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THE CRITICAL SIZE OF A BARE SPHERICAL REACTOR WITH ANISOTROPIC DIFFUSION

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Abstract

Using oblate spheroidal co-ordinates, the critical equation for a bare sphere with unequal axial and radial diffusion coefficients is derived. A two group model is used, with the degree of anisotropy the same in both groups.

CONTENTS

1. INTRODUCTION

The presence of axial coolant channels in a reactor means that in the diffusion approximation, the axial and radial diffusion coefficients are different. For cylindrical reactors which permit simple analytical solutions with isotropic diffusion, there is no great difficulty in including the effects of anisotropy.

In studying the possible implications of anisotropy at least on a two group basis, it was found that a comparatively simple analytical solution could be obtained for a bare sphere if the ratio of axial to radial diffusion coefficient is the same in both groups. This analysis is of some interest as use is made of the comparatively unfamiliar spheroidal wave functions.

As far as possible the notation is the same as that of Flammer[1].

2. THEORY

Let $(D_z)_{1,2}$ and $(D_r)_{1,2}$ be the axial and radial two group diffusion coefficients. It is assumed that $D_z > D_r$ and that

$$\left(\frac{D_z}{D_r}\right)_1 = \left(\frac{D_z}{D_r}\right)_2 = 1 + \Delta \qquad (1)$$

Factorization of the two group equations is then possible leading to the equation relevant to a bare core

$$\frac{\partial^2 \phi}{\partial z^2} + \frac{1}{(1+\Delta)} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + K^2 \phi = 0 \qquad (2)$$

with $K^2 = K_o^2/(1 + \Delta)$ and K_o is calculated in the usual way, using $(D_r)_1$ and $(D_r)_2$. From (2) it follows that in the co-ordinates z, r^* $(= r(1 + \Delta)^{\frac{1}{2}})$.

$$(\nabla^2 + K^2)\phi = 0 \qquad (3)$$

This equation would be obtained from the multigroup equations if Δ were independent of energy, so the analysis is not restricted to any definite number of groups. A sphere of radius R becomes, in the (z, r^*, Φ) co-ordinates, an oblate spheroid, obtained by rotating an ellipse of minor

../2

(1) Flammer, C., 1957 - SPHEROIDAL WAVE FUNCTIONS, Stanford University Press.

axis 2R and major axis 2R(1 + Δ) about its minor axis. Since the boundary condition at the surface is to be taken as $\phi = 0$ there are advantages in changing to a system of oblate spheroidal co-ordinates (ξ , η , Φ) defined by

$$z = a\xi\eta \qquad r^* = a(1-\eta^2)^{\frac{1}{2}}(1+\xi^2)^{1/2} \tag{4}$$

with $0 \le \xi < \infty$, $-1 \le \eta \le +1$ and $0 \le \Phi \le 2\pi$.

No further reference need be made to Φ since the system is symmetric about the z axis.

ξ = 0 corresponds to z = 0, $-a \le r^* \le a$

η = 0 " " z = 0, $r^* > a$

$\eta = \pm 1$ " " the $\pm$ z axis.

Surfaces of constant ξ are oblate spheroids.

The values of a and ξ_B corresponding to the surface of a sphere of radius R come from (4), i.e.,

$$R = a\xi_B$$

and $$R(1+\Delta)^{\frac{1}{2}} = a(1+\xi_B^2)^{\frac{1}{2}}$$

so $$a = R\Delta^{1/2}, \quad \xi_B = \frac{1}{\Delta^{1/2}} \tag{5}$$

In the (ξ , η) co-ordinate system (3) becomes

$$\frac{\partial}{\partial\xi}\left[(\xi^2+1)\frac{\partial\phi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\phi}{\partial\eta}\right] + K^2a^2(\xi^2+\eta^2)\phi = 0 \tag{6}$$

With $C = Ka = K_0R(\Delta/(1+\Delta))^{1/2}$ a solution of (6) is:

$$\phi = S_{on}(-ic,\eta)\,R_{on}(-ic, i\xi) \tag{7}$$

where $$\left(\frac{d}{d\xi}\left[(1+\xi^2)\frac{d}{d\xi}\right] + c^2\xi^2 - \lambda_{on}\right)R_{on} = 0 \tag{8}$$

$$\left(\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}\right] + c^2\eta^2 + \lambda_{on}\right)S_{on} = 0 \tag{9}$$

../3

The separation constants λ_{on} are the eigenvalues which give convergent series solutions, and the suffix o refers to an axially symmetric system. Thus λ_{on} is a function of c^2. These spheroidal functions are obtained as expansions.

$$S_{on}(-ic,\eta) = \sum_r d_r^{on}(-ic)\, P_r(\eta) \tag{10}$$

$$R_{on}(-ic, i\xi) = \sum_r d_r^{on}(-ic)\, i^r j_r(c\xi) \Big/ \sum_r d_r^{on}(-ic) \tag{11}$$

The series are taken over r even or odd as n is even or odd. $P_r(\eta)$ is a Legendre polynomial, and $j_r(c\xi)$ is a spherical Bessel function,

$$j_r(c\xi) = \sqrt{\frac{\pi}{2c\xi}}\, J_{r+\frac{1}{2}}(c\xi) \tag{12}$$

where the Bessel functions are given by

$$J_{r+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\sin\left(x-\frac{r\pi}{2}\right)\sum_{s=0}^{\leq r/2}\frac{(-1)^s (r+2s)!}{(2s)!\,(r-2s)!\,(2x)^{2s}} + \cos\left(x-\frac{r\pi}{2}\right)\sum_{s=0}^{\leq \frac{1}{2}(r-1)}\frac{(-1)^s (r+2s+1)!}{(2s+1)!\,(r-2s+1)!\,(2x)^{2s+1}}\right. \tag{13}$$

In this problem only the lowest eigenvalue λ_{oo} is of interest and from symmetry r must be even. Thus, introducing an arbitrary constant A, the required solution is

$$\phi(\xi,\eta) = A\sum_{r=0,2,\ldots}^{\infty} d_r^{oo}(-ic)\,P_r(\eta)\sum_{r=0,2,\ldots}^{\infty}(-1)^{\frac{r}{2}} d_r^{oo}(-ic)\, j_r(c\xi) \tag{14}$$

If this expression is substituted into the basic equation (6), a three term recurrence formula is obtained for the coefficients d_r^{oo} and the condition that $d_{r+2}^{oo}/d_r^{oo} \to 0$ as $r \to \infty$ suffices to determine the eigenvalue.

Trying for a solution to (8) in the above form of a series of spherical Bessel functions, and using the fact that

$$\frac{d^2}{d\xi^2} j_r(c\xi) = c^2\left\{Q_r^- j_{r-2}(c\xi) - Q_r^0 j_r(c\xi) + Q_r^+ j_{r+2}(c\xi)\right\}$$

$$Q_r^- = \frac{r(r-1)}{(2r-1)(2r-3)}$$

$$Q_r^0 = \frac{r^2}{4r^2-1} + \frac{(r+1)^2}{(2r+1)(2r+3)}$$

$$Q_r^+ = \frac{(r+1)(r+2)}{(2r+3)(2r+5)}$$

gives

$$\frac{d_r^{oo}}{d_{r-2}^{oo}} = \frac{-c^2 Q_r^-}{\lambda_{oo} - r(r+1) + c^2 Q_r^0 + c^2 Q_r^+ \left(\frac{d_{r+2}^{oo}}{d_r^{oo}}\right)} \qquad (15)$$

$$\lambda_{oo} = -\frac{c^2}{3}\left(1 + \frac{2}{5}\frac{d_2^{oo}}{d_o^{oo}}\right) \qquad (16)$$

Equations (15) and (16) and the requirement that $\frac{d_{r+2}^{oo}}{d_r^{oo}} = \frac{c^2}{4r^2}$ for large r, determines λ_{oo} the lowest eigenvalue. When $\Delta \ll 1$, c^2 will be small and adequate solutions of (15) and (16) can be obtained by taking $\frac{d_4^{oo}}{d_2^{oo}} = \frac{c^2}{16}$ so,

$$\frac{d_2^{oo}}{d_o^{oo}} = \frac{-\left(\lambda_{oo} + \frac{c^2}{3}\right)}{2c^2/15} = \frac{-2c^2/3}{\lambda_{oo} - 6 + \frac{11c^2}{21} + \frac{c^4}{84}}$$

This quadratic gives

$$\lambda_{oo} \approx -\frac{c^2}{3}\left(1 + \frac{2c^2}{45}\right) \qquad (17)$$

$$d_2^{oo}/d_o^{oo} \approx \frac{c^2}{9} \qquad (18)$$

Tables for the more accurate calculations of eigenvalues and coefficients are given by Flammer

The condition that $\phi = 0$ at $R = R_c$ gives the equation for the critical radius R_c in terms of $C = K_o R_c\ (\Delta/(1+\Delta))^{1/2}$ from (12) and (14).

$$\sum_{r=0,2,\ldots}^{\infty} (-1)^{\frac{r}{2}} \frac{d_r^{00}(-ic)}{d_0^{00}(-ic)} \sqrt{\frac{\pi\Delta^{1/2}}{2c}}\, J_{r+\frac{1}{2}}\left(\frac{c}{\Delta^{1/2}}\right) = 0 \qquad (19)$$

As ϕ cannot be negative, the lowest eigenvalue C_o for given Δ and K_o determines R_c.

When Δ is small, then from (13), (18) and (19) the condition for criticality is

$$\sin\left(\frac{c}{\Delta^{1/2}}\right) - \frac{c^2}{9}\left\{\left(\frac{3\Delta}{c^2}-1\right)\sin\left(\frac{c}{\Delta^{1/2}}\right) - \frac{3\Delta^{1/2}}{c}\cos\left(\frac{c}{\Delta^{1/2}}\right)\right\} = 0 \qquad (20)$$

The lowest root will occur at $c/\Delta^{1/2} = \pi + \gamma$, and from (20)

$$\gamma \approx \frac{-\pi\Delta}{3}\left(1 - \frac{\pi^2\Delta}{9}\right)$$

Thus
$$K_o R_c \approx \pi\left(1 + \frac{\Delta}{6}\right) \qquad (21)$$

The ratio of the critical masses with and without anisotropic diffusion is then

$$M(\Delta)/M(0) = 1 + \frac{\Delta}{2} \qquad (22)$$

This approximation is valid to about $\Delta = 10\%$. It is easily shown that for a bare cylinder with aspect ratio (height/diameter ratio) of 1.0

$$M(\Delta)/M(0) = 1 + 0.64\,\Delta$$

while for a bare cylinder of minimum mass, aspect ratio = 0.9238,

$$M(\Delta)/M(0) = 1 + 0.75\,\Delta$$

The addition of top and side reflectors should not seriously alter the magnitude of these estimates.

It might be concluded therefore that the error in the critical mass of a cylinder with aspect ratio in the vicinity of 1, due to ignoring anisotropy in the calculations, would be of the order of $\Lambda/2$.

3. CONCLUSION

Oblate spheroidal co-ordinates appear to have some application in the study of neutron diffusion in anisotropic media. Further study of reflected anisotropic spherical systems for the particular case of the same degree of anisotropy in all regions and all groups does not, however, appear to be worthwhile in view of the work involved and the somewhat artificial nature of the problem.

4. NOTATION

a	Semi-interfocal distance of ellipses
C	Parameter of spheroidal equations
D_z, D_r	Axial and radial diffusion coefficients
r, z, Φ	Cylindrical co-ordinates
ξ, η, Φ	Oblate spheroidal co-ordinates
K^2	Reactor buckling based on Dz_1 and Dz_2
K_0^2	Reactor buckling based on Dr_1 and Dr_2
R	Radius of sphere
R_c	Radius of critical sphere
R_{on}	n'th radial eigenfunction for axial symmetry
S_{on}	n'th angle eigenfunction for axial symmetry
λ_{on}	n'th eigenvalue for axial symmetry
d_r^{on}	Coefficient of the r'th term in the expansion of S_{on} and proportional to the r'th coefficient in the expansion of R_{on}.