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THE SIMPLIFIED THEORY OF MEASUREMENT OF HEAT TRANSFER COEFFICIENTS WITH TRANSIENT TECHNIQUES

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July 1967

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ABSTRACT

An outline is given of a simplified approach to the theory of transient convective heat transfer.

Analytical solutions for some cases of substantial interest in surface heat transfer coefficient measurement are obtained. Comparisons with results of previous workers indicate that this new approach leads to exact solutions.

CONTENTS

1. INTRODUCTION

The surface heat transfer coefficient is usually defined as

$$h = (Q/S)\,(\theta_s - T)^{-1}$$

and is calculated from the measured heat flux and the surface-to-coolant temperature difference under steady state conditions. The more complex a nuclear fuel element becomes, the more difficult it becomes to simulate nuclear heating electrically and to measure surface temperature. This is true, for example, for the fuel elements in the random packed pebble bed reactor, for which an alternative method is required.

The heat transfer in a system in which fluid flows past an assembly of surfaces depends on the geometry, flow conditions, physical properties and the heat transfer coefficient. From the measured response of the system to a given transient the heat transfer coefficient can be deduced. If a change in upstream fluid temperature is used as the impressed transient, and the corresponding change in downstream fluid temperature is used as a measure of the response characteristic, the heat transfer coefficient can be determined without direct heating of the sample surface or measurement of its temperature.

Because both fluid heating and temperature measurement are straightforward, transient techniques are experimentally simple and the measurement is precise, irrespective of the test specimen geometry. Unfortunately, interpretation of the measurements in terms of an average heat transfer coefficient is not necessarily straightforward.

The suitable analytical solution of the heat transfer problem in a randomly distributed porous medium is possible provided surface heat transfer is the controlling mechanism, and provided radiation, conduction and dispersion can be neglected or successfully approximated. Then transient response will be a sensitive measure of the heat transfer.

A step change in upstream fluid temperature technique, based on a solution by Schumann (1929), has been used by several investigators including Furnas (1930), Saunders and Ford (1940), Löf and Hawley (1948) and Coppage and London (1956). More recently the method of cyclic temperature variation, based on the work of Bell and Katz (1949) and extended by Dayton et al. (1952) and Bain (Meek 1962) has tended to replace the older technique because of its greater precision and flexibility. Although any periodic function could be used, the sine wave has advantages analytically, because the mathematical simplicity allows extension to more complex systems. It also has experimental advantages, because a single measurement of either amplitude attenuation or phase shift suffices.

This report gives an outline of the theory, and develops a simplified approach. Then analytical solutions for several cases of interest are derived and compared with previous work. The obtained solutions are used in a review of the theory of unsteady state heat transfer, and in an evaluation of the theoretical and experimental conditions for which the heat transfer coefficient can be accurately determined. These are in a separate paper (Hart and Szomanski 1967).

2. THEORY

2.1 Mechanism of Heat Transfer

If the fluid flows past a solid surface and the fluid and surface temperatures are not equal (say the fluid is hotter), then heat is transferred in a direction normal to the surface in two stages:

1. through the fluid boundary layer near the surface, at a rate governed by the heat transfer coefficient h;

2. through the solid, at a rate governed by its thermal conductivity k_s.

The fluid loses heat as it passes a colder solid surface so that a temperature gradient exists in the direction of main flow. Thermal conductivity in the solid controls heat flow in any direction, but in the fluid, in a direction parallel to the surface, heat is transferred by convection (bulk movement of the fluid), and conduction and dispersion (differential movement of the fluid). The last two can be considered as governed by an effective thermal conductivity:

$$k_f' = k_f + D_L \rho_f C_p \quad .$$

With transient techniques only small temperature variations are required in the experiments and as a result the effects of radiant heat transfer are negligible and the physical properties of the system may be considered invariant. The latter is also true for compressible fluids, if pressure differences are small.

2.2 Physical Assumptions

It is assumed in the model that:

1. all cross-sections normal to the direction of the mean flow are uniform,
2. the distribution of fluid velocity in planes normal to the direction of flow is uniform,
3. the surface heat transfer coefficient is independent of the position in the system,
4. if the fluid temperature distribution in planes normal to the direction of mean flow is uniform upstream, it will remain so as the fluid passes through the bed.

These assumptions need be true only in a macroscopic sense; for example a randomly packed bed whose size is large compared to that of the pebble can be regarded as a good approximation of the model.

2.3 Governing Differential Equations

Differential equations governing transient heat transfer are obtained by considering the heat balance in a volume element:

(i) of the fluid,

$$A_f \rho_f C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x}\right) dx = A_f k_f' \frac{\partial^2 T}{\partial x^2} dx - h s (T-\theta_s) dx;$$

and (ii) of the solid,

$$A_s \rho_s C_s \frac{\partial \overline{\theta}}{\partial t} dx = A_s k_s \frac{\partial^2 \theta}{\partial x^2} dx + h s (T-\theta_s) dx \, .$$

The relation between the mean solid bulk temperature $\overline{\theta}$, and the solid surface temperature θ_s, depends on the solid geometry. For instance, if the bed of spheres is considered and the temperature in the solid is assumed to be a function of x , t and r only, then:

$$\overline{\theta}_{(x,t)} = \frac{3}{a^3} \int_o^a r^2 \, \theta(x,t,r) \, dr \quad ,$$

where $\theta(x,t,r)$ satisfies the heat conduction equation:

$$\frac{\partial \theta}{\partial t} = \frac{k_s}{\rho_s C_s} \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r} \frac{\partial \theta}{\partial r}\right) \quad .$$

2.4 Simplification of Differential Equations

These equations are to be solved for the fluid temperature upstream from the bed, T(0,t), as a prescribed function of time. Their analytical solution in general form is not available, therefore it is necessary to simplify them. This leads to a decrease of the generality of solutions, but this is not serious as long as the conditions imposed correspond to experimentally realisable systems.

In all cases considered it is assumed that the effective longitudinal conduction in the fluid can be neglected. As a result the second order term may be deleted from the heat balance in the fluid. In general this assumption appears to be realistic if the fluid is a gas and the flow Reynolds number is high.

Longitudinal conduction in the solid is assumed to be either infinite or zero. Comparison of the respective solutions is then used to delineate the conditions for which it is possible to neglect the longitudinal contribution of the solid to the overall longitudinal conduction in the matrix.

A Biot number $Bi = h\,a/k_s$ is an indication of the relative significance of surface heat transfer and thermal conductivity in the solid in determining the rate of heat exchange between the fluid and solid. It is clear that the condition of $Bi \to \infty$ is not acceptable if the transient response is to be used to measure h, and the condition $Bi \to 0$ should be approached to obtain maximum sensitivity.

2.5 Method of Solution

The analytical solution is greatly simplified if the $\frac{\partial T}{\partial t}$ term can be deleted from the heat balance of the fluid. The conditions under which this is permissible are not immediately obvious, but may be associated tentatively with the large solid thermal capacity. It will be shown in Section 3.1.1 that for gas – solid systems only, and these are of main interest, the ratio $\rho_f C_p/\rho_s C_s$ is small and the change in solid temperature during the passage of a fluid temperature wave-front through the specimen can be neglected. As a result the heat balance in the fluid can be integrated with respect to x only, and in the solid with respect to t only, if infinite radial and longitudinal conductivity is assumed.

Having obtained a solution for infinite longitudinal conductivity in the solid, zero longitudinal conductivity can be obtained by letting the specimen consist of N layers, each Δx long, in each of which the longitudinal conductivity is infinite but between which it is zero, and finding the limit as $\Delta x \to 0$. The intermediate case in which Δx is finite is also of some practical interest and is referred to as a 'layer' model.

3. SOLUTIONS FROM THE SIMPLIFIED THEORY

3.1 Solid With Infinite Thermal Conductivity in Radial and Longitudinal Direction

3.1.1 Step change in upstream fluid temperature

Let the fluid temperature $T(x,t)$ and solid temperature $\theta(t)$ be constant and equal to T_o for all $t < 0$. For all $t \geqslant 0$, let $T(0,t) = T_f$. At $t = \infty$ the fluid and solid temperatures are again constant but equal now to T_f. The magnitude of the step change is:

$$\Delta = T_f - T_o .$$

The simplified heat balance in the fluid is:

$$A_f \rho_f u C_p \, dT = - h s (T - \theta) dx ,$$

and

$$A_s \rho_s C_s \, d\theta = + h s (T - \theta) dt \text{ in the solid.}$$

Assuming that the solid temperature wave-front passes through the specimen of length l, then the initial fluid temperature downstream from the specimen is T(l,0) where:

$$A_f \rho_f u C_p \int_{T_f}^{T(l,0)} \frac{dT}{T-\theta} = - \int_o^l h s \, dx ,$$

so $$\frac{T(1,0) - T_o}{T_f - T_o} = \exp\left\{ - \frac{h\,s\,l}{A_f \rho_f u\, C_p} \right\},$$

and $$T(1,0) = T_o + \Delta e^{-p} .$$

The overall heat balance for the time interval $0 \leqslant t \leqslant l/u$ is:

$$A_f \rho_f u\, C_p \left[T_f - T(1,0)\right] \frac{l}{u} = A_s\, l \rho_s C_s \Delta\theta .$$

Note that the time scale of the downstream fluid temperature response is actually displaced by the interval (l/u) relative to the upstream temperature.

The maximum rise in solid temperature $\Delta\theta$ occurs if p is large, that is, $e^{-p} \to 0$, hence $[T_f - T(1,0)] \sim (T_f - T_o) = \Delta$. Therefore, the maximum fractional change in solid temperature is:

$$\frac{\Delta\theta}{\Delta} = \frac{A_f \rho_f C_p}{A_s \rho_s C_s} .$$

If the fluid is air at atmospheric pressure this fraction is negligible (10^{-4}), hence the assumption that the solid temperature remains constant during the passage of a fluid temperature wave-front through the specimen is a reasonable approximation.

Integration of the heat balance in the fluid between the limits T_f and $T(1,t)$ yields:

$$T(1,t) = T_f e^{-p} + \theta(t) \left[1 - e^{-p}\right] .$$

Differentiating,

$$\frac{d\theta}{dt} = \frac{1}{1 - e^{-p}} \frac{dT(1)}{dt} .$$

Substitution for $T(1,t)$ or $\frac{d\theta}{dt}$ in the overall heat balance:

$$A_f \rho_f u\, C_p \left[T_f - T(1,t)\right] dt = A_s\, l \rho_s C_s\, d\theta$$

yields: $$\frac{d\theta}{dt} + \frac{\theta}{\tau} = \frac{T_f}{\tau} ,$$

and $$\frac{dT(1)}{dt} + \frac{T(1)}{\tau} = \frac{T_f}{\tau} ,$$

where $$\tau = \frac{A_s\, l\, \rho_s C_s}{A_f \rho_f u\, C_p (1 - e^{-p})} .$$

The solutions are:

$$\theta - T_f = C \exp(-t/\tau) ,$$

with the boundary condition $\theta(0) = T_o$,

so that $$\theta = T_f - \Delta \exp(-t/\tau) ;$$

and $$T(1) - T_f = C \exp(-t/\tau) ,$$

with the boundary condition $T(1,0) = T_o + \Delta e^{-p}$,

so that $$T(1) = T_f - \Delta (1 - e^{-p}) \exp(-t/\tau) .$$

These responses are conveniently expressed in dimensionless form as 'accomplished temperature fractions'.

$$[\theta] = \frac{\theta - T_o}{\Delta} = 1 - \exp(-t/\tau)$$

$$[T] = \frac{T(1) - T_o}{\Delta} = 1 - (1 - e^{-p}) \exp(-t/\tau) \ . \qquad (1)$$

Note also that if p is small ($p < 0.1$) then $p \sim [1 - e^{-p}]$. Hence $\tau = A_s \rho_s C_s / h\, s$, and the solution for $[\theta]$ is identical with the integrated form of Newton's law of cooling for a body in surroundings at uniform temperature.

3.1.2 Periodic pulse in upstream fluid temperature

Let $T(0,t) = T_f$, if $2nk\tau \leqslant t < (2n+1)k\tau$,

and $T(0,t) = T_o$, if $(2n+1)k\tau \leqslant t < (2n+2)k\tau$,

where n is an integer $\geqslant 0$.

Also $k = \pi/\tau\omega_o$,

so that $2k\tau$ is the period of a square wave, angular frequency ω_o, and τ is the thermal time constant of the system as previously defined.

Consider the interval $0 \leqslant t < k\tau$.

The initial boundary conditions are:

$$T(0,0) = T_f$$

$$\theta(0) = T_o$$

$$T(1,0) = T_o + \Delta e^{-p},$$

and the responses in the interval are described by the equations previously derived:

$$T(1) = T_f - \Delta [1 - e^{-p}] \exp(-t/\tau)$$

$$\theta = T_f - \Delta \exp(-t/\tau) \ .$$

At time $t = k\tau$ a new set of boundary conditions for the interval $k\tau \leqslant t < 2k\tau$ can now be written:

$$T(0,k\tau) = T_o$$

$$\theta(k\tau) = T_f - \Delta e^{-k}$$

$$T(1,k\tau) = T_f - \Delta e^{-p}(1 - e^{-k}) - \Delta e^{-k} \ ,$$

and the responses in the interval are now

$$T(1) = T_o + \Delta(1 - e^{-p})(1 - e^{-k}) \exp(-t/\tau)$$

$$\theta = T_o + \Delta(1 - e^{-k}) \exp(-t/\tau) \ .$$

Proceeding in this way for successive time intervals, $\Delta t = k\tau$, it may be shown that in the $2n^{th}$ interval,

$$T(1) = T_o + \Delta(1 - e^{-p}) \exp(-t/\tau) \ \{1 - e^{-k} + e^{-2k} - \ldots\ldots - e^{-(2n-1)k}\} \ ,$$

and in the $(2n+1)^{th}$ interval that:

$$T(1) = T_f - \Delta(1-e^{-P})\exp(-t/\tau)\,\{1 - e^{-k} + e^{-2k} - \ldots\ldots + e^{-2nk}\}\ .$$

The steady state is approached as $n \to \infty$, and the maximum steady state downstream fluid temperature is therefore:

$$T(1)_{max} = T_f - \Delta(1-e^{-P})\,e^{-k}\,\{\sum^{\infty}(1 - e^{-k} + e^{-2k} - \ldots\ldots\ldots)\}$$

$$= T_f - \Delta\left\{\frac{(1-e^{-P})\,e^{-k}}{1+e^{-k}}\right\}\ .$$

Similarly, the minimum steady state downstream fluid temperature is:

$$T(1)_{min} = T_o + \Delta\left\{\frac{(1-e^{-P})\,e^{-k}}{1+e^{-k}}\right\}\ .$$

The downstream fluid temperature amplitude is therefore:

$$T(1)_{max} - T(1)_{min} = \Delta\left\{1 - \frac{2(1-e^{-P})\,e^{-k}}{1+e^{-k}}\right\}\ .$$

The ratio of downstream to upstream fluid temperature amplitudes is:

$$r_T = 1 - \frac{2(1-e^{-P})\,e^{-k}}{1+e^{-k}}\ . \qquad (2)$$

It may also be shown that the ratio of solid temperature amplitude to upstream fluid temperature amplitude is:

$$r_\theta = \frac{1-e^{-k}}{1+e^{-k}} = \tanh(k/2)\ .$$

3.1.3 Sinusoidal upstream fluid temperature

Let $T(0,t) = T_o + \Delta\sin(\omega_o t)$.

Integration of the heat balance in the fluid yields:

$$\theta = \frac{1}{1-e^{-P}}\,T(1) - \frac{e^{-P}}{1-e^{-P}}\,[T_o + \Delta\sin(\omega_o t)]\ ,$$

and after differentiation,

$$\frac{d\theta}{dt} = \frac{1}{1-e^{-P}}\,\frac{dT(1)}{dt} - \frac{\Delta\,\omega_o\,e^{-P}}{1-e^{-P}}\,\cos(\omega_o t)\ .$$

Substitution for $\frac{d\theta}{dt}$ in the overall heat balance yields:

$$\frac{dT(1)}{dt} + \frac{T(1)}{\tau} = \frac{T_o}{\tau} + \frac{\Delta\sin(\omega_o t)}{\tau} - \Delta\,\omega_o\,e^{-P}\cos(\omega_o t)\ .$$

The solution is:

$$T(1) = T_o + \frac{\Delta\,\tau}{1+\tau^2\omega_o^2}\left\{\frac{\sin(\omega_o t)}{\tau} - \omega_o\cos(\omega_o t)\right\} -$$

$$- \frac{\Delta\, \omega_o\, e^{-P}\, \tau^2}{1 + \tau^2\, \omega_o^2} \left\{ \frac{\cos(\omega_o t)}{\tau} + \omega_o \sin(\omega_o t) \right\} + C \exp(-t/\tau) \quad .$$

The term $C \exp(-t/\tau)$ decays with time and can therefore be omitted from the steady state solution. Rearranging the remaining terms and making the substitution:

$$m = \frac{A_f\, \rho_f\, u\, C_p}{A_s\, l\, \rho_s\, C_s\, \omega_o} = \frac{1}{\tau(1 - e^{-P})\, \omega_o} \quad ,$$

we get:

$$T(l) = T_o + \Delta \left\{ \frac{m^2 (1 - e^{-P})^2 + e^{-P}}{m^2 (1 - e^{-P})^2 + 1} \sin(\omega_o t) - \frac{m (1 - e^{-P})^2}{m^2 (1 - e^{-P})^2 + 1} \cos(\omega_o t) \right\} \quad .$$

Now let

$$\alpha \sin(\omega_o t) - \beta \cos(\omega_o t)$$
$$= \sqrt{\alpha^2 + \beta^2}\, \sin(\omega_o t - \gamma) \quad ,$$

where

$$\alpha = \frac{m^2 (1 - e^{-P})^2 + e^{-P}}{m^2 (1 - e^{-P})^2 + 1} \quad ,$$

$$\beta = \frac{m (1 - e^{-P})^2}{m^2 (1 - e^{-P})^2 + 1} \quad ,$$

and $$\sin \gamma = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \quad .$$

By comparison with the upstream fluid temperature amplitude Δ, the downstream fluid temperature amplitude is attenuated by the factor:

$$r_T = \sqrt{\alpha^2 + \beta^2}$$

Hence $$r_T = \sqrt{\frac{e^{-2P} + m^2 (1 - e^{-P})^2}{1 + m^2 (1 - e^{-P})^2}} \quad , \qquad (3)$$

and is shifted in phase by the angle:

$$\gamma = \arc\sin \left\{ - \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right\} + \frac{\omega_o\, l}{u} \quad .$$

It may also be shown that the ratio of solid temperature amplitude to upstream fluid temperature amplitude is:

$$r_\theta = \sqrt{\frac{m^2 (1 - e^{-P})^2}{1 + m^2 (1 - e^{-P})^2}} \quad ,$$

and the phase shift is:

$$\gamma_\theta = \arc\sin \left\{ \sqrt{\frac{1}{1 + m^2 (1 - e^{-P})^2}} \right\} \quad .$$

3.2 Solid With Thermal Conductivity Infinite in Radial and Zero in Longitudinal Direction

3.2.1 Step change in upstream fluid temperature

Consider first a specimen consisting of N identical layers normal to the direction of mean flow, each of thickness Δx. Let the longitudinal solid thermal conductivity be infinite within each layer but zero between layers (layer model).

For the fluid temperature downstream from the first layer the response to a step change in upstream fluid temperature is given by the previously derived equation (1):

$$[T]_1 = 1-(1-e^{-p'})\exp(t/\tau') ,$$

where
$$p' = \frac{h\, s\,(\Delta x)}{A_f\, \rho_f\, u\, C_p} ,$$

and
$$\tau' = \frac{A_s\,(\Delta x)\,\rho_s\, C_s}{A_f\,\rho_f\, u\, C_p\,(1-e^{-p'})} ,$$

This fluid temperature response now becomes the upstream fluid temperature transient with respect to the second layer, for which the boundary conditions are:

$$T(0,0) = T_o + \Delta\, e^{-p'}$$

$$\theta(0) = T_o ,$$

and
$$T(\Delta x,0) = T_o + \Delta\, e^{-2p'} .$$

The fluid temperature response downstream from the second layer is then:

$$[T]_2 = 1-(1-e^{-p'})\ \{(1+e^{-p'}) + (1-e^{-p'})\ \tfrac{t}{\tau'}\ \}\exp(-t/\tau') .$$

Similarly, the fluid temperature responses downstream from the third, fourth and so on layers can be derived successively. Finally, the fluid temperature downstream from the N^{th} layer is:

$$[T]_N = 1-(1-e^{-p'})\exp(-t/\tau')\left\{\sum_{n=1}^{N}\frac{(1-e^{-p'})^{n-1}}{(n-1)!}\ a_{n-1}\left(\frac{t}{\tau'}\right)^{n-1}\right\} , \qquad (4)$$

where
$$a_o = \sum_{n=1}^{N} e^{-(n-1)\,p'}$$

$$a_1 = \sum_{n=1}^{N-1}\left(\sum_{n=1}^{n} 1\right) e^{-(n-1)\,p'}$$

$$a_2 = \sum_{n=1}^{N-2}\left[\sum_{n=1}^{n}\left(\sum_{n=1}^{n} 1\right)\right] e^{-(n-1)\,p'}$$

$$a_3 = \sum_{n=1}^{N-3}\left\{\sum_{n=1}^{n}\left[\sum_{n=1}^{n}\left(\sum_{n=1}^{n} 1\right)\right]\right\} e^{-(n-1)\,p'} ,$$

and so on.

It may also be shown that the solid temperature response in the N^{th} layer is:

$$[\theta]_N = 1-\exp(-t/\tau')\left\{1+(1-e^{-p'})\ \frac{t}{\tau'}\left[\sum_{n=1}^{n-1}\frac{(1-e^{-p'})^{n-1}}{n!}\ b_{n-1}\left(\frac{t}{\tau'}\right)^{n-1}\right]\right\} ,$$

where $$b_o = \sum_{n=1}^{N-1} e^{-(n-1)p'} ,$$

$$b_1 = \sum_{n=1}^{N-2} \left(\sum_{n=1}^{n} 1 \right) e^{-(n-1)p'} ,$$

$$b_2 = \sum_{n=1}^{N-3} \left[\sum_{n=1}^{n} \left(\sum_{n=1}^{n} 1 \right) \right] e^{-(n-1)p'} ,$$

and so on.

The coefficients of the terms in the series represented by a_{n-1} and b_{n-1} are set out in the following table, where each number is the sum of the preceding numbers in the same row and the same column respectively.

	1	2	3	4	5	6	7	8	9
Term	1	$e^{-p'}$	$e^{-2p'}$	$e^{-3p'}$	$e^{-4p'}$	$e^{-5p'}$	$e^{-6p'}$	$e^{-7p'}$	$e^{-8p'}$
$(t/\tau')^0$	1	1	1	1	1	1	1	1	1
$(t/\tau')^1$	1	2	3	4	5	6	7	8	9
$(t/\tau')^2$	1	3	6	10	15	21	28	36	45
$(t/\tau')^3$	1	4	10	20	35	56	84	120	165
$(t/\tau')^4$	1	5	15	35	70	126	210	330	495
$(t/\tau')^5$	1	6	21	56	126	252	462	792	1287
$(t/\tau')^6$	1	7	28	84	210	462	924	1716	3003
$(t/\tau')^7$	1	8	36	120	330	792	1716	3432	6435

The last term in each series is to the right of a diagonal for a_{n-1} and to the left for b_{n-1}; for example, the diagonal for N = 9 is marked on the table.

Consider now the behaviour of the expressions for $[T]_N$ and $[\theta]_N$ as $\Delta x \to 0$, that is, $N \to \infty$:

$$p' = \frac{h\ s\ (\Delta x)}{A_f \rho_f\ u\ C_p} .$$

If the total specimen length is l,

$$N = \frac{1}{\Delta x} ,$$

and hence $$N\ p' = \frac{h\ s\ l}{A_f \rho_f\ u\ C_p} = p .$$

Also, as $\Delta x \to 0$, $p' \to 0$, and for small values of p'

$$p' \sim 1 - e^{-p'} .$$

Substitution for $1 - e^{-p'}$ gives

$$\frac{t}{\tau'} = \frac{A_f\,\rho_f\,u\,C_p\,p'\,t}{A_s(\Delta x)\,\rho_s\,C_s} = \frac{h's\,t}{A_s\,\rho_s\,C_s} = M\ .$$

Furthermore $\left(\sum_{n=1}^{n} 1\right) = n$,

$$\mathop{\mathrm{Lt}}_{n\to\infty}\left[\sum_{n=1}^{n}\left(\sum_{n=1}^{n} 1\right)\right] = \frac{n^2}{2!}\ ,$$

$$\mathop{\mathrm{Lt}}_{n\to\infty}\left\{\sum_{n=1}^{n}\left[\sum_{n=1}^{n}\left(\sum_{n=1}^{n} 1\right)\right]\right\} = \frac{n^3}{3!}\ ,$$

and so on.

Hence by expanding the coefficients of terms in t/τ' from the expression for $[T]_N$ as:

$$(1 - e^{-p'})\sum_{n=1}^{n} e^{-(n-1)p'} = 1 - e^{-Np'}\ ,$$

$$(1 - e^{-p'})^2\sum_{n=1}^{N-1}\left(\sum_{n=1}^{n} 1\right)e^{-(n-1)p'} = 1 - e^{-(N-1)p'}\left\{1 + (1 - e^{-p'})\left(\sum_{n=1}^{N-1} 1\right)\right\}\ ,$$

$$\frac{(1 - e^{-p'})^3}{2!}\sum_{n=1}^{N-2}\left[\sum_{n=1}^{n}\left(\sum_{n=1}^{n} 1\right)\right]e^{-(n-1)p'} = \frac{1}{2!}\left[1 - e^{-(N-2)p'}\left\{1 + (1 - e^{-p'})(N-2) - \right.\right.$$
$$\left.\left. - (1 - e^{-p'})\sum_{n=1}^{N-2}\left(\sum_{n=1}^{N-2} 1\right)\right\}\right]\ ,$$

and so on, by letting $N \to \infty$, and by making the substitutions suggested above, it may be shown that

$$[T] = 1 - e^{-M}\left\{[1 - e^{-p}] + [1 - e^{-p}(1 + p)]\,M + \frac{1}{2!}\left[1 - e^{-p}\left(1 + p + \frac{p^2}{2!}\right)\right]M^2 + \ldots\right\}$$

$$= 1 - e^{-M}\sum_{n=o}^{\infty}\frac{M^n}{n!} + e^{-M-p}\sum_{n=o}^{\infty}\frac{M^n}{n!}\left\{\sum_{i=o}^{n}\frac{p^i}{i!}\right\}\ .$$

Hence $$[T] = e^{-M-p}\sum_{n=o}^{\infty}\frac{M^n}{n!}\left\{\sum_{i=o}^{n}\frac{p^i}{i!}\right\}\ . \qquad (5)$$

Similarly,

$$[\theta] = e^{-M-p}\sum_{n=1}^{\infty}\frac{M^n}{n!}\left\{\sum_{i=o}^{n}\frac{p^i}{p!}\right\}\ .$$

Since l may be any specimen length, these equations specify the time–temperature distribution in any plane distant x from the upstream face if:

$$p = \frac{h\,s\,x}{A_f\,\rho_f\,u\,C_p}\ .$$

3.2.2 Sinusoidal upstream fluid temperature

It was shown previously that when a sinusoidal fluid temperature wave passes through a specimen in which the solid longitudinal conductivity is infinite, the wave-form is attenuated and shifted in phase but otherwise unaltered. It therefore follows that if the specimen consists of N identical layers normal to the direction of mean flow, each of thickness Δx, then the attenuation downstream from the N^{th} layer is:

$$r_T = \left\{ \frac{e^{-2p'} + (m')^2 (1 - e^{-p'})^2}{1 + (m')^2 (1 - e^{-p'})^2} \right\}^{N/2} , \qquad (6)$$

and the phase shift is:

$$\gamma_T = N \sin^{-1} \left\{ \frac{(m')^2 (1 - e^{-p'})^4}{[(m')^2 (1 - e^{-p'})^2 + e^{-p'}]^2 + (m')^2 (1 - e^{-p'})^4} \right\}^{\frac{1}{2}} + \frac{\omega_o N (\Delta x)}{u} ,$$

where

$$p' = \frac{h\, s\, (\Delta x)}{A_f\, \rho_f\, u\, C_p}$$

$$m' = \frac{A_f\, \rho_f\, u\, C_p}{A_s\, (\Delta x)\, \rho_s\, C_s\, \omega_o} .$$

Rewriting the expression for r_T as

$$r_T = \left\{ 1 + \frac{e^{-2p'} - 1}{1 + (m')^2 (1 - e^{-p'})^2} \right\}^{N/2} ,$$

and expanding in a binomial series,

$$r_T = 1 + \frac{(N/2)(e^{-2p'} - 1)}{1 + (m')^2 (1 - e^{-p'})^2} + \frac{(N/2)(N/2 - 1)(e^{-2p'} - 1)^2}{2!\,[1 + (m')^2 (1 - e^{-p'})^2]^2} + \ldots .$$

Now let $\Delta x \to 0$.

Since $p' \to 0$, $p' \sim (1 - e^{-p'})$

and $-2p' \sim (e^{-2p'} - 1)$.

Also since $N = 1/(\Delta x)$

$$N p' = p = \frac{h\, s\, l}{A_f\, \rho_f\, u\, C_p}$$

and $$m' p' = m p = \frac{h\, s}{A_s\, \rho_s\, C_s\, \omega_o} .$$

Hence

$$\mathop{Lt}_{N \to \infty} r_T = 1 - \alpha + \frac{\alpha^2}{2!} - \frac{\alpha^3}{3!} + \ldots\ldots\ldots$$

$$= e^{-\alpha}$$

where $$\alpha = \frac{p}{1 + m^2 p^2} ,$$

$$\therefore \quad r_T = \exp - \left\{ \frac{p}{1 + m^2 p^2} \right\} . \qquad (7)$$

By making the approximation:

$$\sin \gamma = \gamma, \quad \text{as } \gamma \to 0$$

it may also be shown that the phase shift downstream from a bed in which the longitudinal conductivity of the solid is zero is:

$$\gamma_T = \frac{m\, p^2}{1 + m^2 p^2} + \frac{\omega_o l}{u} \quad .$$

Similarly, the ratio of the solid temperature amplitude in the plane distance x from the upstream face to the fluid temperature amplitude upstream from the specimen is:

$$r_\theta = \left\{ \exp\left(- \frac{p\,(x/l)}{1 + m^2 p^2} \right) \right\} \left\{ \frac{m^2 p^2}{1 + m^2 p^2} \right\}^{1/2} \quad ;$$

and the corresponding phase shift is:

$$\gamma_\theta = \left\{ \frac{m\, p^2 (x/l)}{1 + m^2 p^2} \right\} + \left\{ \frac{1}{1 + m^2 p^2} \right\} + \frac{\omega_o x}{u} \quad .$$

4. DISCUSSION OF RESULTS

An analytical solution of a step change in upstream fluid temperature, when the longitudinal conductivity in the solid is zero, was first obtained by Schumann (1929). He used as a time variable:

$$z = \frac{h\, s}{A_s \rho_s C_s} \left(t - \frac{x}{u} \right) \quad .$$

Since for gas flows of interest the (x/u) term is usually small, it may be neglected and z may be replaced by:

$$M = \frac{h\, s\, t}{A_s \rho_s C_s}$$

with only a very small error in the time zero point. Then Schumann's result, quoted in Jakob (1957) may be written as:

$$[T] = \exp(-p-M) \sum_{n=o}^{\infty} M^n \phi_n(p\,M) \quad , \qquad (8)$$

where $\phi_o(p M) = J_o(2i\sqrt{p\,M}) = 1 + pM + \dfrac{(p\,M)^2}{(2!)^2} + \ldots \quad .$

It is easily shown by expansion and rearrangement of terms that in Equation (8):

$$\sum_{n=o}^{\infty} M^n \phi_n(p M) = \sum_{n=o}^{\infty} \frac{M^n}{n!} \left\{ \sum_{i=o}^{n} \frac{p^i}{i!} \right\}$$

Therefore for gas flow problems Equations 8 and 5 are in fact identical solutions.

Equation 3 was first obtained by Bell and Katz (1949) as an approximation valid if $(\omega_o l/u)$ is small. They solved the governing differential equations without neglecting the $\frac{\partial T}{\partial t}$ term. As a result the simplified analysis is valid, if both $(\omega_o l/u)$ and $(\rho_f C_p/\rho_s C_s)$ are small, which is the case in most gas-solid systems.

Equation 7 was also first obtained by Bell and Katz (1949). If the specimen is a matrix of spheres it is easily shown that Equation 7 may be transformed to:

$$r_T = \exp\left(-p \frac{X^4}{X^4 + 36\,Bi^2}\right) , \qquad (9)$$

which was obtained independently by Bain (Meek 1962).

These comparisons show that the simplified approach presented leads to exact analytical solutions of the transient heat transfer problems for the case of most practical interest, that is, for matrixes where the model of solid with conductivity infinite in radial and zero in longitudinal direction may be used. Further application to more complex cases is possible and the solution for the periodic pulse and infinite longitudinal conductivity in solid is obtained, as in Equation 2.

The intermediate step in the simplified theory described, that is, the solutions for a layer model, are of practical interest for specimens consisting of parallel woven wire screens normal to the flow, or regular arrays of high conductivity spheres.

The simplified analysis has not been extended to the case of finite thermal conductivity in the radial direction. This may be of practical interest for the systems with high flow Reynolds numbers and relatively low k_s, for example ceramic pebbles. In this case the analytical solutions of earlier workers (Bell and Katz 1949), (Dayton et al. 1952), (Meek 1962) must be used. These were obtained only for the sinusoidal fluid temperature wave.

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APPENDIX

NOTATION

(Symbols used only once are defined as they occur in the text)

A_f	area of cross section open to flow
A_s	area of cross section occupied by solid
a	radius of sphere
Bi	Biot number ($h\ a/k_s$)
C_p	Fluid specific heat at constant pressure
C_s	Solid specific heat
D_L	Longitudinal dispersion coefficient in fluid
d	Sphere diameter
h	Coefficient of surface heat transfer
i	An integer
k	$\pi/\tau\ \omega_o$
k_f	Fluid thermal conductivity
k_f'	Effective fluid thermal conductivity ($k_f + D_L\ \rho_f C_p$)
k_s	Solid thermal conductivity
l	Length of specimen in direction of flow
M	$h\ s\ t/A_s\ \rho_s C_s$
m	$A_f \rho_f\ u\ C_p/A_s\ l\ \rho_s\ C_s\ \omega_o$
m'	$A_f\ \rho_f\ u\ C_p/A_s(\Delta x)\ \rho_s C_s\ \omega_o$
N	Number of layers in specimen $l/(\Delta x)$
n	An integer
p	$h\ s\ l/A_f \rho_f\ u\ C_p$
p'	$h\ s\ (\Delta x)/A_f\ \rho_s\ u\ C_p$
Q	Total heat transferred per unit time
q	$h/[2\ k_s\ \rho_s\ C_s\ \omega_o]^{\frac{1}{2}}$
Re	Reynolds number ($d\ \rho_f\ u_o/\mu$)
r	Radial coordinate normal to solid surface
r_T	Fluid temperature attenuation ratio

(continued)

r_θ	Solid temperature attenuation ratio
S	Total surface area
s	Surface area per unit length of specimen
T	Fluid temperature
T_f	Final temperature after step change
T_o	Initial or base temperature
[T]	Accomplished temperature fraction $(T-T_o)/\Delta$
t	Time
u	Fluid velocity through specimen in direction of mean flow
u_o	Superficial velocity based on total area of cross section
X	$[2 a^2 \rho_s \omega_o C_s/k_s]^{\frac{1}{2}}$
x	Longitudinal coordinate in direction of mean flow
Δx	Length of section in which solid longitudinal conductivity is infinite
γ	Phase shift
μ	Fluid viscosity
ρ_f	Fluid density
ρ_s	Solid density
θ	Solid temperature
θ_s	Solid surface temperature
$[\theta]$	Accomplished temperature fraction $(\theta - T_o)/\Delta$
Δ	Amplitude of upstream fluid temperature change
τ	Thermal time constant $A_s\, l\, \rho_s C_s/A_f \rho_f u C_p(1 - e^{-P})$
τ'	$A_s \Delta x \rho_s C_s/A_f \rho_f u C_p(1 - e^{-P'})$
ω_o	Angular frequency