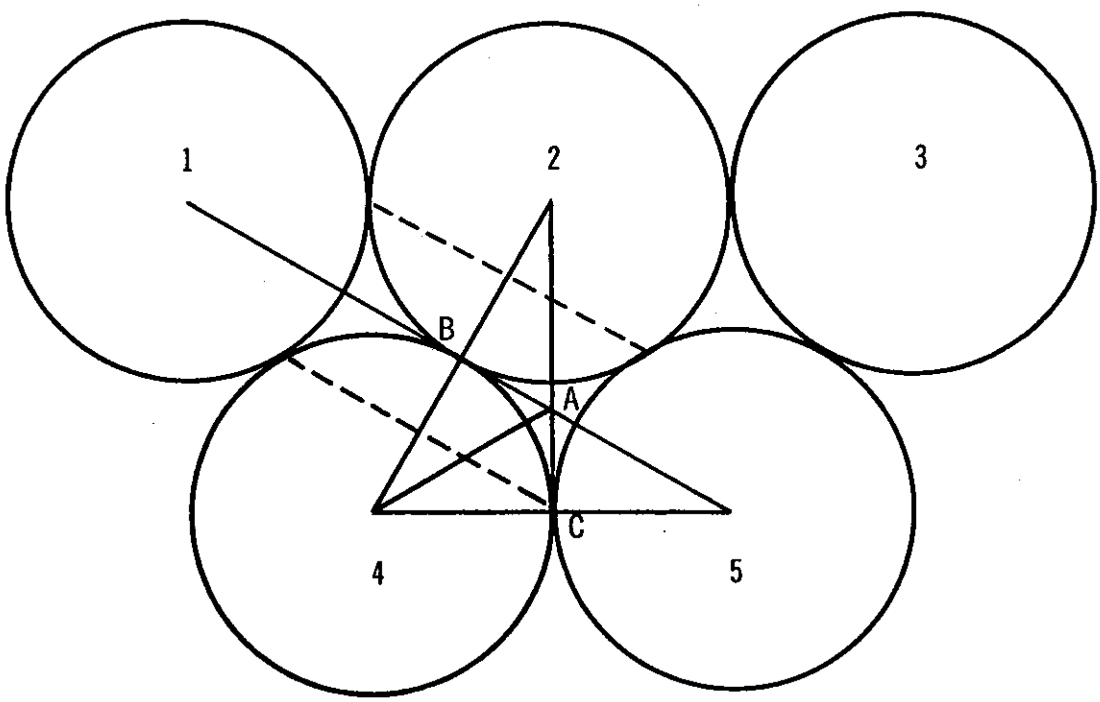
(i) Rhombohedral array in semi-infinite vessel

The intrusion distance is $1 - \sqrt{2/3}$, hence

$$V_{SSEG} = \frac{\pi}{3} \left(1 - \sqrt{\frac{2}{3}}\right)^2 \left(0.5 + \sqrt{\frac{2}{3}}\right) = 0.04642$$

(ii) Cylindrically ordered packing in infinite cylinder

The wavelike structure of the rings gives rise to cyclic variations in the radial positions of the intruding spheres. The following diagram shows a group of five spheres against the cylindrical wall, viewed radially.



The maximum intrusion distance is $1 - \sqrt{2/3}$, the same as for the rhombohedral array, and occurs when the intruding sphere is in one of the 'cusps', for example over point A and in contact with spheres 2, 4 and 5. The minimum is $1 - \sqrt{3/2}$, and occurs when the intruding sphere is halfway between two 'cusps', for example over point B, the point of contact between spheres 2 and 4.

APPENDIX 5 (continued)

If a sphere is placed over B and rolled while maintaining contact with spheres 2 and 4, the points of contact lie on circles represented by the dotted lines in the diagram. When the sphere is over point A, the line joining its centre to B has rotated through the angle arsin(1/3) = 0.3398. In the process it has occupied every possible position, and if they are given equal weight with respect to the plane of the paper, the mean value* of the intrusion distance is given in terms of the angular position, θ, by

$$d_I = \frac{\int_0^{\mathrm{arsin}(1/3)} \left(1 - \frac{\sqrt{3}}{2}\cos\theta\right) \frac{\sqrt{3}}{2}\cos\theta \, d\theta}{\int_0^{\mathrm{arsin}(1/3)} \frac{\sqrt{3}}{2}\cos\theta \, d\theta}$$

$$= 1 - \frac{\frac{\sqrt{3}}{4}\left[\sin\theta\cos\theta + \theta\right]_0^{\mathrm{arsin}(1/3)}}{\left[\sin\theta\right]_0^{\mathrm{arsin}(1/3)}}$$

$$= 1 - \frac{1}{\sqrt{6}} - \frac{3\sqrt{3}}{4}\,\mathrm{arsin}(1/3)$$

$$= 1 - 0.8497 = 0.1503.$$

The mean volume of the intruding spherical segment is given by

$$V_{SSEG} = \frac{\pi}{3} \cdot \frac{\int_0^{\mathrm{arsin}(1/3)} \left(1 - \frac{\sqrt{3}}{2}\cos\theta\right)^2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\cos\theta\right) \cdot \frac{\sqrt{3}}{2}\cos\theta \, d\theta}{\int_0^{\mathrm{arsin}(1/3)} \frac{\sqrt{3}}{2}\cos\theta \, d\theta}$$

$$= \frac{\pi}{3} \cdot \frac{\int_0^{\mathrm{arsin}(1/3)} \left(\frac{9}{16}\cos^4\theta - \frac{9\sqrt{3}}{16}\cos^3\theta + \frac{\sqrt{3}}{4}\cos\theta\right) d\theta}{\frac{\sqrt{3}}{6}}$$

$$= \frac{2\pi}{\sqrt{3}} \left[\sin\theta\left(\frac{9}{64}\cos^3\theta - \frac{3\sqrt{3}}{16}\cos^2\theta + \frac{27}{128}\cos\theta - \frac{\sqrt{3}}{8}\right) + \frac{27}{128}\theta\right]_0^{\mathrm{arsin}(1/3)}$$

$$= 0.03216.$$

The value calculated from the mean value of d_I is 0.03193, a discrepancy of 0.7 per cent.

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* If they are given equal weight with respect to the angular position, the computed values agree within 0.04 per cent.

APPENDIX 5 (continued)

(iii) Cylindrically ordered packing in finite cylinder with one sphere in central ring

The same considerations apply as for the infinite cylinder, but corrections are necessary to account for the curvature of the wall. Both the shape of the segment and the intrusion distance vary with D_v/D_p, but only the latter is corrected for in the following.

Referring to the diagram, one effect of the curvature is to increase the radial distance of point C from the wall by an amount

$$\frac{D_v - 1}{2} - \sqrt{\left(\frac{D_v - 1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

(See Appendix 1 of Part I for derivation).

The intrusion distance at A therefore decreases by two thirds of this amount, which for 1 in. spheres in an 8.71 in. cylinder gives 0.0217.

Similarly the intrusion distance at B decreases by an amount

$$\frac{D_v - 1}{2} - \sqrt{\left(\frac{D_v - 1}{2}\right)^2 - \left(\frac{1}{4}\right)^2}$$

which for 1 in. spheres in an 8.71 in. cylinder is 0.0081.

The decrease in intrusion distance is assumed to vary linearly between the two points, and the mean volume of the intruding spherical segment is computed by numerical integration.

The value so determined for 1 in. spheres in an 8.71 in. cylinder is 0.02633.

(iv) Cylindrically ordered packing in finite cylinder with four spheres in a central ring

The same considerations apply as for the previous case, except that the mean radial spacing of the rings is greater, and hence the intrusion distance is smaller.

The mean radial spacing is obtained by eliminating ν from the second and third equations of Appendix 3, giving

$$\frac{\overline{\Delta R}_v}{D_p} = \frac{0.8497 \dfrac{D_v}{D_p} - 2.051}{\dfrac{D_v}{D_p} - 2.699}$$

The intrusion distance is reduced accordingly, and the mean volume of the intruding spherical segment is computed by numerical integration.

The value of $\overline{\Delta R}_v/D_p$ so determined for 1 in. spheres in an 8.71 in. cylinder is 0.8900, an increase of 0.0403 over the previous case. The mean volume of the intruding spherical segment is 0.01346.

APPENDIX 6

DERIVATION OF FACTOR F_{IIU} FOR FLOATING INTRUDING SPHERES

When $OF_W = 0.615$, $V_{WU} = 0.3718$ and $V_{IIU} = 0.09144$. If 4.4 per cent of the spheres throughout the packing are floating spheres, then the volume of the floating spheres capable of touching the wall per unit area is

$$V_{WU}(1/0.956 - 1) - 0.046V_{WU} = 0.01711 \ .$$

On the basis of all such spheres being counted as intruding spheres, and all having receded a distance $D_p/8$ from the wall, the volume of the portion of each sphere in the inner half of the outer region is 0.6406 of the volume of the sphere.

Hence $V_{IIU} = 0.09144 + (0.6406 \times 0.01711) = 0.1024$.

This is an increase of 11.99 per cent, compared with 4.6 per cent for OF_C .

Hence $F_{IIU} = 1 + (11.99/4.6)(F_C - 1) = 1 + 2.61\ (F_C - 1)$.

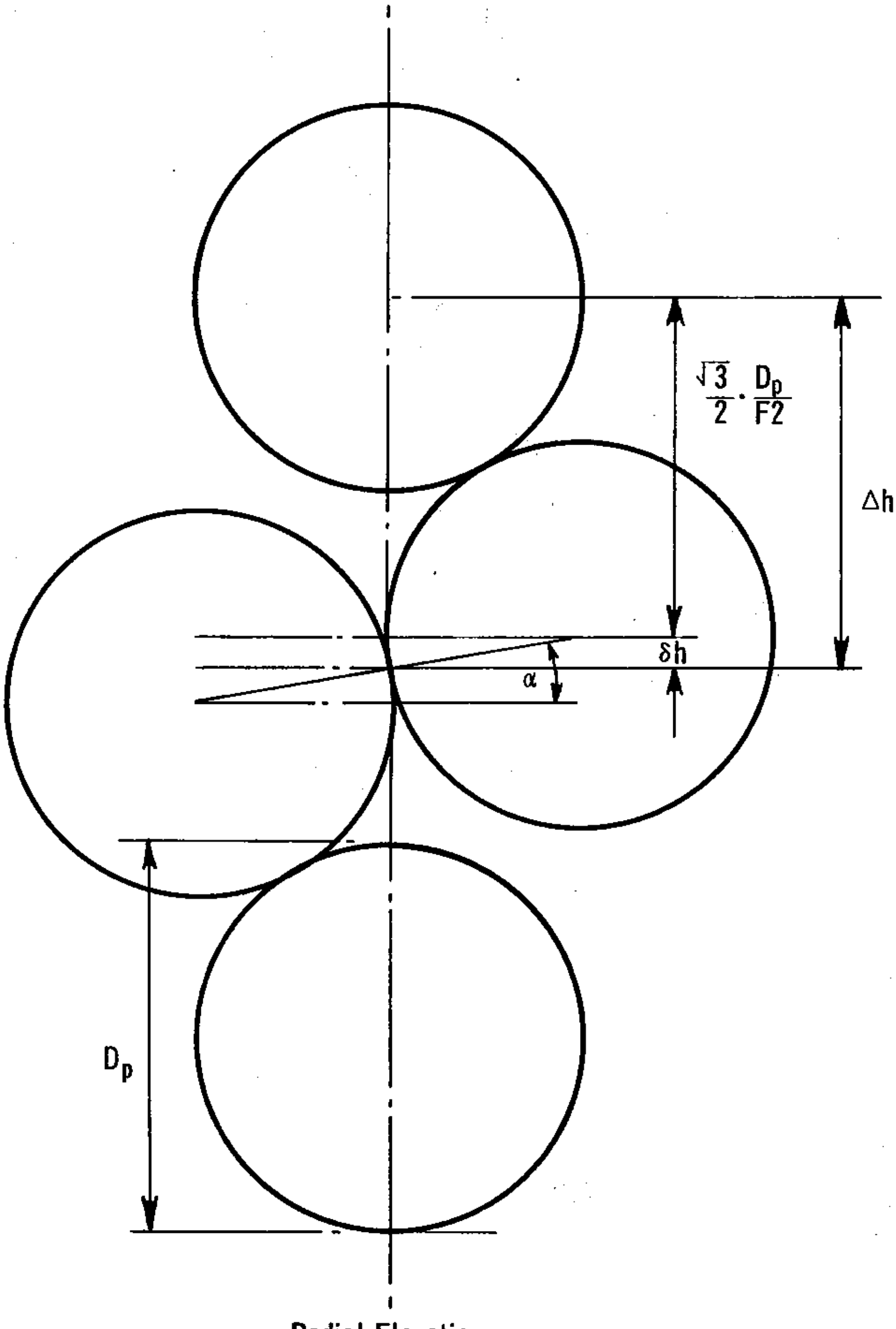

FIGURE 1. EFFECT OF VERTICAL WAVE DISPLACEMENT ON AXIAL SPACING OF SPHERES

(After Rocke 1971)

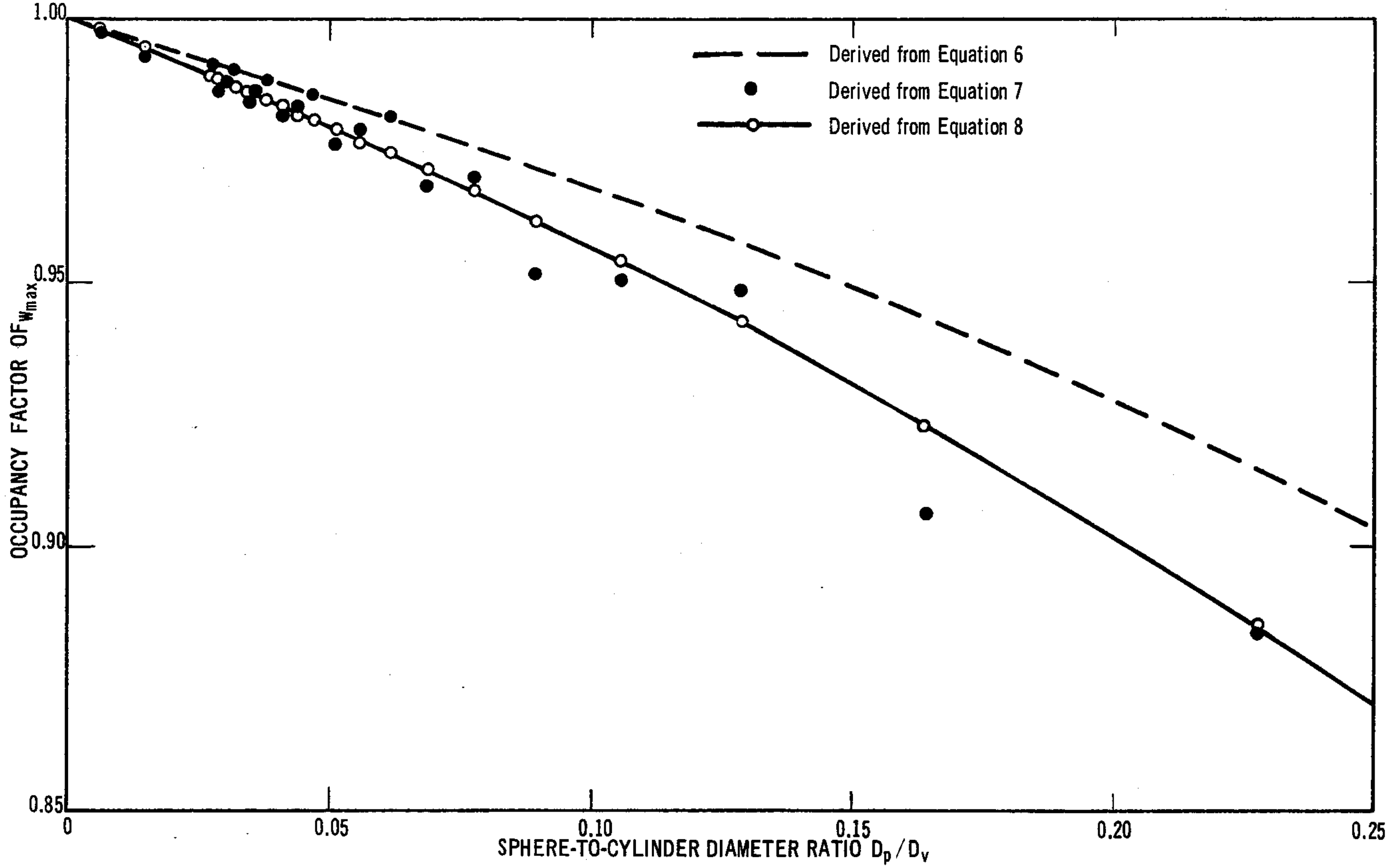

FIGURE 2. OCCUPANCY FACTORS OF SPHERES TOUCHING WALL IN DENSEST PACKINGS

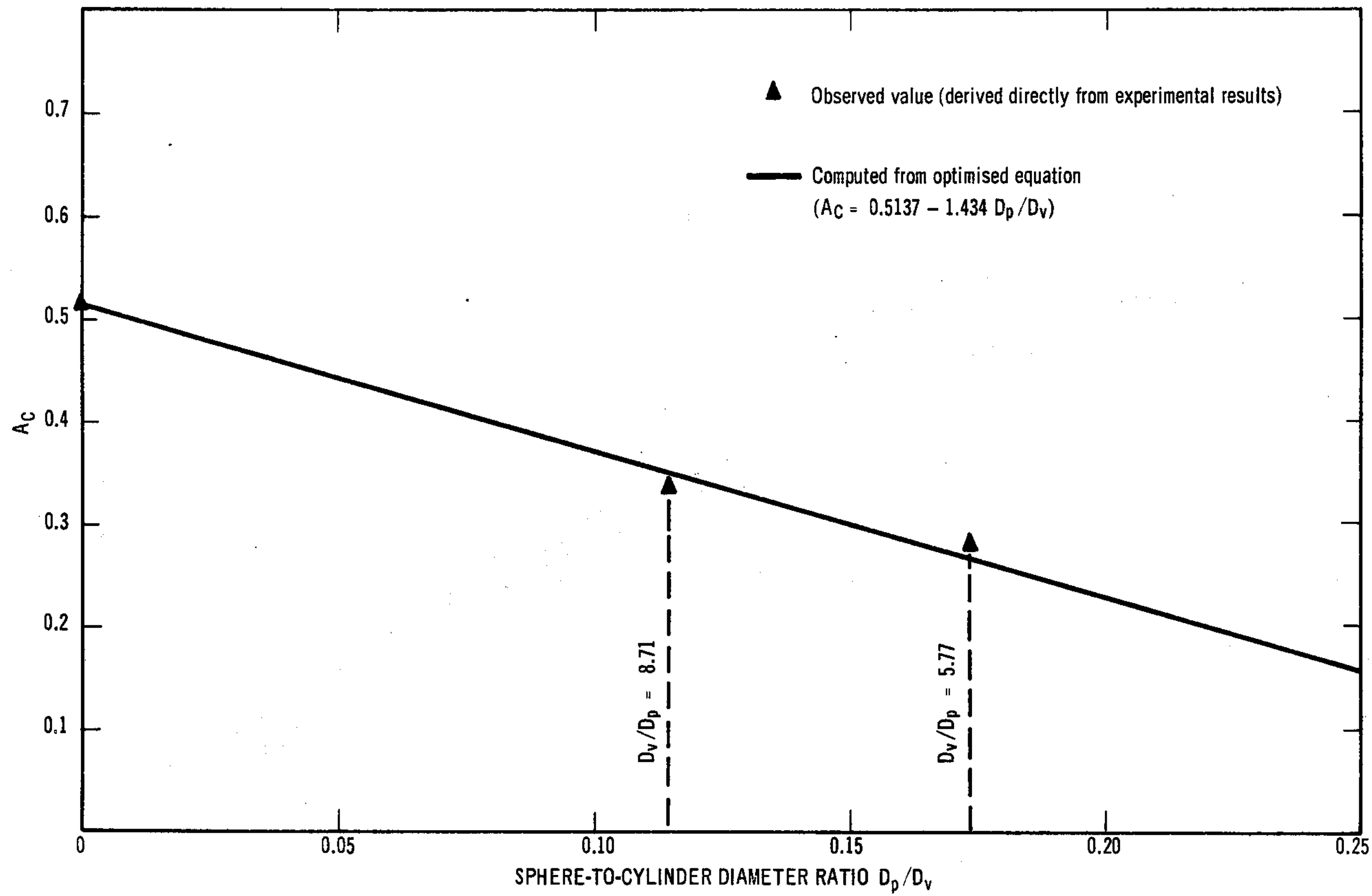

FIGURE 3. VALUES OF COEFFICIENT A_C FOR EQUATION 3

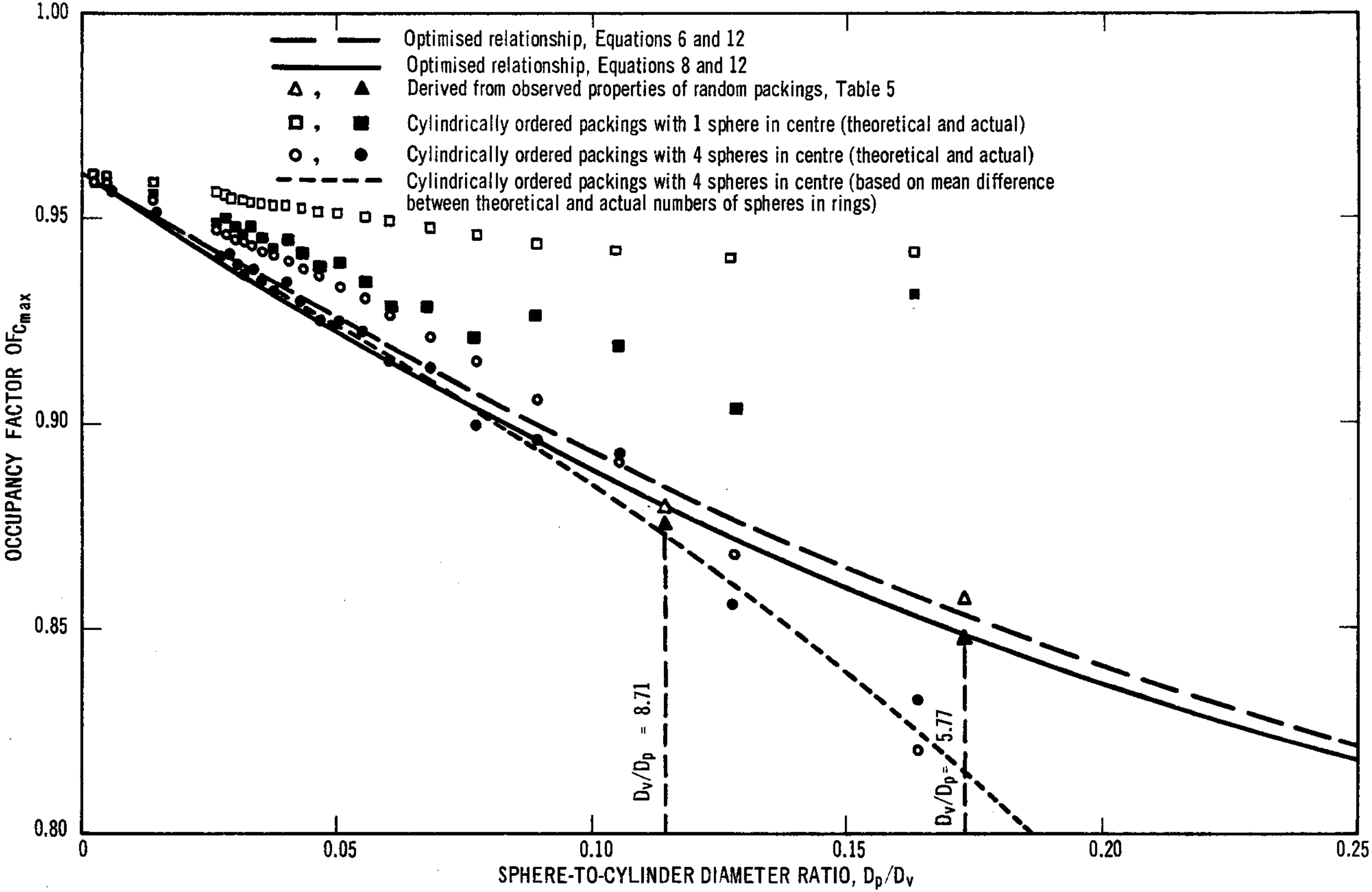

FIGURE 4. OCCUPANCY FACTORS OF CENTRAL REGION OF DENSEST PACKINGS

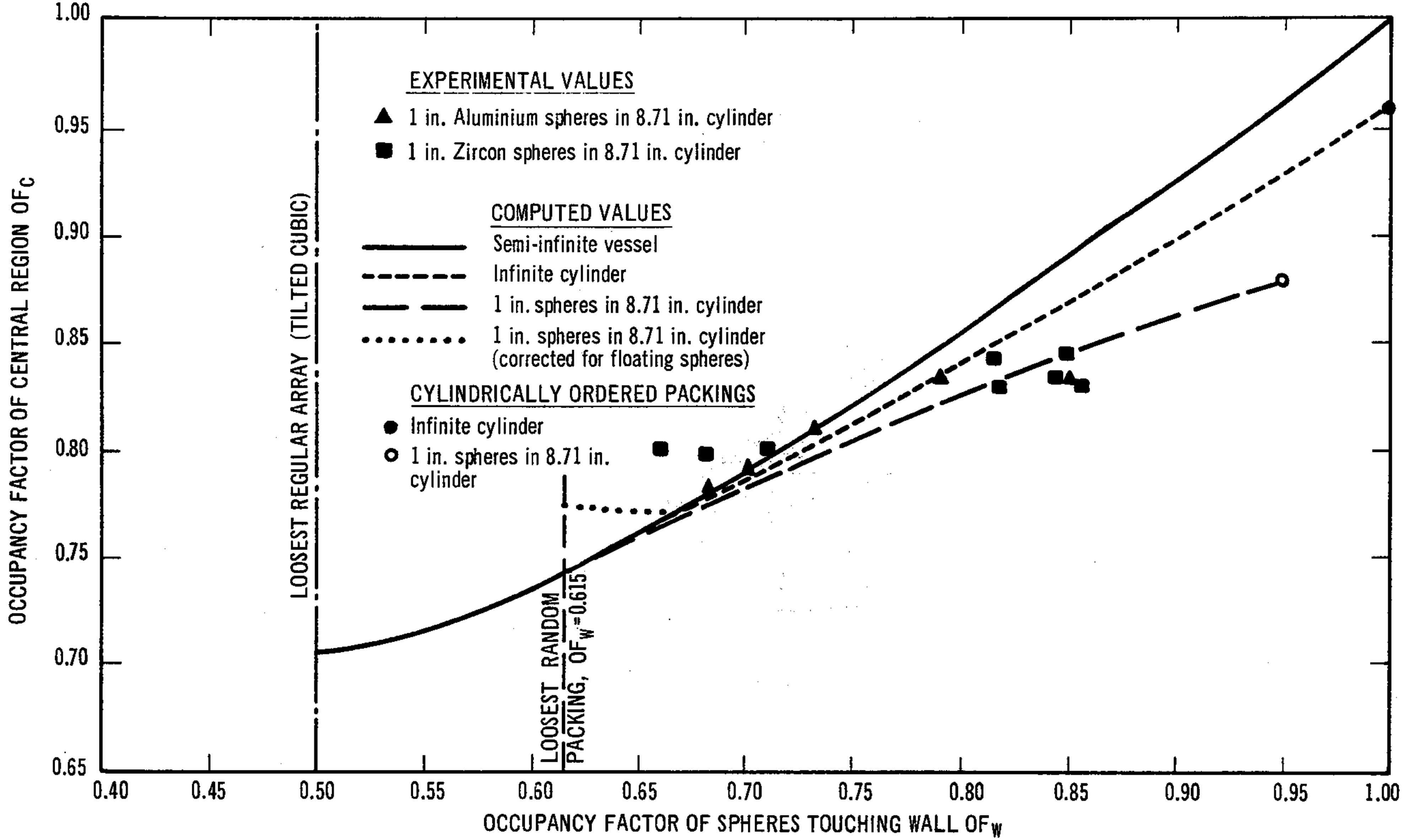

FIGURE 5. OBSERVED AND COMPUTED OCCUPANCY FACTORS OF CENTRAL REGION

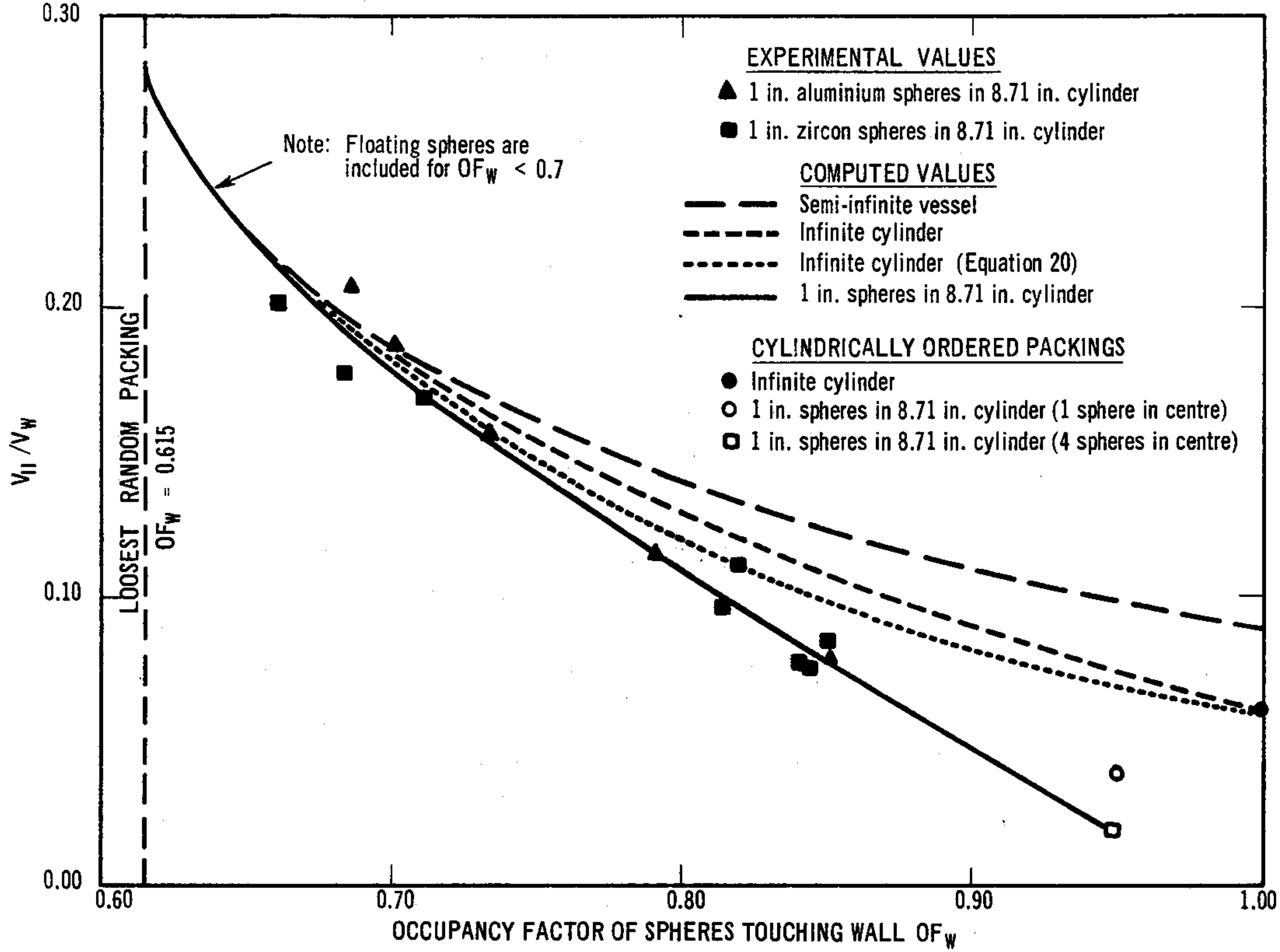

FIGURE 6. OBSERVED AND COMPUTED VOLUMES OF INTRUDING SPHERES IN INNER PART OF OUTER REGION

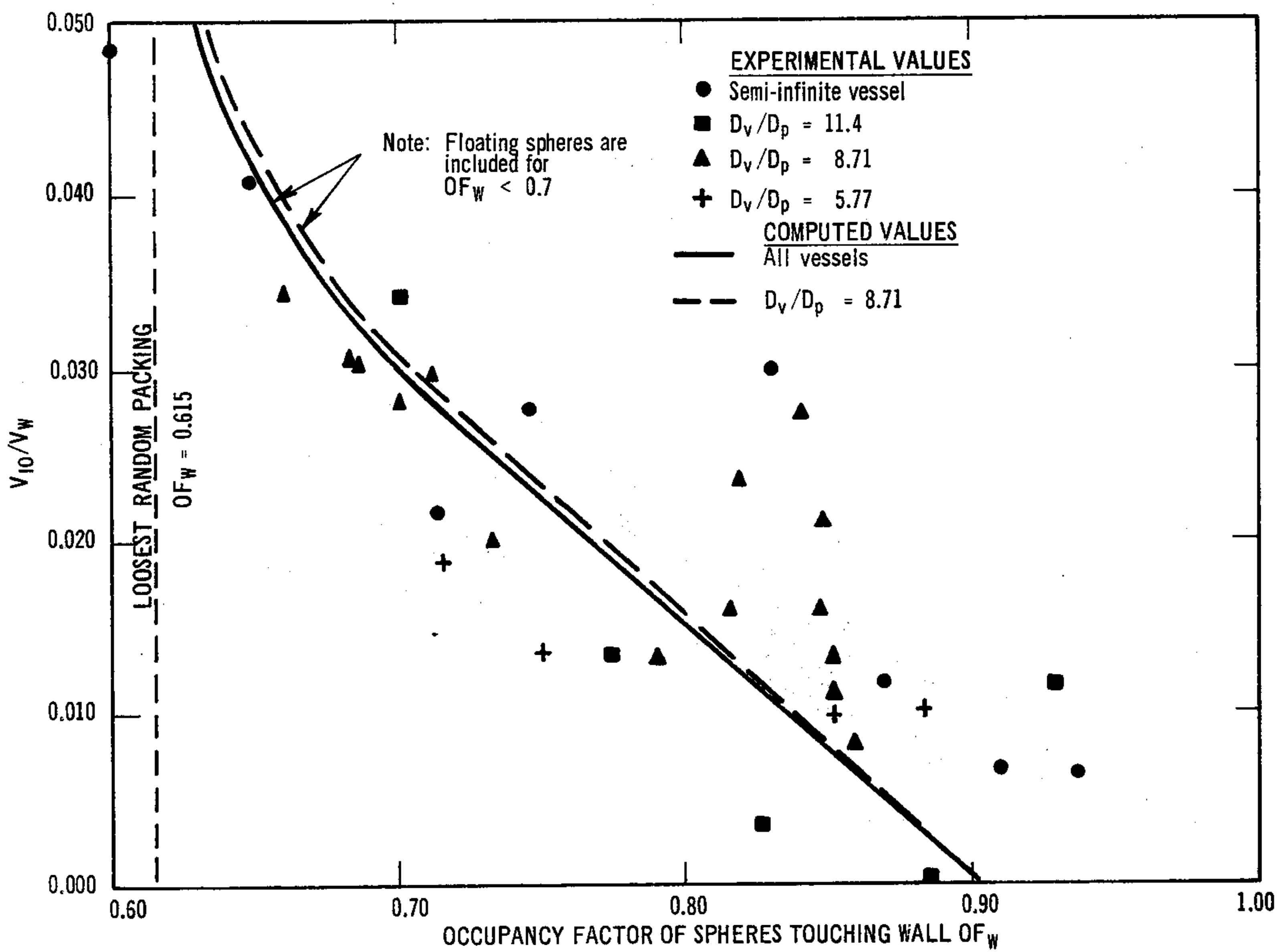

FIGURE 7. OBSERVED AND COMPUTED VOLUMES OF INTRUDING SPHERES IN OUTER PART OF OUTER REGION

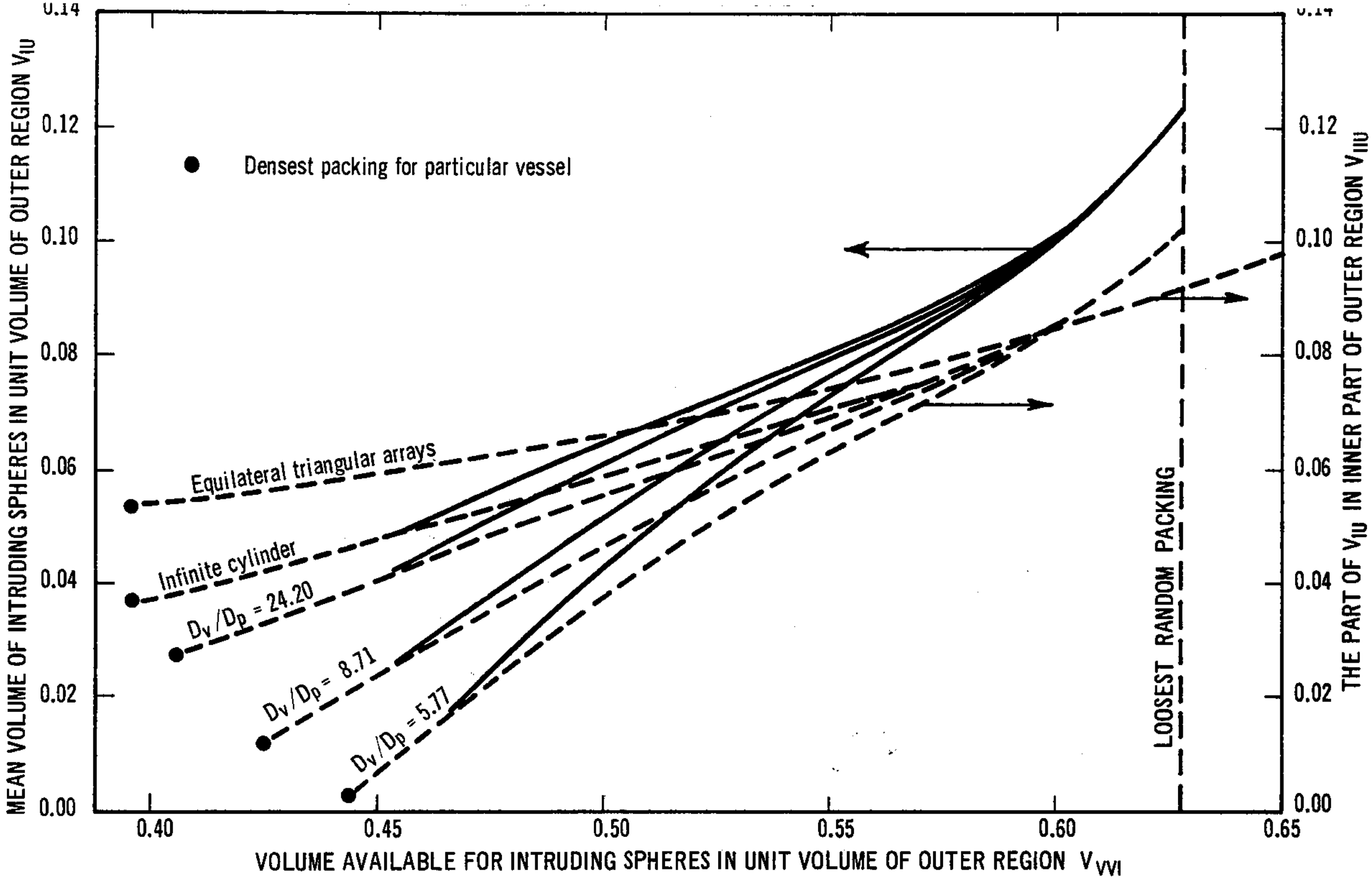

FIGURE 8. COMPUTED RELATIONSHIPS BETWEEN VOLUMES OF INTRUDING SPHERES AND AVAILABLE VOID VOLUME OF OUTER REGION