AAEC/E223

AAEC/E223

AUSTRALIAN ATOMIC ENERGY COMMISSION
RESEARCH ESTABLISHMENT
LUCAS HEIGHTS

SOME GEOMETRICAL PROPERTIES OF PACKINGS OF EQUAL SPHERES IN CYLINDRICAL VESSELS
PART IV - EXTENSION OF MODEL TO OUTER REGION OF SEMI-INFINITE VESSEL WITH PLANE WALL

by

G.A. TINGATE

October 1971

ISBN 0 642 99443 9

AUSTRALIAN ATOMIC ENERGY COMMISSION

RESEARCH ESTABLISHMENT

LUCAS HEIGHTS

SOME GEOMETRICAL PROPERTIES OF PACKINGS OF EQUAL SPHERES IN CYLINDRICAL VESSELS

PART IV – EXTENSION OF MODEL TO OUTER REGION OF SEMI-INFINITE VESSEL WITH PLANE WALL

by

G. A. TINGATE

ABSTRACT

A range of random packings has been prepared in a prismatic vessel with a plane vertical transparent wall, simulating a semi-infinite vessel. Observations have been made in the outer region of the packings, using and extending the experimental method described in Part I of the series.

The three-dimensional model presented in Part III has been extended to the outer region of unbiased packings in such a vessel. Equations from Part III are used, together with some of the properties of the regular arrays not previously considered.

The rhombohedral and cubic arrays are shown to be members of a family of regular arrays, some of whose properties agree closely with experimentally determined properties of unbiased random packings over the range normally obtained in practice.

The model is also shown to be in fair agreement with observed properties in the outer region of the loosest random packings prepared in the laboratory, enabling estimates to be made of some of their properties in the central region which have not so far been determined experimentally.

The two-dimensional model presented in Part III is also extended to the outer region, and the computed results are supported by a limited experimental study.

National Library of Australia card number and ISBN 0 642 99443 9

PREFACE

This report is a part of a series on "Some Geometrical Properties of Packings of Equal Spheres in Cylindrical Vessels" as follows:

Part I	Exploratory Study of Random Packings in Small Vessels. G. A. Tingate, AAEC/E208.
Part II	The Cylindrically Ordered Packing. F. A. Rocke, A.A.E.C. report in preparation.
Part III	Basic Model away from the Influence of Wall Effects. N. W. Ridgway, G.A. Tingate, AAEC/E202.
Part IV	Extension of Model to Outer Region of Semi-infinite Vessel with Plane Wall. G. A. Tingate, this report, AAEC/E223.
Part V	Adaptation of Model to Packings in Cylindrical Vessels. G. A. Tingate, A.A.E.C. report in preparation.
Part VI	Discussion and Conclusions. N. W. Ridgway, F. A. Rocke, and G. A. Tingate, A.A.E.C. report in preparation.

CONTENTS

Page

1. INTRODUCTION 1

2. FIRST APPROACH – MODEL BASED ON EQUATIONS FROM PART III 3

2.1 Basic Concept 3
2.2 Application of Basic Concept 3
2.3 Experimental Testing of Model for Semi-Infinite Vessel 8

3. SECOND APPROACH – MODEL BASED ON PROPERTIES OF REGULAR ARRAYS 10

4. THIRD APPROACH – COMPOSITE MODEL 12

5. INVESTIGATION OF DISCREPANCIES BETWEEN OBSERVATION AND COMPUTATION 12

5.1 $OF_C - OF_W$ Relationship 12
5.2 $V_I/V_W - OF_W$ Relationship 12
5.3 $\epsilon_C - n_C$ Relationship 13
5.4 $n_W - OF_W$ Relationship 13

6. TWO-DIMENSIONAL MODEL 13

7. DISCUSSION 14

7.1 Relationships Between the Various Occupancy Factors and Derived Properties 14

7.2 Relationships Between Mean Numbers of Points of Contact and Other Properties 15

7.3 Two-Dimensional Case 16

7.4 Future Experimental Work 17

8. SUMMARY AND CONCLUSIONS 18

9. ACKNOWLEDGEMENTS 19

10. NOTATION 19

11. REFERENCES 20

APPENDIX 1 Some properties of the Equilateral Triangular Arrays

APPENDIX 2 Some Properties of the Two-Dimensional Equispaced Arrays

APPENDIX 3 Values Computed by Composite Model

Figure 1 Packings of equal spheres in semi-infinite and cylindrical vessels (Showing two regions and three sphere categories)

Figure 2 Properties of three-dimensional packings away from wall effects

Figure 3 Properties of two-dimensional packings away from edge effects

(continued)

CONTENTS (continued)

Figure 4 Observed and computed occupancy factors of central region (first approach)

Figure 5 Observed and computed volume ratios between intruding spheres and spheres touching wall (first approach)

Figure 6 Observed and computed numbers of points of contact between spheres touching wall

Figure 7 Typical outer layer of heavily vibrated packing in prismatic vessel (looking through transparent vertical plane wall)

Figure 8 Typical horizontal layer of heavily vibrated packing in prismatic vessel (looking vertically down inner face of transparent plane wall)

Figure 9 Tilted cubic array on triangular base

Figure 10 Two of the blocked passage equilateral triangular arrays on hexagonal bases

Figure 11 Clear passage equivalent of tilted cubic array on triangular base

Figure 12 Two of the clear passage equilateral triangular arrays on hexagonal bases

Figure 13 Comparison of computed occupancy factors of central region

Figure 14 Observed and computed volume ratios between intruding spheres and spheres touching wall (before adjustment)

Figure 15 Observed and computed volume ratios between intruding spheres and spheres touching wall (after adjustment)

Figure 16 Structure resulting from flow in two-dimensional packing

1. INTRODUCTION

In 1966 the Australian Atomic Energy Commission completed a feasibility study of a high temperature gas-cooled reactor system based on a pebble bed concept, using spheres of fuelled beryllium oxide. The work reported here is part of a study of the structure of random packings of equal spheres made in support of this project.

Part I of this series gives an account of an exploratory study of such packings, in terms of two regions and three sphere categories (see Figure 1). Most of the supporting experiments were with a cylinder-to-sphere diameter ratio (D_v/D_p) of 5.77, but equations were derived for estimating some of the properties of packings with larger ratios. The study showed that a continuous range of unbiased random packings is possible. With large D_v/D_p ratios, the mean void fraction of the central region can range from 0.36 to 0.42, corresponding to a range of occupancy factor* of the spheres in the central region OF_C of 0.86 to 0.78. Variations in the outer region are much greater. The range for the spheres touching the wall OF_W in unbiased packings is 0.88 to 0.72, and for the intruding spheres OF_I it is 0.79 to 1.50.

In Part III a model was developed for the central region of packings in a semi-infinite vessel, on the basis that they are generated by an expansion process starting with the rhombohedral array. Equations were derived for the properties of interest, including one for the mean void fraction ϵ_C in terms of the mean number of points of contact n_C, namely:

$$1 - \epsilon_C = \frac{\pi}{\sqrt{3\left(12 - \frac{n_C}{2}\right)}} \qquad (1)$$

This equation and various points of interest are plotted in Figure 2.

A model was also developed in Part III for two-dimensional packings (see Figure 3). The observed properties of the spheres touching the wall in the three-dimensional packings have been checked against this model, but it is quite inadequate to account for the observed range. The two-dimensional model gives a minimum value of OF_W of 0.866, that of the square array, whereas values down to 0.7 can be readily obtained with three-dimensional random packings. This would imply that the properties of the outer region are functions of the central region, and that a common expansion mechanism holds throughout the packing.

In the following, therefore, the model for the central region is extended to the outer region of a semi-infinite vessel. Several guidelines for such an extension can be obtained directly from the model itself, and it is appropriate to summarise them here.

The model is completely defined at the dense end of the range in terms of the properties of the rhombohedral array. At the loose end it is partly defined, in that it gives the mean void fraction of the cubic packing, 0.4764, when $n_C = 0$. However this does not correspond to a real packing, nor is it likely that this void fraction can be approached with random packings, partly because a high degree of ordering would have to develop and partly because the cubic array is structurally unstable. Although further experimental studies might provide a good estimate of the properties at the loose limit of the range, the desired definition would still be lacking unless the theoretical properties of the loosest possible random packing were known.

---

* occupancy factor = $\dfrac{\text{observed volume of spheres}}{\text{volume calculated for densest possible packing}}$

No mention of such a packing or its measured or calculated properties could be found in the available literature, and it is beyond the scope of this study to attempt to establish them by direct means. However, two alternative criteria can be postulated for the route leading to this limiting condition. The first is that the properties of the central region of all unbiased random packings conform with Equation 1. The second, based on stability considerations, is that the number of points of contact should remain at or close to 6 when ϵ_C is less than 0.3954.

Neither criterion offers any guidance for the determination of a practical end to the expansion process. However, if the first alternative were to hold, the mean number of points of contact would be significantly less than 6, and perhaps an integer. Integral values of n_C less than 4 would not be expected, so it would appear from Table 1 that the most likely limit of ϵ_C would be 0.4264 on this basis.

TABLE 1

MEAN VOID FRACTIONS COMPUTED FROM EQUATION 1

n_C	ϵ_C
6	0.3954
5	0.4115
4	0.4264
3	0.4402
2	0.4531

Further guidelines are provided by the experimental results from Part I, for both the central and outer regions, even though they are for small cylindrical packings. The correction for the curvature of a cylindrical wall is only about 1 per cent for a D_t/D_p ratio of 5.77, and so these results provide a fair check for any extension of the model to the outer region of a semi-infinite vessel. From the results given in Part I, the limiting value of OF_W should be substantially less than 0.7, while from the two-dimensional model the mean number of points of contact between spheres touching the wall should be less than 3 and possibly an integer.

In the following, two distinct approaches have been adopted. Each leads to a model which accounts for some of the properties or trends observed for the two regions and the three sphere categories over the range of unbiased random packings prepared in the laboratory. The first approach is to adapt some of the three-dimensional equations from Part III to the essentially two-dimensional outer region. The resulting model is in fair agreement with observed values. It also gives a practical end to the expansion process in the outer region, and enables some of the properties of the loosest possible random packing to be estimated. The second approach makes use of relationships between properties of the rhombohedral and cubic arrays not considered in Part III. The relationships are also found to hold over a continuous range of intermediate regular arrays. Some of the properties computed for these arrays are found to agree closely with those observed in unbiased random packings. A composite model is then developed from the two, with a final adjustment to conform with observed properties in the outer region of the loosest packings prepared in the laboratory. This enables some of the properties of the central region of the loosest possible random packing to be estimated.

The further extension of the composite model to packings in cylindrical vessels is reported in Part V.

2. FIRST APPROACH – MODEL BASED ON EQUATIONS FROM PART III

2.1 Basic Concept

A dense regular array is assumed to expand by the progressive breaking of points of contact and the further separation of the spheres where they have broken contact. It is assumed initially that there is no distinction between sphere-to-sphere and sphere-to-wall points of contact, to the extent that breaking and separation occur at the same rates.

A common expansion mechanism would give rise to quite different effects in the two regions of the packing. The breaking of a point of contact in the central region would have no effect on OF_C, and the affected spheres would remain in the same category, while any increase in the mean distance between the centres of adjacent spheres would have a distance-cubed effect on OF_C. However, the breaking of a point of contact at the wall would result in the sphere becoming an intruding sphere, affecting both OF_W and OF_I*. The breaking of a point of contact between adjacent spheres touching the wall would have no effect on OF_W or that sphere category, while any increase in the mean distance between their centres would have a distance-squared effect on OF_W.

No criterion could be postulated on this basis for the termination of the expansion process for the spheres in the central region or for the spheres touching the wall. In the case of the intruding spheres, however, two opposing effects govern changes in OF_I. At the start of the process the effect of expansion is rudimentary, so that the spheres breaking contact with the wall give rise to a net increase in OF_I. As more and more points of contact break, the expansion process compounds, so a limiting value of OF_I must eventually be reached where the two effects counterbalance. If the mechanism were to hold throughout the packing, and all three sphere categories were to reach the loosest possible condition together, then a model developed on this basis would enable the properties of the loosest possible random packing to be calculated. Even if the rates of breaking and separation in the two regions were not the same, it would be possible to adjust for this after making the appropriate experimental observations over the practical range of random packings.

2.2 Application of Basic Concept

2.2.1 Initial conditions

The first terms of the model are given in Table 2. They are derived directly from the properties of the rhombohedral array by considering a portion with 10,000 spheres of unit diameter touching the (plane) wall in a close packed triangular array.

2.2.2 Expansion process

Expansion is treated as taking place in steps, such that with each step one sphere breaks contact with the wall, and the other points of contact break at the same rate throughout the packing. In this way related properties of interest can be computed at intervals over the range of packings so represented. In the following, the i-th terms (resulting from i – 1 such steps) are derived on this basis.

The number of spheres touching the wall is given by

$$n_i = 10{,}000 - (i - 1) = 10{,}001 - i \quad . \tag{2}$$

In the central region we have

$$n_{C_i} = 12 \left(\frac{n_i}{10{,}000}\right) = 0.0012\, n_i \quad . \tag{3}$$

* The physical existence of this part of the expansion process can be demonstrated by vibrating a loose random packing in a transparent vessel. Intruding spheres can be seen to move outwards and many of them eventually touch the wall. The mathematical treatment presented here is the reverse of this process, so chosen because the dense end of the range offers a completely defined starting point.

TABLE 2

FIRST TERMS OF MODEL

PROPERTY	SYMBOL	VALUE OR EXPRESSION
Mean number of points of contact between spheres in central region	n_{C_1}	12
Mean void fraction of central region	ϵ_{C_1}	$1 - \frac{\pi}{3\sqrt{2}} = 0.259520$
Mean sphere volume in unit volume of central region (or occupancy fraction)	V_{CU_1}	$\frac{\pi}{3\sqrt{2}} = 0.740480$
Occupancy factor of central region	OF_{C_1}	1.0
Mean radial distance between centres of adjacent spheres in central region	ρ_{C_1}	1.0
Volume of unit cell in central region (i.e. volume associated with each sphere)	V_{UCC_1}	$\frac{1}{\sqrt{2}} = 0.707107$
Mean number of points of contact between spheres touching wall	n_{W_1}	6
Mean volume of spheres touching wall in unit volume of outer region	V_{WU_1}	$\frac{\pi}{3\sqrt{3}} = 0.604600$
Occupancy factor of spheres touching wall	OF_{W_1}	1.0
Mean radial distance between centres of adjacent spheres touching wall	ρ_{W_1}	1.0
Volume of unit cell in outer region (i.e. volume associated with each sphere touching wall)	V_{UCO_1}	$\frac{\sqrt{3}}{2} = 0.866025$
Volume of sphere	V_p	$\frac{\pi}{6} = 0.523599$
Number of spheres touching wall	n_i	10,000
Volume of spheres touching wall	V_{W_1}	$n_1 \cdot V_p = 10{,}000\ V_p$
Volume of outer region	V_{OR_1}	$n_1 \cdot V_{UCO_1} = 10{,}000\ V_{UCO_1}$
Volume of intruding spheres	V_{I_1}	$\left(\frac{4}{3}\sqrt{\frac{2}{3}} - 1\right) V_{W_1} = 0.088662\ V_{W_1}$
Mean volume of intruding spheres in unit volume of outer region	V_{IU_1}	0.053605
Occupancy factor of intruding spheres	OF_{I_1}	1.0
Mean void volume available for intruding spheres in unit volume of outer region	V_{VIU_1}	$1 - V_{WU_1} = 0.395400$
Filling ability of intruding spheres	FA_{I_1}	$V_{IU_1}/V_{VIU_1} = 0.135572$

The term ϵ_{C_i} is obtained from Equation 1, while the mean radial distance ρ_{C_i} between the centres of adjacent spheres is obtained by rearranging terms in Equation 11 of Part III to give

$$\rho_{C_i} = \left(2 - \frac{n_{C_i}}{12}\right)^{\frac{1}{6}} . \tag{4}$$

Expressions for the other properties of interest in the central region are:

$$OF_{C_i} = \frac{1 - \epsilon_{C_i}}{1 - \epsilon_{C_1}} \tag{5}$$

$$V_{UCC_i} = \frac{V_{UCC_1}}{OF_{C_i}} \tag{6}$$

and

$$V_{CU_i} = \frac{\pi}{6\ V_{UCC_i}} . \tag{7}$$

In the outer region several possibilities present themselves, for example whether the vacancies left by spheres breaking contact with the wall are subject to expansion or contraction. The basic model was derived in the simplest possible terms, so the vacancies were treated as being subject to expansion.

Even on this basis the equations for n_W and ρ_W are not similar in form to Equations 3 and 4, since they are governed by the effects of both sphere-to-sphere and sphere-to-wall points of contact. Instead they are determined indirectly by introducing the terms n_{WE} and ρ_{WE}, obtained by two-dimensional expansion alone. The equations for these two terms are the equivalents of Equations 3 and 4, namely:

$$n_{WE_i} = 6\left(\frac{n_i}{10{,}000}\right) = 0.0006\ n_i \tag{8}$$

and

$$\rho_{WE_i} = \left(2 - \frac{n_{C_i}}{12}\right)^{\frac{1}{6}} = \left(2 - \frac{n_i}{10{,}000}\right)^{\frac{1}{6}} . \tag{9}$$

The volume of the outer region associated with the spheres touching the wall increases with the ρ^2_{WE}, giving

$$V_{OR_i} = V_{OR_1}\left(\frac{\rho_{WE_i}}{\rho_{WE_1}}\right)^2 . \tag{10}$$

Similarly, if it is assumed that the intruding spheres are governed by the three-dimensional expansion process once contact has been broken with the wall, then

$$V_{I_i} = V_{I_{i-1}}\left(\frac{\rho_{C_{i-1}}}{\rho_{C_i}}\right)^3 + V_p . \tag{11}$$

The decrease in n_W is the product of two effects, the breaking of points of contact at the wall and the breaking of points of contact between the spheres remaining in contact with the wall. Since the two effects are equal,

$$n_{W_i} = n_{W_1}\left(\frac{n_i}{10{,}000}\right)^2 . \tag{12}$$

Expressions for the other properties of interest in the outer region are:

$$V_{W_i} = n_i \cdot V_p \tag{13}$$

$$V_{UCO_i} = \frac{V_{OR_i}}{n_i} \tag{14}$$

$$V_{WU_i} = \frac{\pi}{6\, V_{UCO_i}} \tag{15}$$

$$OF_{W_i} = \frac{V_{UCO_1}}{V_{UCO_i}} \tag{16}$$

$$\rho_{W_i} = \sqrt{\frac{OF_{W_1}}{OF_{W_i}}} \tag{17}$$

$$V_{IU_i} = \frac{V_{I_i}}{V_{OR_i}} \tag{18}$$

$$OF_{I_i} = \frac{V_{IU_i}}{V_{IU_1}} \tag{19}$$

$$V_{VIU_i} = 1 - V_{WU_i} \tag{20}$$

and

$$FA_{I_i} = \frac{V_{IU_i}}{V_{VIU_i}} \quad . \tag{21}$$

This was the basis for computations, with enough steps to cover the known range of random packings and to go well beyond any loose practical packing which could be postulated.

The computed values of OF_C were excessive for given values of OF_W particularly at the loose end of the range. The spheres in the central region must therefore expand at a greater rate than assumed, implying that there must be a distinction between the sphere-to-sphere and the sphere-to-wall points of contact. The values of OF_I were also excessive for given values of OF_W, hence the intruding spheres must also expand at a greater rate than assumed. Limiting values were not obtained for V_I or for any related term, but this is to be expected when the expansion rate is underestimated.

2.2.3 Modification of basic concept

The relative rates of expansion of the three sphere categories were made to vary by introducing to the appropriate equations coefficients whose best values could be computed by trial and error. Many possibilities had to be considered, and limited experimental information was available to compare with the computed results. The study was therefore restricted to constant coefficients at this stage, whereas they were subsequently found to vary over the range. Even so it was possible to improve the model, by making the following changes.

Spheres touching wall: One shortcoming of the basic concept is that the vacancies left by spheres which have broken contact with the wall are made to expand at the same rate as the rest of the packing. Inspection of the outer region of loose random packings indicates that the vacancies contract. This offers an independent method of computing a limit to the expansion process. At the start, changes in OF_W are dominated by the (two-dimensional) expansion, since the effect of the contraction of vacancies is rudimentary, The contraction effect compounds until a limit is eventually reached where the two effects counterbalance.

These effects were computed by treating V_{OR}, the volume of the outer region, in two parts, V_{ORE} subject to expansion and V_{ORC} subject to contraction. Thus initially

$$V_{ORE_1} = V_{OR_1} = 10{,}000\, V_{UCO_1} \tag{22}$$

and

$$V_{ORC_1} = 0 \quad . \tag{23}$$

The first computations were made on the basis of V_{ORC} contracting at the same rate as V_{ORE} expanded. However, OF_W failed to reach a limiting value. The rate of contraction was therefore increased by introducing a coefficient C_W and replacing Equation 10 by the three equations

$$V_{ORE_i} = V_{ORE_{i-1}} \left(\frac{\rho_{WE_i}}{\rho_{WE_{i-1}}}\right)^2 - V_{UCO_1} \tag{24}$$

$$V_{ORC_i} = V_{ORC_{i-1}} \left(\frac{1 - C_W(\rho^2_{WE_i} - 1)}{1 - C_W(\rho^2_{WE_{i-1}} - 1)}\right) + V_{UCO_1} \tag{25}$$

and

$$V_{OR_i} = V_{ORE_i} + V_{ORC_i} \quad . \tag{26}$$

The modified equations gave a limiting value of OF_W of 0.6464 when $C_W = 6.0$. The physical significance attributed to this is that each sphere receding from the wall makes way for six neighbouring spheres. Since it is physically impossible for all six spheres surrounding a vacancy to move inwards, the implication is that the effect is felt at least two sphere diameters from the vacancy.

It should be noted that there are several other interpretations of the criterion that V_{ORC} contracts at six times the rate that V_{ORE} expands. Seven of these were computed. Six failed to give a limiting value of OF_W in the range 1.0 to 0.5, while the seventh gave a limiting value of 0.7, which was shown by subsequent experiments to be too high. Equation 25 and the other six gave values of OF_W in close agreement over the range 1.0 to 0.7, which covers the range of random packings normally obtained in practice.

At this point several trial adjustments were made to this part of the model, some of which might be said to make it more realistic, but all were unsuccessful. For instance, it might be thought that the contraction factor for the vacancies at the wall should diminish progressively with the mean number of sphere-to-sphere contact points. However, when computations were repeated with a contraction factor which was made to diminish at the appropriate rate, OF_W failed to reach a limiting value. The many variations considered to this point had therefore been narrowed down to one possibility, giving, as anticipated, a limiting value of OF_W significantly less than 0.7.

Spheres in central region: The rate of expansion of the spheres in the central region was altered by making their points of contact break at a rate greater than that of the sphere-to-wall points of contact. A coefficient C_C was introduced into Equation 3, to give

$$n_{C_i} = 0.0012\, C_C\, n_i \tag{27}$$

and the properties of the central region were recalculated on this basis.

As stated in the Introduction, the most likely limiting value of OF_W at the loose end of the range was considered to be 0.4264. This value, corresponding to the limiting value of OF_W of 0.6464, was obtained by putting $C_C = 1.389$.

Intruding spheres: Computations were based on the intruding spheres expanding at various rates greater than that represented in Equation 11. A range of limiting values of OF_I was obtained, but all fell within the known range of random packings. The assumption that the intruding spheres are subject to the three-dimensional expansion process was therefore discarded. Even if the three-dimensional process were known to hold, its application to the outer region would not be as straightforward as is implied by Equation 11. This is because the effect would appear partly as a two-dimensional expansion process and partly as migration to the central region as the spheres receded from the wall. In any event, the intruding spheres in close proximity to the wall are clearly constrained by the spheres touching the wall. It is therefore more appropriate to regard the intruding spheres as being governed initially by the same two-dimensional expansion process as the spheres touching the wall and progressively escaping this influence as they move further into the central region.

Equation 11 was therefore replaced by

$$V_{I_i} = V_{I_{i-1}}\left(1 - \left(1 - \frac{V_{UCO_{i-1}}}{V_{UCO_i}}\right) / C_I\right) + V_P \qquad (28)$$

The coefficient C_I is the fraction of the total expansion attributable to the two-dimensional expansion process, the remainder being attributable to migration to the central region. Equation 28 overcomes the further deficiency that V_I is not directly related to V_{OR} and V_{UCO} in Equation 11.

None of the computations with various values of C_I gave a limiting value of OF_I. Use was therefore made of the experimental results for a D_v/D_p ratio of 5.77, reported in Part I. The values of OF_W and V_I/V_W of the loosest packing were respectively 0.721 and 0.183 and the values for a similar packing in a semi-infinite vessel were expected to be much the same. The model was therefore adjusted to give these values, the value of C_I being 0.1396.

2.2.4 Check of model against available experimental data

Experimental values were not available to check the model directly, so the results from Part I for a D_v/D_p ratio of 5.77 were plotted in Figures 4 and 5. From Figure 4 it can be seen that the values of OF_C computed for the semi-infinite vessel fall well away from the experimental points. This is in part because the rhombohedral array is not the appropriate starting point for the expansion process in a cylindrical vessel, even for an infinite D_v/D_p ratio. The starting point is the cylindrically ordered packing, which forms the subject of Part II (Rocke 1972). The properties vary with D_v/D_p, as indicated in the following table.

TABLE 3

COMPARISON OF PROPERTIES OF RHOMBOHEDRAL ARRAY AND

TWO CYLINDRICALLY ORDERED PACKINGS

	OF_W	OF_C	V_I/V_W
Rhombohedral array	1.00	1.00	0.089
Cylindrically ordered packing			
D_v/D_p infinite	1.00	0.96	0.061
D_v/D_p = 5.77	0.99*	0.86*	0.036*

The computations were repeated on this basis, still being made to agree with the semi-infinite model at the loose end of the range. This was achieved by putting C_C = 0.64 and C_I = 0.15. The computed relationships are plotted in Figures 4 and 5. Agreement with observed values was considered sufficiently close to justify an experimental programme to check the semi-infinite model.

2.3 Experimental Testing of Model for Semi-Infinite Vessel

2.3.1 Central region

The model is based on Equations 1 and 4 and makes use of them directly and unchanged for computing the properties of the central region of random packings. This part of the model therefore agrees with experimentally determined values as reported in Part III**.

* Preliminary estimates only; corrected values are derived in Part V.

** This does not imply that the computed relationship between OF_C and OF_W is correct.

A detailed experimental investigation of the central region was not attempted. This was partly due to the time and effort required with the prismatic vessels used to represent the semi-infinite vessel, but chiefly because a detailed experimental investigation was already under way with cylindrical vessels. Experimental determinations of the properties of the central region of packings in large prismatic vessels, large, that is, with respect to sphere diameter, were therefore limited to the overall properties of the loosest obtainable packing. A Perspex vessel 5 in. x 5 in. x 3½ in. high was prepared for use with 3 mm precise glass spheres. A loose packing was prepared by inserting a liner of internal dimensions 4 in. x 4 in. x 4½ in. high, filling with spheres, and withdrawing to allow the spheres to 'expand' into the vessel. This was repeated several times, and the loosest packing so prepared had a mean void fraction in the central region of 0.423. This is in close agreement with the value of 0.422 reported in Part I for cylindrical vessels.

2.3.2 Outer region

The model was readily amenable to experimental testing in the outer region, along the lines of the method reported in Part I. A prismatic vessel 24 in. x 6 in. x 24 in. high was prepared for use with 1 in. diameter spheres. The front plane wall was transparent, giving a test area 16 in. x 16 in. away from edge effects. It was marked with a 4 in. square grid, on the inner face to minimise parallax errors. Two types of sphere were used in turn; machined aluminium and bonded zircon sand. Most of the experiments were carried out with the aluminium spheres because of their superior precision and uniformity of shape. The zircon spheres provided confirmation that surface roughness had little or no effect on the structure.

Random packings were prepared over as wide a range as possible. The loosest packings were prepared by placing a 1 in. thick board against the inside face of the transparent wall, pouring in the spheres until the vessel was slightly overfilled, and withdrawing the board. The denser packings were obtained by vibrating down such packings.

In each case a visual determination was made of OF_W, V_I and n_W. It was found essential to use a spotlight and a magnifying glass to distinguish between contacts and near contacts. Selected packings were then unpicked carefully for a determination of V_I as described in Part I. During unpicking it was usually found that a few of the spheres previously judged to be touching the transparent wall were in fact just clear of it, and the record was adjusted accordingly.

The average value of OF_W of the loosest packings was found to be 0.615. A determination of V_I was made for the loosest of these, with a value of OF_W of 0.600. Observed values of V_I/V_W are plotted against OF_W in Figure 5, where it can be seen that the model is in fair agreement for values of OF_W down to about 0.7.

Experimentally determined values of n_W are plotted against OF_W in Figure 6. It can be seen that they lie well away from the computed curve except at the loose end of the range, and run into the two-dimensional curve* at about n_W = 4.

* This is derived as follows:

$$1 - \epsilon_C' = \frac{\pi}{2\sqrt{5 - \frac{n_W}{3}}} \quad \text{(from Figure 3)}$$

$$OF_W = \frac{1 - \epsilon_C'}{(1 - \epsilon_C')_{\text{Rhombohedral}}} \quad \text{(by definition)}$$

$$\text{Hence} \quad OF_W = \frac{\sqrt{5 - \frac{6}{3}}}{\sqrt{5 - \frac{n_W}{3}}} = \sqrt{\frac{3}{5 - \frac{n_W}{3}}}$$

$$\text{from which} \quad n_W = 3\left(5 - \frac{3}{OF_W^2}\right).$$

The scatter of the experimental points at $n_W = 4$ is probably due to the smaller test area in the case of the denser packings. It was found impossible to prepare dense packings over the full test area, even with combined heavy vibration and poking. It was just possible to prepare the denser packings by vibration when the vessel was half filled, permitting observations over the bottom quarter of the test area only. Scatter is unavoidable due to the comparatively small number of spheres touching the wall, about 70, and also to the tendency for a single near-regular cluster to dominate over the reduced test area (see Figure 7). With large test areas several clusters can be seen to form with different orientations and slightly different occupancy factors. It might be possible to prepare packings with n_W greater than 4 by jolting hand packed rhombohedral arrays, but this was not attempted. However, visual examination of the structure of packings with $n_W = 4$ and greater (see Figures 7 and 8) left little doubt that the two-dimensional curve would be followed between $n_W = 4$ and $n_W = 6$. The intruding spheres had all been forced either to the wall or to a distance from the wall approximately the same as for the rhombohedral array. The spheres touching the wall were clearly approaching the density and pattern of the rhombohedral array. They would therefore behave essentially as a two-dimensional packing as densification proceeded to its conclusion. The number of intruding spheres was the same as the number of spheres touching the wall, as is the case with the rhombohedral array.

The compliance with the two-dimensional curve at the dense end of the range is at variance with the assumed common expansion mechanism. The implication is that the spheres touching the wall spread initially in two dimensions, thereafter the sphere-to-wall points of contact break and the assumed common expansion mechanism holds. The model can be adjusted for this by making it conform with the two-dimensional curve from $n_W = 6$ to $n_W = 4$, without changing the relationships between the other terms.

These points indicated the need for adjustments to the model on the basis of the experimental results. Before doing so a new analytical approach was investigated in the hope of finding more definite relationships over the whole range. This led to a second model, which did not call for a knowledge of the mean numbers of points of contact between spheres.

3. SECOND APPROACH – MODEL BASED ON PROPERTIES OF REGULAR ARRAYS

The relationships established in Part III for the central regions of the regular and random packings were considered to be sufficiently well matched to justify resuming the search for similar relationships in the outer region. There was still no reason to change from an expansion process starting with the known properties of the two regions at the rhombohedral end of the range, but it was clear that observed properties of random packings could not be explained in terms of the properties of the cubic array considered so far. The key to the problem was found to lie in some of its other properties. All previous considerations are based on the conventional orientation, which is but one of two axisymmetric orientations. These can be visualised by considering a 2 x 2 x 2 element of a cubic array containing 8 spheres. If it is placed on a horizontal surface in the conventional orientation, two horizontal planes may be drawn through the centres of the spheres, each plane containing 4 centres. If the element is rotated through 45° about a horizontal edge, it assumes a position which is not axisymmetric. Three horizontal planes may now be drawn through the centres, containing 2, 4 and 2 centres respectively. The distance between adjacent planes is $D_p/\sqrt{2}$. If the element is next suspended from a corner, it assumes the second axisymmetric orientation. Four horizontal planes may now be drawn through the centres, containing 1, 3, 3 and 1 centres respectively. The distance between adjacent planes is $D_p/\sqrt{3}$.

We call this configuration the tilted cubic array. (See Figure 9). The spheres in a given horizontal layer are in an equilateral triangular lattice and the distance between the centres of adjacent spheres is $D_p/\sqrt{2}$. When a semi-infinite tilted cubic array is placed with its outer layer against a plane wall, OF_W is 0.5. This is consistent with the fact, noted in the Introduction, that OF_W in practical three-dimensional packings can be much less than that of the now irrelevant square array.

An interesting physical relationship exists between the rhombohedral and tilted cubic arrays. If the layers of the blocked passage* rhombohedral array are visualised as being squeezed together,

* Two variations of the rhombohedral array exist, commonly designated clear passage and blocked passage. With the former the structure repeats itself every two layers; with the latter, every three layers.

with the spheres in each layer maintaining an equilateral triangular configuration as they separate, the tilted cubic array will eventually be reached. The process can be continued further, finally terminating when spheres which are vertically above one another make contact, that is, when the spacing of the layers is one-third of the sphere diameter (see Figure 10). The mean void fraction of this array is 0.3198. In the process a continuous range of regular arrays has been generated, which we will call the blocked passage equilateral triangular arrays. The properties of these arrays can be readily computed over the whole range, in terms of any convenient parameter, for example the spacing of the layers, as can be seen from column 6 of Appendix 1.

When the same process is applied to the clear passage rhombohedral array, a continuous range of clear passage equilateral triangular arrays is generated. An array is eventually reached which is equivalent to the tilted cubic array. Many of its properties are the same, though it has a different structure, as can be seen from Figure 11. The process can also be continued further, terminating when the spacing of the layers is half the sphere diameter (see Figure 12). The mean void fraction of the central region of this array is 0.4626.

It should be noted that these arrays are all structurally stable. The unstable nature of the cubic array in its conventional orientation is not, therefore, necessarily a factor limiting the expansion of random packings.

Interesting relationships exist between the geometrical properties of the rhombohedral and tilted cubic arrays. All are functions of a variable m, where m = 2 for the rhombohedral array and m = 4 for the tilted cubic array. This gives rise to the possibility that the geometrical properties of random packings in a semi-infinite vessel might also be functions of m*.

General expressions in terms of m are given in column 7 of Appendix 1. They are derived directly from the expressions in column 6. From inspection of the values of ϵ_C in columns 3, 4 and 5, it might be assumed that m would be 3 for the 'normal' random packing. This is not so; as can be seen from column 7 the general expression is

$$1 - \epsilon_C = \frac{\pi}{3\sqrt{3}} \cdot \frac{2}{m} \cdot \frac{1}{\sqrt{1-\frac{m}{6}}} \tag{29}$$

not

$$1 - \epsilon_C = \frac{\pi}{3\sqrt{m}} \quad . \tag{30}$$

The value of m for the 'normal' random packing is found by solving the equation

$$\frac{\pi}{3\sqrt{3}} \cdot \frac{2}{m} \cdot \frac{1}{\sqrt{1-\frac{m}{6}}} = \frac{\pi}{3\sqrt{3}} \quad , \tag{31}$$

from which

$$m^2\left(1 - \frac{m}{6}\right) = 4 \text{ , which gives } m = 2.694593....$$

Equation 29 correctly gives a minimum value of 0.5236, corresponding to the cubic array (m = 4), while Equation 30 does not.

The $OF_C - OF_W$ relationship for the equilateral triangular arrays is plotted in Figure 13. The relationship computed from the first model is also plotted, and the two are in general agreement, though requiring further reconciliation.

*Note: Although the mean void fraction of the central region of the 'normal' random packing is the same as that of the orthorhombic array, the latter is not one of the equilateral triangular arrays, and equivalent relationships do not apply in the wall region.

The V_I/V_W – OF_W relationship for the equilateral triangular arrays is plotted in Figure 14. The relationship computed by the first model is also plotted, together with the experimentally determined points. Agreement is good at the dense end of the range, but the experimental points fall above the curves over most of the loose end of the range.

4. THIRD APPROACH – COMPOSITE MODEL

The OF_C – OF_W and V_I/V_W – OF_W relationships for the equilateral triangular arrays were incorporated in the first model, enabling the (variable) values of C_W, C_C and C_I to be computed directly by rearranging terms in Equations 25, 27 and 28. The adjustments outlined at the end of Section 2 were also incorporated. OF_W was made to reach its limit at the observed value 0.615, by expressing C_W as a linear function of i, and retaining the value 6.0 at the dense end of the range. The computed values of the terms of main interest are tabulated in Appendix 3. It can be seen that C_W remains almost constant over the range, as was indicated under Section 2.2.3. It can also be seen that the maximum value of C_I is only 0.166, early in the expansion process, and diminishes to zero at the loose end of the range. This adds weight to the argument that changes in OF_I are dominated by migration to the central region, with a comparatively small initial contribution from two-dimensional expansion in the outer region. It is also apparent that C_I must fall to zero for a limiting value of V_I/V_W to be obtained. This accounts for the failure of any constant value of C_I to give a limiting value, as reported at the end of Section 2.2.3. It is also of interest to note that the computed values of FA_I vary by about 11 per cent over the range and only 4 per cent over the range of random packings normally obtained in practice. This agrees with the observation, reported in Part I, that V_I increases linearly with decrease in V_W.

The computed n_W – OF_W relationship is plotted as the broken curve in Figure 6. There is no obvious discrepancy between observation and calculation within the accuracy corresponding to the scatter of the experimental points.

5. INVESTIGATION OF DISCREPANCIES BETWEEN OBSERVATION AND COMPUTATION

5.1 OF_C – OF_W Relationship

The most unexpected result arising from the composite model was that the computed value of ϵ_C of the loosest random packing was 0.45, compared with 0.423 observed. Several more loose packings of 1 in. spheres were prepared in the prismatic vessel and examined more carefully. In each case it was found that several of the spheres adjacent to the wall were floating, playing no part in the overall structure of the packing. When the vessel was rocked gently the floating spheres could be seen to move freely while the rest of the packing remained remarkably stable. Other floating spheres could be seen further into the packing. For a total of 888 spheres touching the wall, 38 floating spheres were observed in close proximity to the wall. The floating spheres thus represented 4.1 per cent of the total of the spheres touching or capable of touching the wall.

With a vertical plane wall a floating sphere is unlikely to touch the wall since it would then be in an unstable position. The observed minimum value of OF_W of 0.615 is therefore taken as not including any floating spheres. In the central region, however, all floating spheres would be included in the observed value of the mean void fraction of 0.423. For the mean void fraction of the structure proper to be 0.45, the floating spheres would have to be 4.7 per cent of the total, which is in good agreement with the observed value of 4.1 per cent in close proximity to the wall.

Floating spheres were also observed in slightly denser packings prepared by vibrating such packings, but not in the range normally obtained in practice, namely where OF_W is greater than about 0.70.

5.2 V_I/V_W – OF_W Relationship

The second discrepancy, already noted, is that the observed values of V_I/V_W at the loose end of the range exceed the computed values. This can be adjusted over most of the range on the following basis.

In the equilateral triangular arrays, the intruding spheres do not come within half a sphere diameter of the wall over the range corresponding to the random packings. The intruding sphere material is therefore confined to the inner half of the outer region. In the random packings part of the intruding sphere material occupies the outer half of the outer region, its contribution to V_I being quite significant at the loose end of the range.

The volume of the intruding sphere material in the inner half of the outer region was computed for the random packings from the experimental results, and the values are plotted in Figure 15. They agree closely with computed values over the range of random packings normally encountered in practice, but a discrepancy still remains at the loose end of the range. About one third of this discrepancy is accounted for by the floating spheres, which were included as intruding spheres during the experimental determinations. The various corrected points are presented in Figure 15.

5.3 ϵ_C – n_C Relationship

The model does not help to clarify or extend the ϵ_C–n_C relationship established in Part III; in particular it does not establish whether n_C remains substantially equal to 6 or decreases in accordance with Equation 1 as ϵ_C decreases below 0.395.

The equilateral triangular arrays provide little guidance, though the following is worthy of note. Each sphere in the rhombohedral array has 12 points of contact, of which the six around the equator are not load bearing points. Immediately the layers are squeezed closer together contact is broken at the equatorial points, but the six load bearing points remain throughout the range. The cubic array can, however, be visualised as expanding slightly so that all six points of contact just break, while ϵ_C remains at 0.4764. There are then two values of n_C at each end of the range, 12 and 6 when ϵ_C = 0.2595, 6 and 0 when ϵ_C = 0.4764. The first and fourth are the terminal values of the curve in Figure 2.

An experiment was carried out to determine the ϵ_C– n_C relationship in the central region of the loosest obtainable packing. Several packings of 0.25 in. nylon spheres were prepared in an 8 in. diameter Perspex tube by filling and withdrawing a central load tube. The observed values of ϵ_C were all in close agreement, the average being 0.420. From Table 1 it can be seen that n_C would have the value of 4.4 if Equation 1 were appropriate and no allowance were made for floating spheres. If allowance were made, ϵ_C would be 0.45 and n_C would have the improbable value of 2.2. A determination of n_C was made for one such packing. The vessel was topped up with a dilute ink solution, drained, and dried out by a gentle stream of air at ambient temperature over a period of many hours. When spheres were removed from the packing, points of contact were clearly identified by a small ring and close contacts by a small spot. Samples were taken from various parts of the central region by each of three observers and counts made. All were in close agreement and the average was 6.3 points of contact per sphere. It therefore appears unlikely that Equation 1 holds in practice for values of ϵ_C greater than 0.3954.

5.4 n_W – OF_W Relationship

The agreement between observation and calculation in Figure 6 indicates that the points of contact between adjacent spheres touching the wall break closely at the same rate as the sphere-to-wall points of contact.

6. TWO–DIMENSIONAL MODEL

The composite model does not directly cast new light on the ϵ'_C -- n'_C relationship. However a brief exploratory study was made of the two-dimensional case for possible parallels with the three-dimensional case.

As stated in Part III, the equation given in Figure 3 enables the following values to be calculated:

when $n'_C = 0, \quad \epsilon'_C = 1 - \frac{\pi}{2\sqrt{5}} \quad = 0.2975$ (significance unknown);

when $n'_C = 3, \quad \epsilon'_C = 1 - \frac{\pi}{2\sqrt{4}} \quad = 0.2146$ (same ϵ'_C as circumscribed square);

when $n'_C = 6, \quad \epsilon'_C = 1 - \frac{\pi}{2\sqrt{3}} \quad = 0.0931$ (circumscribed hexagon).

The study was extended by considering the two-dimensional equivalents of the equilateral triangular arrays. The starting point in this case is the densest possible packing of equal discs, the hexagonal array, in a semi-infinite vessel with a horizontal straight edge forming the base. The array is oriented to give horizontal rows of touching discs. If the rows are visualised as being squeezed together, while the discs in each row remain equispaced as they separate, a continuous range of regular arrays is generated. We call these the two-dimensional equispaced arrays. They include the square array, and the process can be continued further, finally terminating when discs which are vertically above one another make contact. This is merely a return to the hexagonal array, oriented at right angles to the original.

As in the three-dimensional case, relationships exist between the geometrical properties of the densest and loosest arrays. All are functions of a variable m', where $m' = 1$ for the original hexagonal array, $m' = 2$ for the square array, and $m' = 3$ for the final orientation of the hexagonal array. Properties of interest are tabulated in Appendix 2.

The two-dimensional rig referred to in Part III was used for an experimental determination of the properties of loose random packings in the outer region. Packings were prepared by placing two strips of wood 2½ in. wide against one vertical straight edge of the vessel, filling it with spheres, and withdrawing the strips quickly in turn. The properties of interest were calculated from the observations in terms of the equivalent discs.

The experiment was repeated seven times, giving a total of 107 spheres of 0.9902 in. diameter touching the edge over a total length of 150 in. Only one floating sphere was observed. The overall occupancy factor of the equivalent discs touching the edge was 0.71, compared with 0.7071 calculated. The ratio of the intruding disc area to that of the discs touching the edge was 0.268, compared with 0.244 calculated. When corrected by subtracting the area of the intruding spheres in the outer half of the outer region, the ratio was reduced to 0.223.

These results indicate that the outer and central regions reach the loosest possible condition together, despite the potential for a much smaller occupancy factor of the discs touching the edge. Although the agreement between computation and observation is close, it must be emphasised that this was but one spot check of the two-dimensional model. In particular an exhaustive study of other preparation methods was not attempted, so that it is not certain that the loosest possible packing was achieved. A more detailed study would be of interest, but is outside the scope of this study.

7. DISCUSSION

7.1 Relationships Between the Various Occupancy Factors and Derived Properties

The calculated occupancy factors and derived properties of the equilateral triangular arrays agree generally with experimentally determined properties of random packings. The agreement is particularly close in the outer region over the range of random packings normally encountered in practice, after an allowance has been made for the intruding sphere material in the outer half of the outer region. If the allowance is not made, a discrepancy of about 17 per cent is introduced to the volume of the intruding sphere material, representing about 3 per cent of the total volume of sphere material in the outer region.

As can be seen from Figure 15, after making this allowance, and a further allowance for the floating spheres, the computed and observed values of V_I/V_W are 0.307 and 0.260 respectively. This unresolved discrepancy of 18 per cent represents about 3 per cent of the total volume of sphere material in the outer region.

It is not known whether this is due to experimental error or to the model. However, there is no known reason to anticipate a breakdown in the model as the loosest possible condition is approached, since both the loosest random packing and the cubic array fall well short of the limiting condition. This point can only be resolved by further experiments, or by finding precise quantitative criteria governing the loosest possible condition.

The unresolved discrepancy in the values of V_I/V_W at the loose end of the range does not detract from the good agreement between observed and computed values over the range of random packings relevant to a recirculating pebble bed nuclear reactor, that is, in the vicinity of the 'normal' random packing, for which $OF_W = 0.742$.

The composite model indicates that the three sphere categories reach the loosest possible condition together in unbiased random packings. The model based on the properties of the regular two-dimensional arrays, and supported by a limited experimental investigation, indicates that this also applies in the two-dimensional case. Radial bias should therefore only arise in practical random packings when the packing method gives rise to variable consolidation effects.

An interesting implication of the points of agreement between the properties of the equilateral triangular arrays and the unbiased random packings is that the ability of the three sphere categories to fill the available space is independent of the degree of ordering of the packing. Further evidence for this is given in Part I, where the strong local radial variations in random packings in cylindrical vessels are found to have no significant effect on the mean void fraction of the central region.

However, it can readily be shown that this is not a universal law. A range of square arrays can be built up in the same way as the equilateral triangular arrays, but the relationships between the computed properties of the three sphere categories are different. The properties of the orthorhombic and tetragonal sphenoidal arrays are different again, even in their stable configurations. These can be built up respectively on a rectangular grid with sides D_p and $\sqrt{3/2}\ D_p$ and on a triangular grid with sides D_p, $\sqrt{7/4}\ D_p$ and $\sqrt{7/4}\ D_p$. Furthermore they have the same value of OF_W but different values of OF_C.

It therefore appears unlikely that the relationships extend beyond the equilateral triangular arrays and the unbiased random packings. It is not possible to say categorically that they must hold even there, in the absence of a formal proof. Even if one were forthcoming, it would not enable the properties of the loosest possible random packing to be computed with a higher degree of certainty than the values computed for this report.

7.2 Relationships Between Mean Numbers of Points of Contact and Other Properties

In the central region, indications are that the $\epsilon_C - n_C$ relationship given in Equation 1 does not hold for ϵ_C greater than 0.3954, the value for the 'normal' random packing, for which $n_C = 6$, but that n_C remains essentially constant at this value. The $OF_C - OF_W$ relationship established by the composite model constitutes a direct relationship between the number of sphere-to-wall contacts per unit area and OF_C (and hence ϵ_C for the central region).

In the outer region, it appeared early in the investigation that the $n_W - OF_W$ relationship would follow the two-dimensional curve over its entire practical range (that is, from $n_W = 6$ to $n_W = 3$), and that n_W would then remain essentially constant at 3 as OF_W decreased further. This would have constituted a two-dimensional parallel to the $\epsilon_C - n_C$ relationship in the central region; however the disposition of the experimental points in Figure 6 offers little support for this possibility.

7.3 Two-Dimensional Case

The relationships between the properties of the two-dimensional equispaced arrays and the random packings help to clarify a point outstanding from Part III. This is the unknown significance of the value of ϵ'_C when $n'_C = 0$ (see Section 6 above).

We now have two expressions for ϵ'_C:

From Figure 3

$$1 - \epsilon'_C = \frac{\pi}{2\sqrt{5 - \frac{n'_C}{3}}} \tag{32}$$

and from Appendix 2

$$1 - \epsilon'_C = \frac{\pi}{4\sqrt{m' - \frac{m'^2}{4}}} \tag{33}$$

Eliminating ϵ_C, we have

$$m'^2 - 4m' + \left(5 - \frac{n'_C}{3}\right) = 0 \quad . \tag{34}$$

Any value of n'_C between 0 and 3 gives a value of m' which is a complex number; so presumably only that part of the curve between $n'_C = 3$ and $n'_C = 6$ has any physical significance. Equation 34 gives the values of m' already established, for example $m' = 2$ when $n'_C = 3$, and $m' = 1$ and 3 when $n'_C = 6$.

One difference between the two- and three-dimensional cases remains unexplained. This is formation of stable two-dimensional packings with less than four points of contact per sphere. One possible explanation is that the spheres in the two-dimensional experiments had a greatly reduced ability to rotate under load compared with the three-dimensional situation. The possible importance of this point can be appreciated by referring to a simple experiment. If say ten equal precise spheres, such as 1 in. ball bearings, are laid in a row in an accurate vee block, and the two end spheres are carefully held to prevent rotation, a substantial axial load can be applied to the row. If one of the end spheres is rotated very slightly while attempting to sustain the load, the load bearing ability of the row breaks down immediately and the spheres fly out of line.

In the formation of a three-dimensional packing, as points of contact are being made and loads are being applied, a sphere has three rotational degrees of freedom; also the loads are applied in random directions. The sphere and its neighbours may therefore have a mobility under inclined forces comparable with that of a sphere on a vee block when rotation is initiated; if so it would have a high probability of evading the clamping action of less than six neighbours. A random packing with a mean of six points of contact per sphere can be densified even further by vibration. When the vessel has a plane base, a substantial volume can form with a rhombohedral structure. This can also be demonstrated by placing several layers of spheres in a shallow rectangular box and shaking horizontally. A rhombohedral structure forms immediately. Both processes can be seen to induce rotation of the spheres, and it appears that this type of mobility is a necessary prerequisite to the formation of dense packings.

In the two-dimensional case, with only one rotational degree of freedom and uniplanar loads, the mobility of a sphere may be so small that it has little ability to adjust its position when it is 'clamped' by three adjacent spheres. However, this is not necessarily so in the outer region of a three-dimensional packing in a semi-infinite vessel. As a loose packing is vibrated down, the movement of intruding spheres may induce a slight rolling action in the spheres touching the wall, enabling them to escape the clamping action of their neighbours and make more contacts for a given value of OF_W. This would account for the observed $n_W - OF_W$ relationship not following the two-dimensional curve between $n_W = 3$ and 4.

This point can be clarified further by studies of recirculated two-dimensional packings. The mean number of points of contact would be expected to be about four, because the clamping forces change and the spheres rotate continually owing to relative movement throughout the packing. An estimate of the mean number of points of contact can be made from Figure 16, which shows the structure resulting from flow in a two-dimensional vessel with two outlets. The vessel was filled initially with bronze spheres and kept topped up with steel spheres of the same diameter as flow proceeded.

The dark spheres in the lower portion of the figure have remained stationary, while the light spheres have reached steady state flow conditions. Unfortunately a direct determination of the mean number of points of contact (of the light spheres) is not possible from the original photograph. However, a value has been deduced from the occupancy factor, which was determined directly from the photograph by two independent observers, avoiding the dense regular areas as far as possible. An average value of 0.933 was obtained, corresponding to a mean number of points of contact of 4.66 (calculated from the equation given in Figure 6).

The vessel had straight vertical sides, encouraging the formation of dense regular areas which tend to move by slipping in blocks. The outlet channels also contribute to this effect. A vessel with an irregular outline, with outlet orifices instead of channels, might give a value of n'_C nearer to four. At least Figure 16 shows conclusively that the value cannot be three.

7.4 Future Experimental Work

The experiments and computations reported here (and also those with cylindrical vessels reported in Parts I and V) were substantially completed before the significance of the effect of floating spheres in loose packings was fully appreciated. Time did not permit more than a few key determinations to be made of this effect. Indications are that it is not significant over the range of random packings relevant to a nuclear reactor but future workers in this area are advised to satisfy themselves on this point by appropriate experimental determinations.

Experience to date indicates that the order of accuracy of the experimental determinations of OF_C, OF_W and OF_I by the methods used is about ±1 per cent, and that this accuracy is unlikely to be improved by any method. This accuracy has been achieved, both experimentally and analytically, except possibly at the loose end of the range. Further experimental evidence would be desirable to clarify the small remaining uncertainties, and this would involve repeating each experiment several times to achieve better accuracy. The use of two different methods and/or two observers would also be advantageous. To finalise the matter, it would also be necessary to establish precise quantitative criteria governing the loosest possible condition, particularly in the central region.

Over the range of unbiased random packings normally encountered in practice, and particularly over the range relevant to a recirculating pebble bed nuclear reactor, the present study provides a complete balance sheet to the order of accuracy stated above.

Several key experiments are suggested for workers interested in the whole range of packings, including the properties of the loosest possible random packing and the criteria governing it –

(i) Determination of the values of ϵ_C and n_C for the 'normal' random packing. This is the packing resulting from recirculation under gravity to the point at which it reaches a steady state condition. It is then envisaged as a stable structure when at rest yet having mobility throughout when gravitational flow is initiated from the bottom. Similar determinations are desirable for the densest possible random packing obtainable in the absence of ordering induced by the containing vessel.

(ii) Experimental determination of all the properties of the loosest random packings obtainable in the laboratory. This would clarify the residual errors at the loose end of the range and enable further refinements to be made to the composite model for a final check of its consistency and validity.

(iii) Repetitions of the determinations of n_W and OF_W in the wall region, and averaging results to reduce the scatter exhibited by the experimental points in Figure 6. It might also be possible to produce denser packings than can be obtained by vibration by jolting hand packed rhombohedral arrays. The composite model can be adjusted if necessary by making the two types of contact break at slightly different rates. This would not affect the rest of the model.

(iv) Determination of the properties of the central region of recirculated two-dimensional random packings, in particular to ascertain whether the mean number of points of contact per sphere is four away from the influence of the edges of the vessel.

8. SUMMARY AND CONCLUSIONS

1. The relationships between random packings and the regular arrays, given in Part III, have been extended to the wall region of unbiased random packings in a semi-infinite vessel.

2. They apply not only to unbiased random packings and the rhombohedral and cubic arrays, but also to a family of regular arrays, the equilateral triangular arrays, to which the rhombohedral and cubic arrays belong.

3. The ability of the three sphere categories to fill the available space is the same for unbiased random packings and the equilateral triangular arrays. This enables the occupancy factors and derived properties of random packings to be calculated, subject to a small correction due to intrusion of sphere material in the outer half of the outer region.

4. The equilateral triangular arrays provide no direct information on the mean number of points of contact in either region of random packings.

5. On the basis of the $n_C - \epsilon_C$ relationship developed in Part III, a model has been developed to account for observed mean properties, including the mean numbers of points of contact, in both regions of random packings.

6. Similar relationships are shown to apply in the two-dimensional case, supported by a limited experimental study.

7. The three-dimensional model enables the properties of the loosest theoretical random packing to be computed, provided any one related property is known. On the basis of the experimentally determined minimum value of OF_W of 0.615, the mean void fraction of the central region of the loosest theoretical random packing is calculated to be 0.450. In very loose practical packings, floating spheres are also present, but they play no part in the structure proper. They are estimated to comprise about 4.7 per cent of the total, which would account for the overall mean void fraction of about 0.423 observed in the loosest packings prepared for the present study.

8. An accuracy of the order of ± 1 per cent has been achieved for the values of the occupancy factors in the present study, both experimentally and theoretically, over the range of unbiased random packings relevant to a pebble bed nuclear reactor. An overall accuracy of the order of 3 per cent has been achieved in the model at the loose end of the range, and further experimental and theoretical investigations are necessary. Several other areas for investigation by future workers are also outlined, including the properties of the 'normal' random packing and the densest possible random packing.

9. Future experiments must include a careful determination of the number of floating spheres, and their mean number of points of contact with the wall or with adjacent spheres.

10. Some latitude exists for the composite model developed in this study to be adjusted to match more precise experimental data should it become available.

9. ACKNOWLEDGEMENTS

Grateful acknowledgement is due to Mr. P.E. Wydeveld who prepared and ran the original computer programme, Mr. F.C. Gatt who systematically covered some of the early variations, Mr. A.D. Tonkin who checked several of the key variations by double precision, Mrs. M.A. Cowper who computed the coefficients for the composite model and assisted in the preparation of the various graphs and tables and Messrs. E.W. Clarke and A.E. Longmore who prepared the vessels and packings and carried out many of the experimental determinations.

10. NOTATION

D,	=	diameter of cylindrical vessel
D_p	=	diameter of sphere
OF_C	=	occupancy factor of spheres in central region
OF_W	=	occupancy factor of spheres touching wall
OF_I	=	occupancy factor of intruding spheres
n_C	=	mean number of points of contact between spheres in central region
ϵ_C	=	mean void fraction of central region
ρ_C	=	mean radial distance between centres of adjacent spheres in central region
V_{CU}	=	mean sphere volume in unit volume of central region (or occupancy fraction)
V_{UCC}	=	volume of unit cell in central region (i.e. volume associated with each sphere)
n_W	=	mean number of points of contact between spheres touching wall
ρ_W	=	mean radial distance between centres of adjacent spheres touching wall
V_{WU}	=	mean volume of spheres touching wall in unit volume of outer region
V_{UCO}	=	volume of unit cell in outer region (i.e. volume associated with each sphere touching wall)
V_p	=	volume of sphere
n	=	number of spheres touching wall
V_W	=	volume of spheres touching wall
V_{OR}	=	volume of outer region
V_I	=	volume of intruding spheres
V_{IU}	=	mean volume of intruding spheres in unit volume of outer region
V_{VIU}	=	mean void volume available for intruding spheres in unit volume of outer region
FA_I	=	filling ability of intruding spheres
n_{WE}	=	mean number of points of contact between adjacent spheres touching wall obtained by two-dimensional expansion alone

ρ_{WE}	=	mean radial distance between centres of adjacent spheres touching wall obtained by two-dimensional expansion alone
V_{ORE}	=	volume of outer region subject to expansion
V_{ORC}	=	volume of outer region subject to contraction
C_C, C_W, C_I	=	coefficients to adjust equations for the three sphere categories
m	=	a dependent variable used in establishing the equations for the equilateral triangular arrays
ϵ'_C	=	mean void fraction of central region of two-dimensional packing
n'_C	=	mean number of points of contact between discs in central region of two-dimensional packing
m'	=	a dependent variable used in establishing the equations for the two-dimensional equi-spaced arrays

11. REFERENCES

The only references in this part are to other parts of the series whose titles are given in the Preface.

References relevant to the overall study are given in Parts I, II and III.

APPENDIX 1

SOME PROPERTIES OF THE EQUILATERAL TRIANGULAR ARRAYS (STANDARDISED TO UNIT SPHERE DIAMETER)

Property	Symbol	Rhombohedral Array (m = 2)	Equivalent of 'Normal' Random Packing (m = 2.69459)	Tilted Cubic Array (m = 4)	General Expression in Terms of Other Properties	General Expression in Terms of m (Selected Examples)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Distance between horizontal layers	d_L	$\sqrt{\frac{2}{3}}$ = 0.816497	0.742227	$\frac{1}{\sqrt{3}}$ = 0.577350	–	$\sqrt{1-\frac{m}{6}}$
Distance between centres of adjacent spheres in same layer	ρ	1.0	1.160731	$\sqrt{2}$ = 1.414214	$\sqrt{3(1-d_L^2)}$	$\sqrt{\frac{m}{2}}$
Mean volume of unit cell in central region (i.e. volume associated with each sphere)	V_{UCC}	$\frac{\sqrt{2}}{2}$ = 0.707107	$\frac{\sqrt{3}}{2}$ = 0.866025	$\frac{\sqrt{4}}{2}$ = 1.0	$\frac{\sqrt{3}}{2}\, d_L \cdot \rho^2$	$\frac{\sqrt{3}}{4} \cdot m \cdot \sqrt{1-\frac{m}{6}}$
Mean volume of unit cell in outer region (i.e. volume associated with each sphere touching wall)	V_{UCO}	$\frac{\sqrt{3}}{2}$ = 0.866025	1.116679	$\sqrt{3}$ = 1.732051	V_{UCC}/d_L	$\frac{\sqrt{3}}{4}\, m$
Mean sphere volume in unit volume of central region (or occupancy fraction)	V_{CU} $(=1-\epsilon_C)$	$\frac{\pi}{3\sqrt{2}}$ = 0.740480	$\frac{\pi}{3\sqrt{3}}$ = 0.604600	$\frac{\pi}{3\sqrt{4}}$ = 0.523599	$\pi/(6\, V_{UCC})$	$\frac{\pi}{3\sqrt{3}} \cdot \frac{2}{m} \cdot \frac{1}{\sqrt{1-\frac{m}{6}}}$
Mean void fraction of central region	ϵ_C	$1-\frac{\pi}{3\sqrt{2}}$ = 0.259520	$1-\frac{\pi}{3\sqrt{3}}$ = 0.395400	$1-\frac{\pi}{3\sqrt{4}}$ = 0.476401	$1-V_{CU}$	$1-\frac{\pi}{3\sqrt{3}} \cdot \frac{2}{m} \cdot \frac{1}{\sqrt{1-\frac{m}{6}}}$
Occupancy factor of spheres in central region	OF_C	1.0	$\sqrt{\frac{2}{3}}$ = 0.816497	$\frac{\sqrt{2}}{2}$ = 0.707107	$V_{CU}/\left(\frac{\pi}{3\sqrt{2}}\right)$	$\sqrt{\frac{2}{3}} \cdot \frac{2}{m} \cdot \frac{1}{\sqrt{1-\frac{m}{6}}}$
Mean volume of spheres touching wall in unit volume of outer region	V_{WU}	$\frac{\pi}{3\sqrt{3}}$ = 0.604600	0.448750	$\frac{\pi}{6\sqrt{3}}$ = 0.302300	$\pi/(6\, V_{UCO})$	$\frac{2\pi}{3\sqrt{3} \cdot m}$
Occupancy factor of spheres touching wall	OF_W	1.0	0.742227	0.5	$V_{WU}/\left(\frac{\pi}{3\sqrt{3}}\right)$	$\frac{2}{m}$
Mean intrusion distance	d_I	$1-\sqrt{\frac{2}{3}}$ = 0.183503	0.257773	$1-\frac{1}{\sqrt{3}}$ = 0.422650	$1-d_L$	$1-\sqrt{1-\frac{m}{6}}$
Mean volume of intruding spherical segment	V_{SSEG}	0.046423	0.086438	0.201533	$\frac{\pi}{3} \cdot (d_I)^2 \cdot (1.5-d_I)$	$\frac{\pi}{3}\left(1-\sqrt{1-\frac{m}{6}}\right)^2 \cdot \left(0.5+\sqrt{1-\frac{m}{6}}\right)$
Mean volume of intruding spheres in unit volume of outer region	V_{IU}	0.053605	0.074082	0.116355	V_{SSEG}/V_{UCO}	
Occupancy factor of intruding spheres	OF_I	1.0	1.381987	2.170601	$V_{IU}/0.053605$	
Mean void volume available for intruding spheres in unit volume of outer region	V_{VIU}	$1-\frac{\pi}{3\sqrt{3}}$ = 0.395400	0.551250	$1-\frac{\pi}{6\sqrt{3}}$ = 0.697700	$1-V_{WU}$	$1-\frac{2\pi}{3\sqrt{3} \cdot m}$
Filling ability of intruding spheres	FA_I	0.135572	0.134388	0.166770	V_{IU}/V_{VIU}	
Ratio, volume of intruding spheres to volume of spheres touching wall	V_I/V_W	0.088662	0.207107	0.384900	V_{IU}/V_{WU}	
Effective distance between centres of adjacent spheres in central region	ρ_{EFF}	1.0	1.069913	1.122462	$(2\, V_{UCC}^2)^{\frac{1}{6}}$	$\left[\frac{3}{8} m^2\left(1-\frac{m}{6}\right)\right]^{\frac{1}{6}}$

APPENDIX 2

SOME PROPERTIES OF THE TWO-DIMENSIONAL EQUISPACED ARRAYS (STANDARDISED TO UNIT DISC DIAMETER)

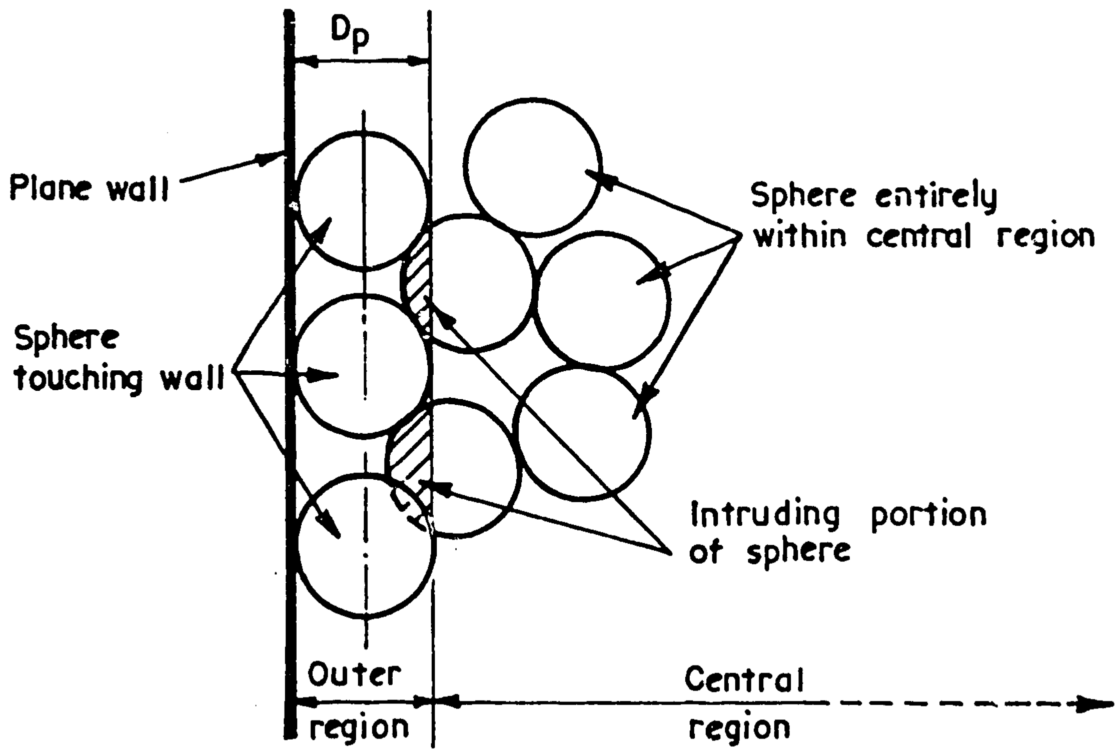
Property	Symbol *	Hexagonal Array ($m' = 1$)	Square Array ($m' = 2$)	Hexagonal Array ($m' = 3$)	General Expression in Terms of Other Properties	General Expression in Terms of m' (Selected Examples)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Distance between horizontal rows	d_R	$\frac{\sqrt{3}}{2} = 0.866025$	$\frac{\sqrt{2}}{2} = 0.707107$	0.5	–	$\sqrt{1-\frac{m'}{4}}$
Distance between centres of adjacent discs in same row	ρ'	1.0	$\sqrt{2} = 1.414214$	$\sqrt{3} = 1.732051$	$2\sqrt{1-d_R^2}$	$\sqrt{m'}$
Mean area of unit cell in central region (i.e. area associated with each disc)	A_{UCC}	$\frac{\sqrt{3}}{2} = 0.866025$	1.0	$\frac{\sqrt{3}}{2} = 0.866025$	$d_R \cdot \rho'$	$\sqrt{m' - \frac{m'^2}{4}}$
Mean area of unit cell in outer region (i.e. area associated with each disc touching edge)	A_{UCO}	1.0	$\sqrt{2} = 1.414214$	$\sqrt{3} = 1.732051$	A_{UCC}/d_R	$\sqrt{m'}$
Mean disc area in unit area of central region (or occupancy fraction)	A_{CU} $(=1-\epsilon_C')$	$\frac{\pi}{2\sqrt{3}} = 0.906900$	$\frac{\pi}{4} = 0.785398$	$\frac{\pi}{2\sqrt{3}} = 0.906900$	$\pi/(4\,A_{UCC})$	$\frac{\pi}{4\sqrt{m' - \frac{m'^2}{4}}}$
Mean void fraction of central region	ϵ_C'	$1-\frac{\pi}{2\sqrt{3}} = 0.093100$	$1-\frac{\pi}{4} = 0.214602$	$1-\frac{\pi}{2\sqrt{3}} = 0.093100$	$1 - A_{CU}$	$1-\frac{\pi}{4\sqrt{m' - \frac{m'^2}{4}}}$
Occupancy factor of discs in central region	OF_C'	1.0	$\frac{\sqrt{3}}{2} = 0.866025$	1.0	$A_{CU}\Big/\left(\frac{\pi}{2\sqrt{3}}\right)$	$\frac{\sqrt{3}}{\sqrt{4m' - m'^2}}$
Mean area of discs touching edge in unit area of outer region	A_{EU}	$\frac{\pi}{4} = 0.785398$	$\frac{\pi}{4\sqrt{2}} = 0.555360$	$\frac{\pi}{4\sqrt{3}} = 0.453450$	$\pi/(4A_{UCO})$	$\frac{\pi}{4\sqrt{m'}}$
Occupancy factor of discs touching edge	OF_E	1.0	$\frac{\sqrt{2}}{2} = 0.707107$	$\frac{\sqrt{3}}{3} = 0.577350$	$A_{EU}\Big/\left(\frac{\pi}{4}\right)$	$\frac{1}{\sqrt{m'}}$
Mean intrusion distance	d_I'	$1-\frac{\sqrt{3}}{2} = 0.133975$	$1-\frac{\sqrt{2}}{2} = 0.292893$	0.5	$1 - d_R$	$1-\sqrt{1-\frac{m'}{4}}$
Mean area of intruding circular segment	A_{CSEG}	0.062690	0.191677	0.392455	$\frac{1}{4}\operatorname{arsin}(2\sqrt{d_R - d_R^2}) - (d_R - 0.5)\sqrt{d_R - d_R^2}$	
Mean area of intruding discs in unit area of outer region	A_{IU}	0.062690	0.135536	0.226584	A_{CSEG}/A_{UCO}	
Occupancy factor of intruding discs	OF_I'	1.0	2.162005	3.614366	$A_{IU}/0.062690$	
Mean void area available for intruding discs in unit volume of outer region	V_{AIU}	0.214602	0.444640	0.546550	$1 - A_{EU}$	$1-\frac{\pi}{4\sqrt{m'}}$
Filling ability of intruding discs	FA_I'	0.292122	0.304822	0.414571	A_{IU}/V_{AIU}	
Ratio, area of intruding discs to area of discs touching edge	A_I/A_W	0.079819	0.244051	0.499689	A_{IU}/A_{EU}	
Effective distance between centres of adjacent discs in central region	ρ'_{EFF}	1.0	1.074570	1.0	$\left(\frac{4}{3}A_{UCC}^2\right)^{\frac{1}{4}}$	$\left(\frac{4}{3}\left(m' - \frac{m'^2}{4}\right)\right)^{\frac{1}{4}}$

* These symbols are the two-dimensional equivalents of those used in Appendix 1

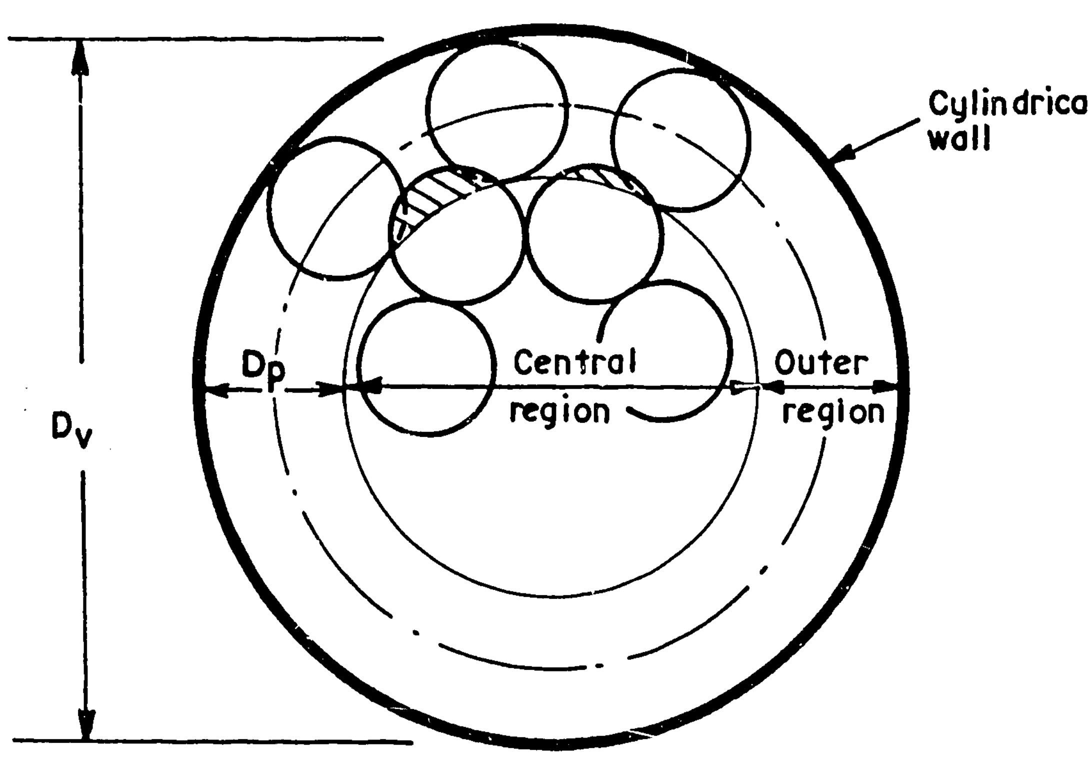
APPENDIX 3

VALUES COMPUTED BY COMPOSITE MODEL

i	n	n_W	n_C	ϵ_C	V_{IU}	OF_W	OF_C	OF_I	FA_I	V_I/V_W	C_W	C_C	C_I
1	10000	6.00	12.0	0.260	0.0536	1.000	1.000	1.00	0.136	0.089	6.00		
101	9900	5.82	11.8	0.265	0.0542	0.990	0.993	1.01	0.135	0.091	6.00		
201	9800	5.63	11.6	0.271	0.0548	0.980	0.985	1.02	0.134	0.092	5.99		
301	9700	5.44	11.4	0.276	0.0554	0.970	0.978	1.03	0.134	0.094	5.99		
401	9600	5.23	11.3	0.282	0.0560	0.960	0.970	1.04	0.133	0.096	5.98		
501	9500	5.03	11.1	0.287	0.0566	0.950	0.963	1.06	0.133	0.099	5.98		
601	9400	4.82	10.9	0.293	0.0573	0.940	0.955	1.07	0.133	0.101	5.97		
701	9300	4.59	10.6	0.298	0.0579	0.930	0.948	1.08	0.132	0.103	5.97		
801	9200	4.37	10.4	0.303	0.0586	0.920	0.941	1.09	0.132	0.105	5.97		
901	9100	4.13	10.2	0.309	0.0593	0.910	0.933	1.11	0.132	0.108	5.96		
1001	9000	3.96	10.0	0.315	0.0601	0.899	0.925	1.12	0.132	0.111	5.96	2.42	0.164
1101	8900	3.87	9.7	0.321	0.0610	0.887	0.917	1.14	0.132	0.114	5.96	2.42	0.165
1201	8800	3.79	9.4	0.328	0.0619	0.875	0.908	1.15	0.131	0.117	5.96	2.41	0.165
1301	8700	3.70	9.2	0.334	0.0628	0.863	0.900	1.17	0.131	0.120	5.95	2.41	0.166
1401	8600	3.62	8.9	0.340	0.0637	0.852	0.892	1.19	0.131	0.124	5.95	2.41	0.166
1501	8500	3.53	8.6	0.345	0.0646	0.841	0.884	1.20	0.131	0.127	5.95	2.40	0.166
1601	8400	3.45	8.4	0.351	0.0655	0.831	0.876	1.22	0.132	0.130	5.94	2.40	0.166
1701	8300	3.37	8.1	0.356	0.0664	0.820	0.869	1.24	0.132	0.134	5.94	2.39	0.166
1801	8200	3.29	7.9	0.362	0.0673	0.810	0.862	1.26	0.132	0.137	5.93	2.39	0.165
1901	8100	3.21	7.6	0.367	0.0682	0.800	0.855	1.27	0.132	0.141	5.93	2.38	0.165
2001	8000	3.13	7.3	0.372	0.0691	0.790	0.849	1.29	0.132	0.145	5.93	2.37	0.164
2101	7900	3.05	7.1	0.376	0.0701	0.781	0.842	1.31	0.133	0.148	5.92	2.36	0.164
2201	7800	2.97	6.8	0.381	0.0710	0.772	0.836	1.32	0.133	0.152	5.92	2.35	0.163
2301	7700	2.90	6.6	0.385	0.0719	0.763	0.830	1.34	0.133	0.156	5.92	2.35	0.162
2401	7600	2.82	6.3	0.390	0.0728	0.754	0.824	1.36	0.134	0.160	5.91	2.34	0.160
2501	7500	2.75	6.1	0.394	0.0737	0.745	0.819	1.38	0.134	0.164	5.91	2.32	0.159
2601	7400	2.68	5.9	0.398	0.0746	0.737	0.813	1.39	0.135	0.167	5.91	2.31	0.157
2701	7300	2.61	5.6	0.402	0.0756	0.729	0.808	1.41	0.135	0.171	5.90	2.30	0.155
2801	7200	2.53	5.4	0.405	0.0765	0.721	0.803	1.43	0.136	0.175	5.90	2.29	0.153
2901	7100	2.46	5.2	0.409	0.0774	0.714	0.798	1.44	0.136	0.179	5.90	2.27	0.150
3001	7000	2.40	4.9	0.412	0.0783	0.706	0.793	1.46	0.137	0.183	5.89	2.26	0.148
3101	6900	2.33	4.7	0.416	0.0792	0.699	0.789	1.48	0.137	0.187	5.89	2.25	0.144
3201	6800	2.26	4.5	0.419	0.0801	0.692	0.785	1.49	0.138	0.191	5.88	2.23	0.141
3301	6700	2.19	4.3	0.422	0.0810	0.685	0.781	1.51	0.138	0.195	5.88	2.21	0.137
3401	6600	2.13	4.1	0.425	0.0818	0.679	0.777	1.53	0.139	0.199	5.88	2.19	0.133
3501	6500	2.07	3.9	0.428	0.0827	0.672	0.773	1.54	0.139	0.203	5.87	2.18	0.129
3601	6400	2.00	3.7	0.430	0.0835	0.666	0.770	1.56	0.140	0.207	5.87	2.16	0.124
3701	6300	1.94	3.6	0.433	0.0844	0.660	0.766	1.57	0.140	0.211	5.87	2.13	0.119
3801	6200	1.88	3.4	0.435	0.0852	0.655	0.763	1.59	0.141	0.215	5.86	2.11	0.113
3901	6100	1.82	3.2	0.437	0.0860	0.649	0.760	1.60	0.142	0.219	5.86	2.09	0.107
4001	6000	1.76	3.1	0.439	0.0867	0.644	0.757	1.62	0.142	0.223	5.86	2.07	0.101
4101	5900	1.70	2.9	0.441	0.0874	0.640	0.755	1.63	0.143	0.226	5.85	2.04	0.094
4201	5800	1.64	2.8	0.443	0.0881	0.635	0.752	1.64	0.143	0.229	5.85	2.01	0.086
4301	5700	1.59	2.7	0.444	0.0888	0.631	0.750	1.66	0.144	0.233	5.84	1.99	0.078
4401	5600	1.53	2.6	0.446	0.0894	0.627	0.748	1.67	0.144	0.236	5.84	1.96	0.069
4501	5500	1.48	2.5	0.447	0.0899	0.624	0.747	1.68	0.144	0.238	5.84	1.93	0.060
4601	5400	1.43	2.4	0.448	0.0904	0.621	0.745	1.69	0.145	0.241	5.83	1.90	0.050
4701	5300	1.37	2.3	0.449	0.0908	0.619	0.744	1.69	0.145	0.243	5.83	1.86	0.039
4801	5200	1.32	2.3	0.450	0.0911	0.617	0.743	1.70	0.145	0.244	5.83	1.83	0.028
4901	5100	1.27	2.2	0.450	0.0913	0.616	0.742	1.70	0.145	0.245	5.82	1.79	0.016
5001	5000	1.22	2.2	0.451	0.0914	0.615	0.742	1.71	0.146	0.246	5.82	1.75	0.003
5101	4900	1.17	2.2	0.450	0.0914	0.615	0.742	1.70	0.146	0.246	5.82	1.70	-0.012
5201	4800	1.13	2.2	0.450	0.0912	0.617	0.743	1.70	0.145	0.245	5.81	1.66	-0.027
5301	4700	1.08	2.3	0.449	0.0907	0.619	0.744	1.69	0.145	0.242	5.81	1.60	-0.044
5401	4600	1.03	2.4	0.448	0.0901	0.623	0.746	1.68	0.145	0.239	5.80	1.54	-0.062



(a) SEMI-INFINITE VESSEL



(b) CYLINDRICAL VESSEL

FIGURE 1. PACKINGS OF EQUAL SPHERES IN SEMI-INFINITE AND CYLINDRICAL VESSELS (SHOWING TWO REGIONS AND THREE SPHERE CATEGORIES)

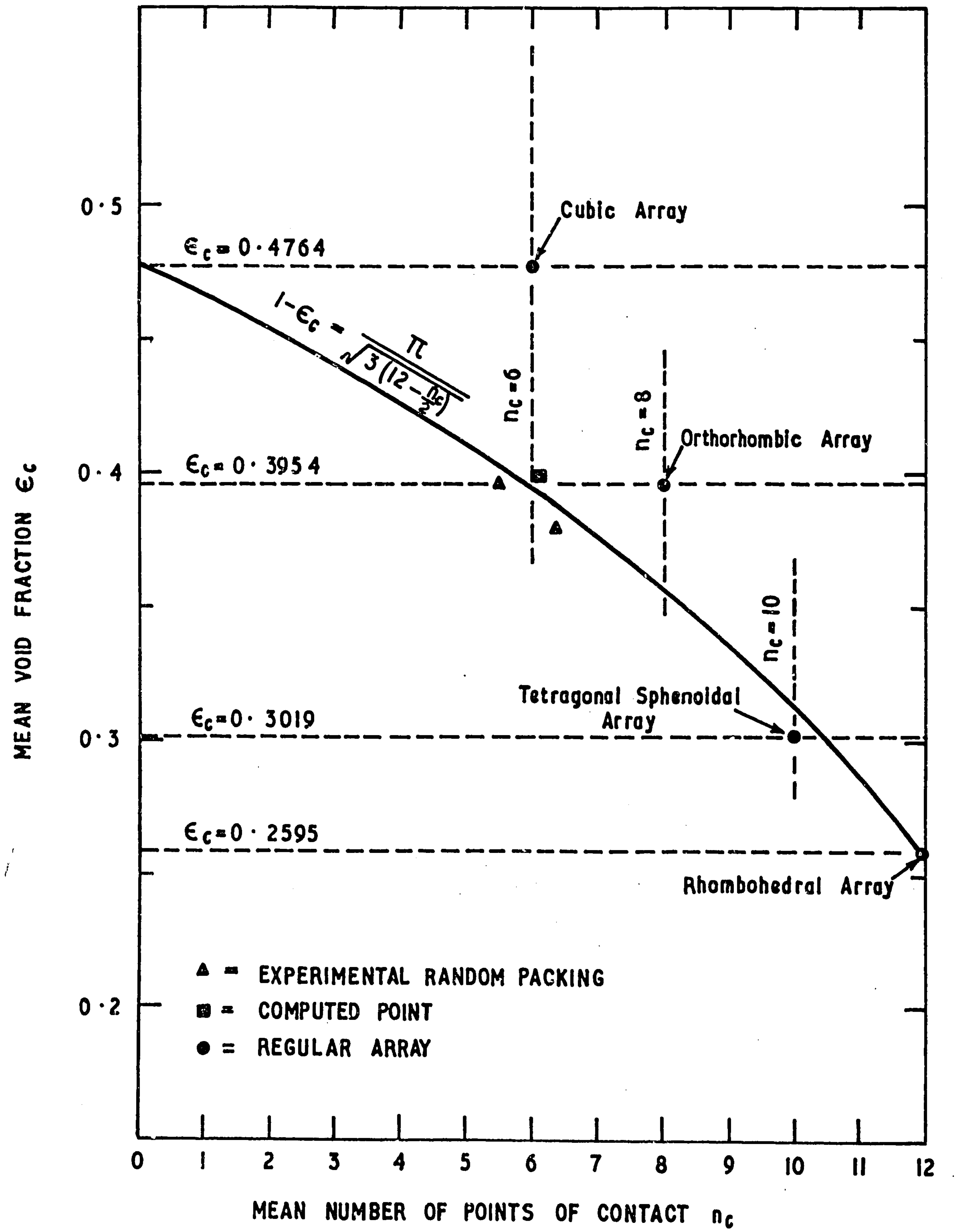


FIGURE 2. PROPERTIES OF THREE-DIMENSIONAL PACKINGS AWAY FROM WALL EFFECTS (After Figure 1 in Part III)

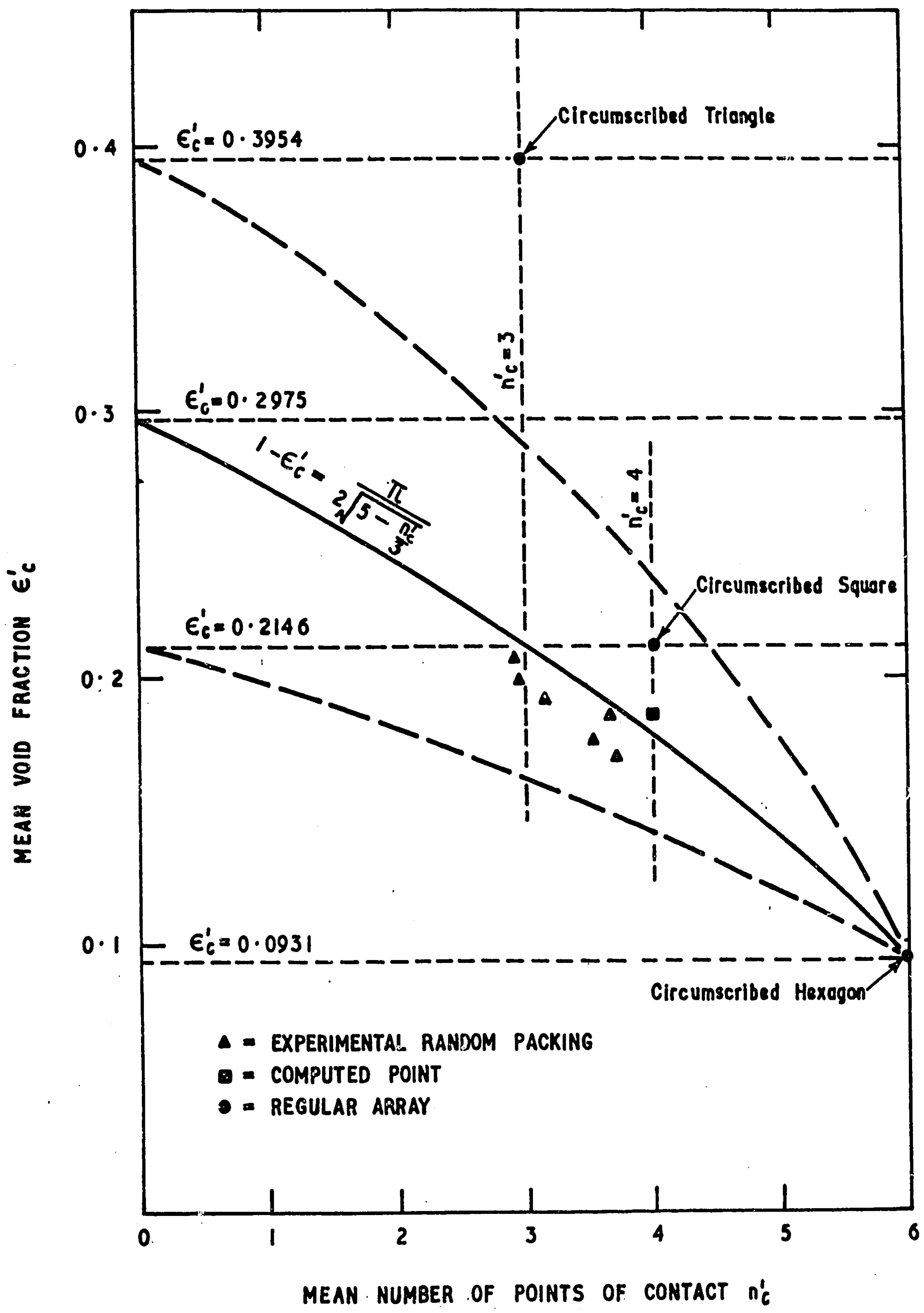


FIGURE 3. PROPERTIES OF TWO-DIMENSIONAL PACKINGS AWAY FROM EDGE EFFECTS (After Figure 3 in Part III)

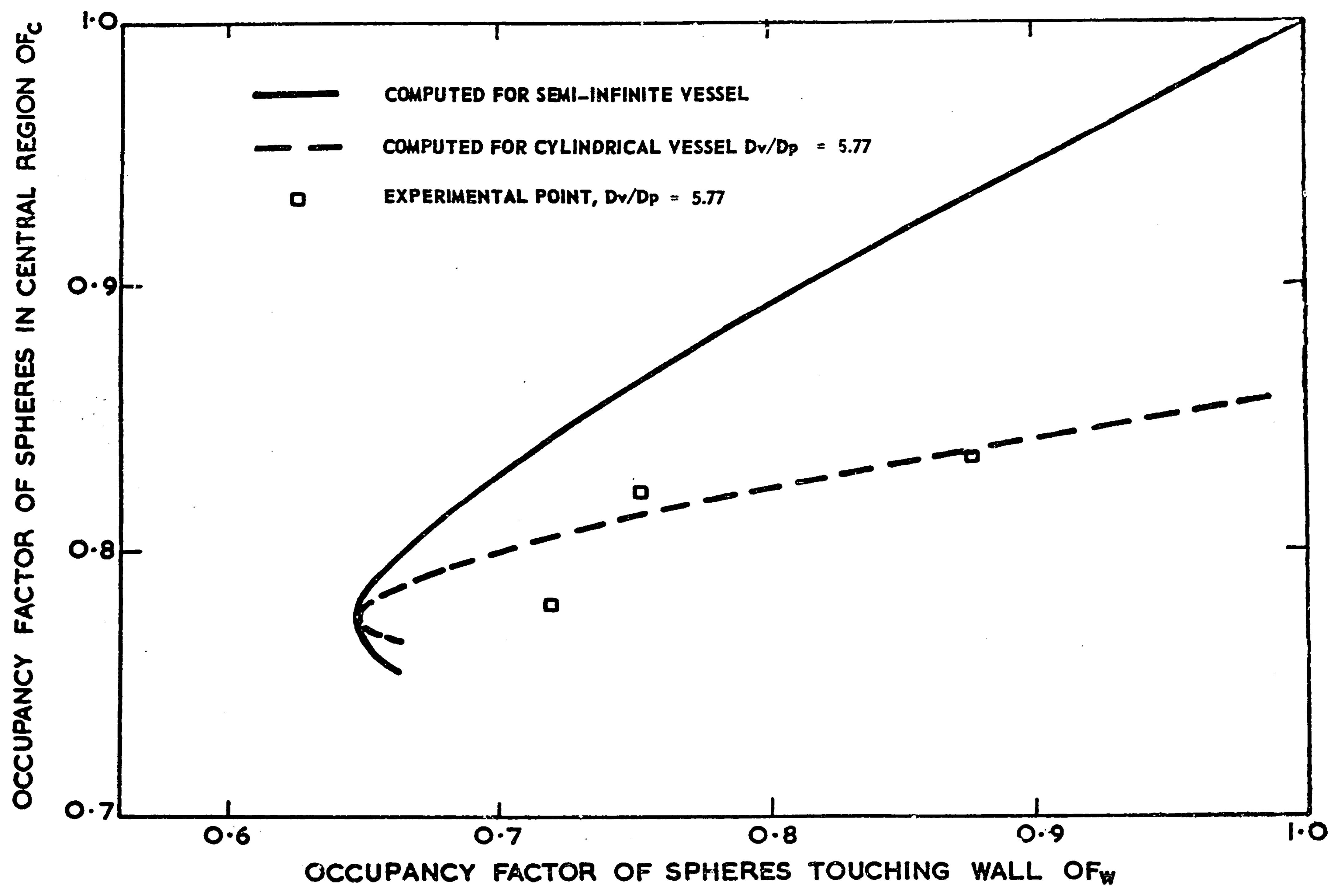


FIGURE 4. OBSERVED AND COMPUTED OCCUPANCY FACTORS OF CENTRAL REGION (FIRST APPROACH)

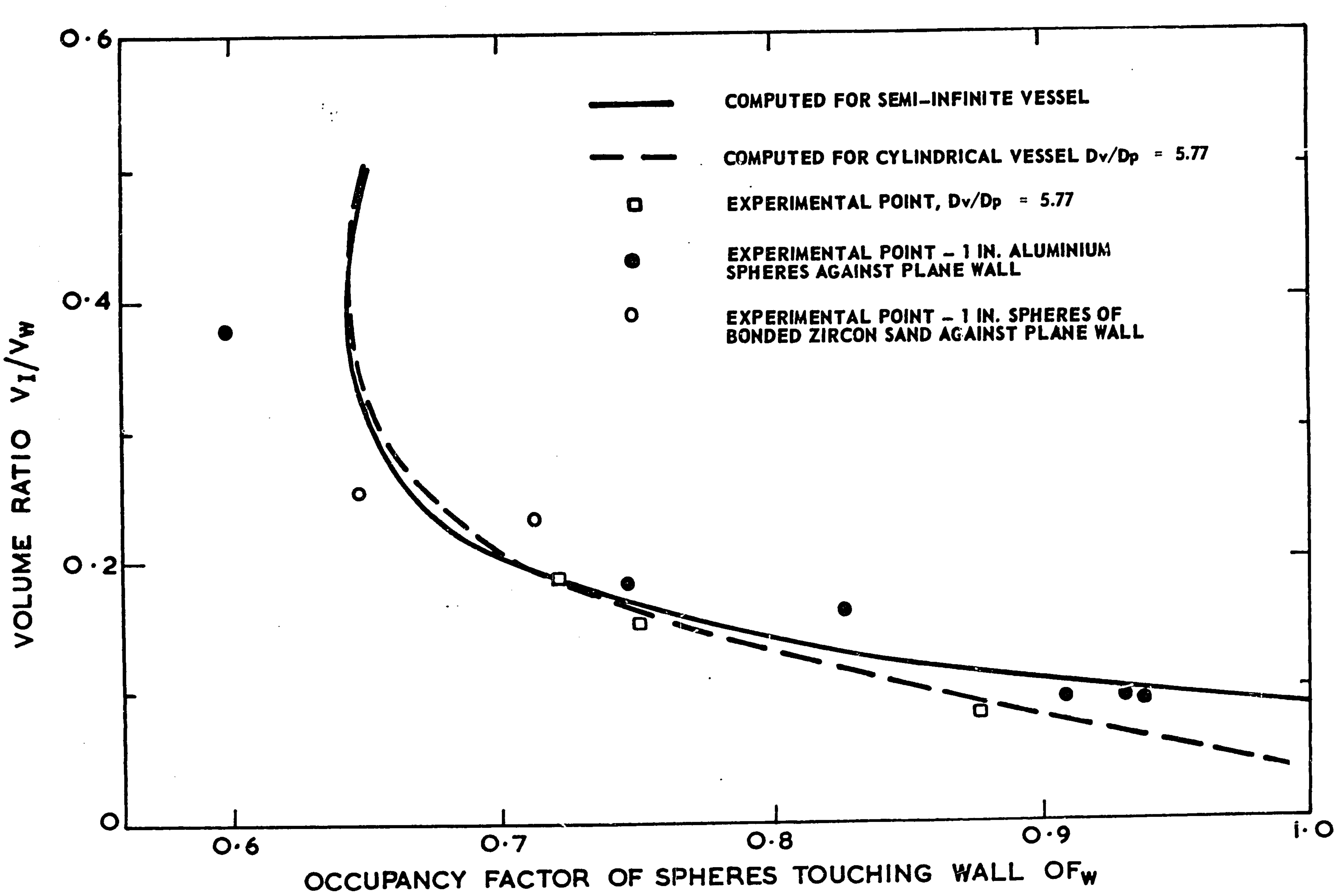


FIGURE 5. OBSERVED AND COMPUTED VOLUME RATIOS BETWEEN INTRUDING SPHERES AND SPHERES TOUCHING WALL (FIRST APPROACH)

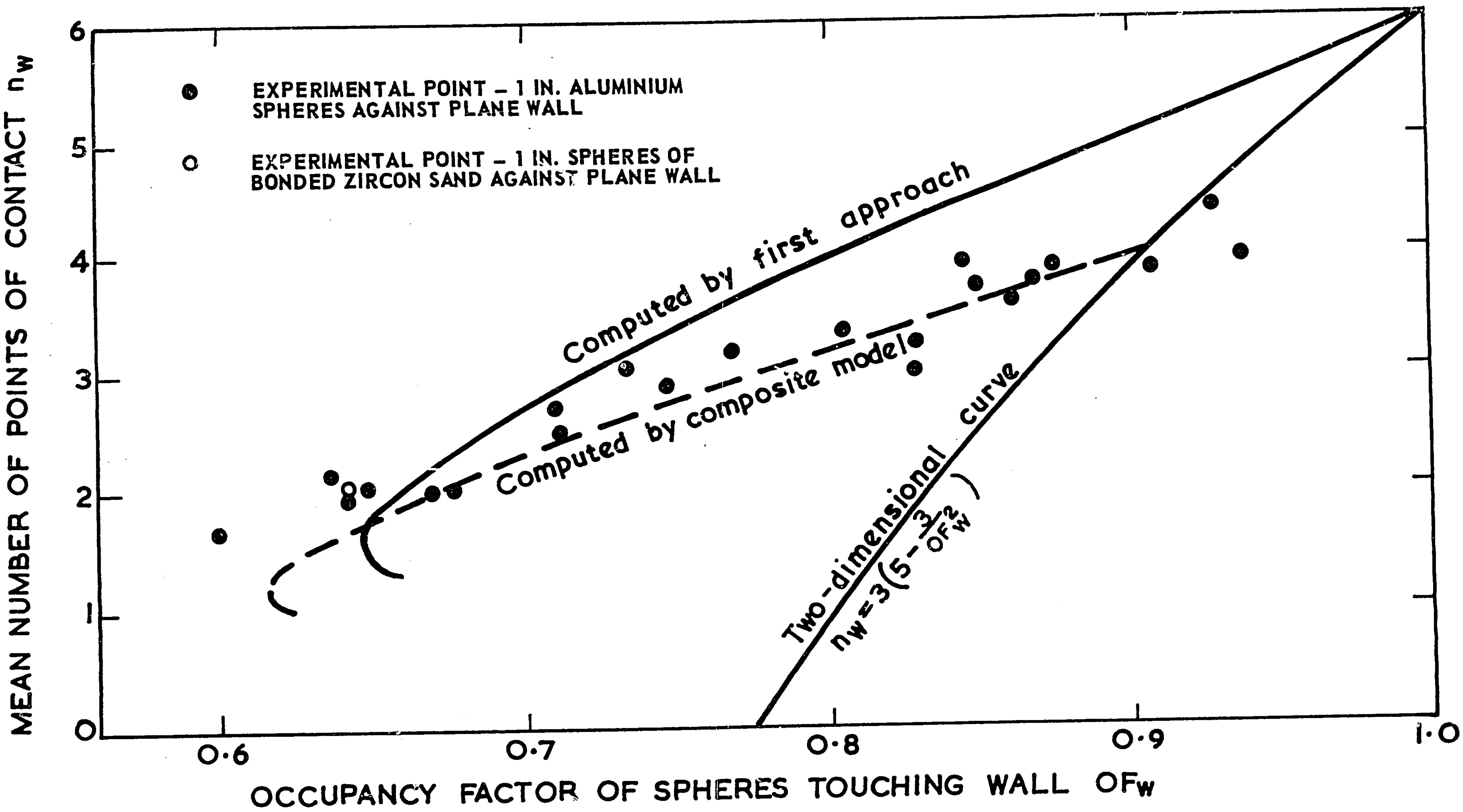


FIGURE 6. OBSERVED AND COMPUTED NUMBERS OF POINTS OF CONTACT BETWEEN SPHERES TOUCHING WALL

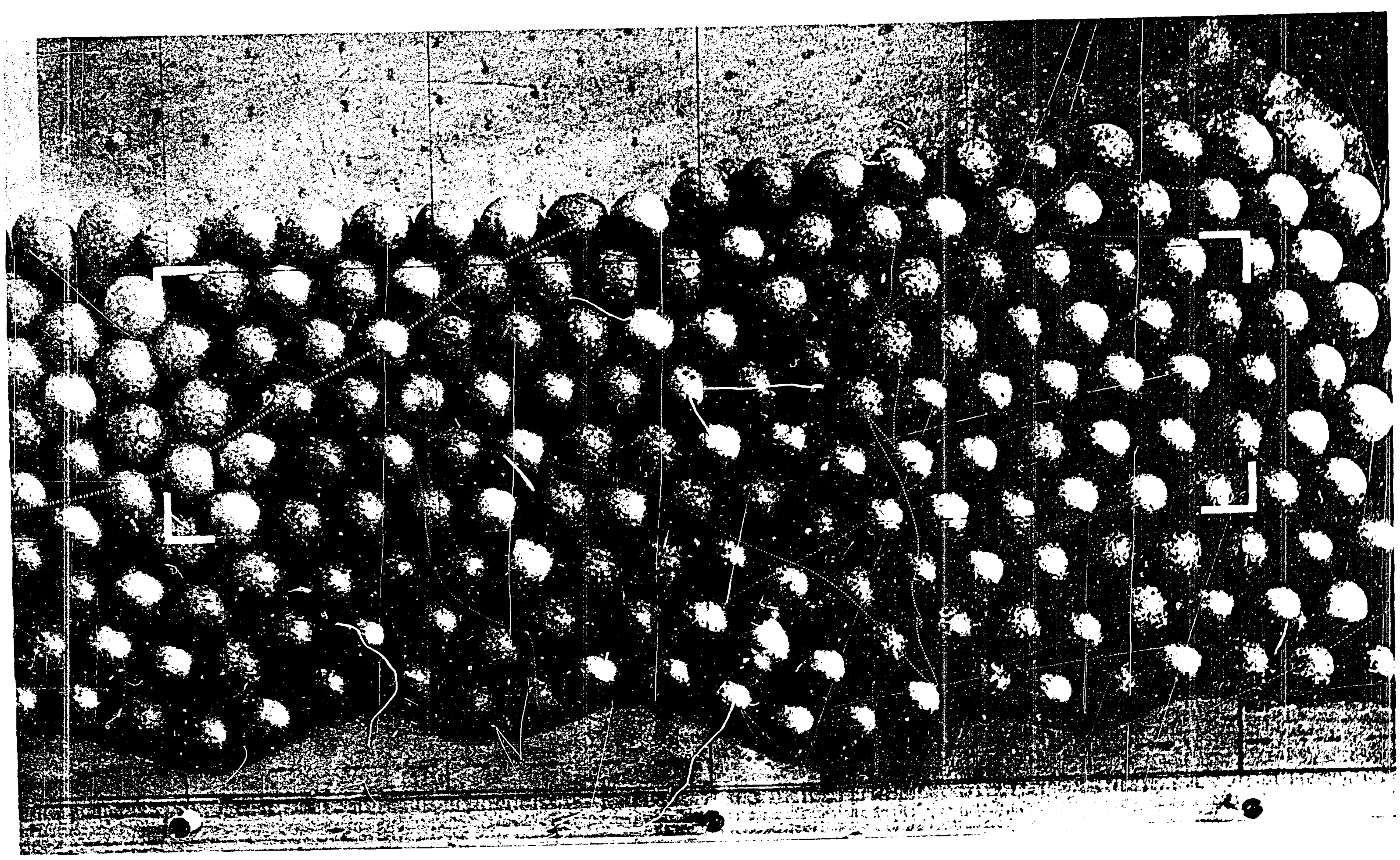


FIGURE 7. TYPICAL OUTER LAYER OF HEAVILY VIBRATED PACKING IN PRISMATIC VESSEL LOOKING THROUGH TRANSPARENT VERTICAL PLANE WALL (Test area outlined)

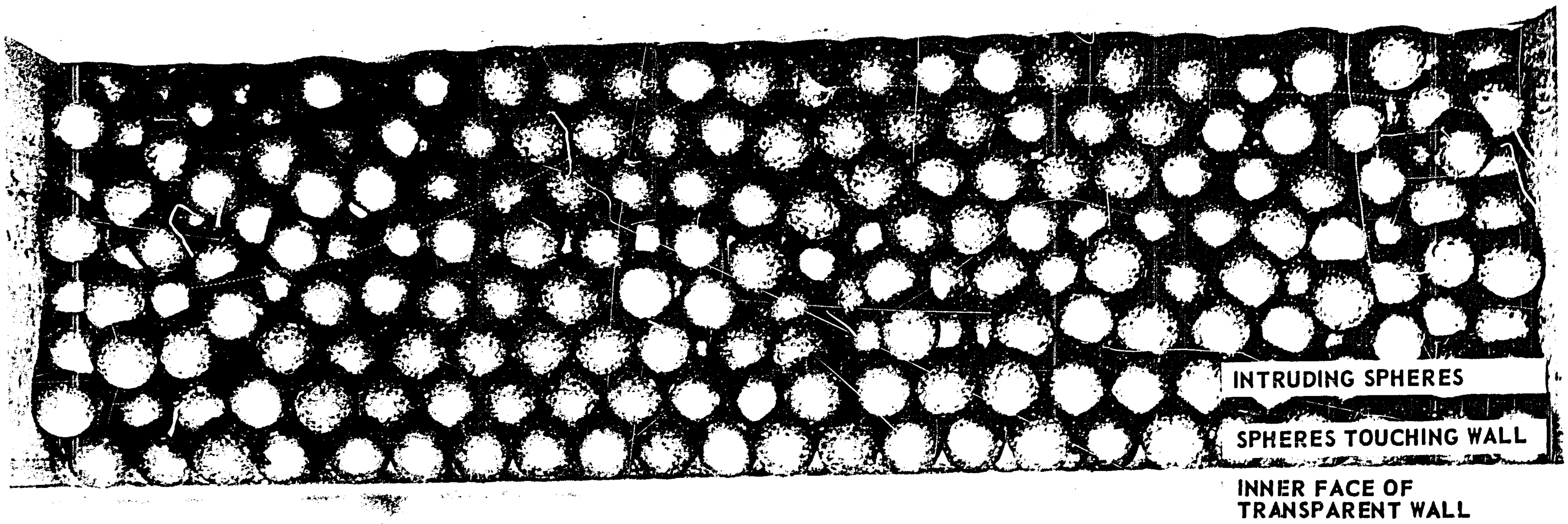


FIGURE 8. TYPICAL HORIZONTAL LAYER OF HEAVILY VIBRATED PACKING IN PRISMATIC VESSEL, LOOKING VERTICALLY DOWN INNER FACE OF TRANSPARENT PLANE WALL

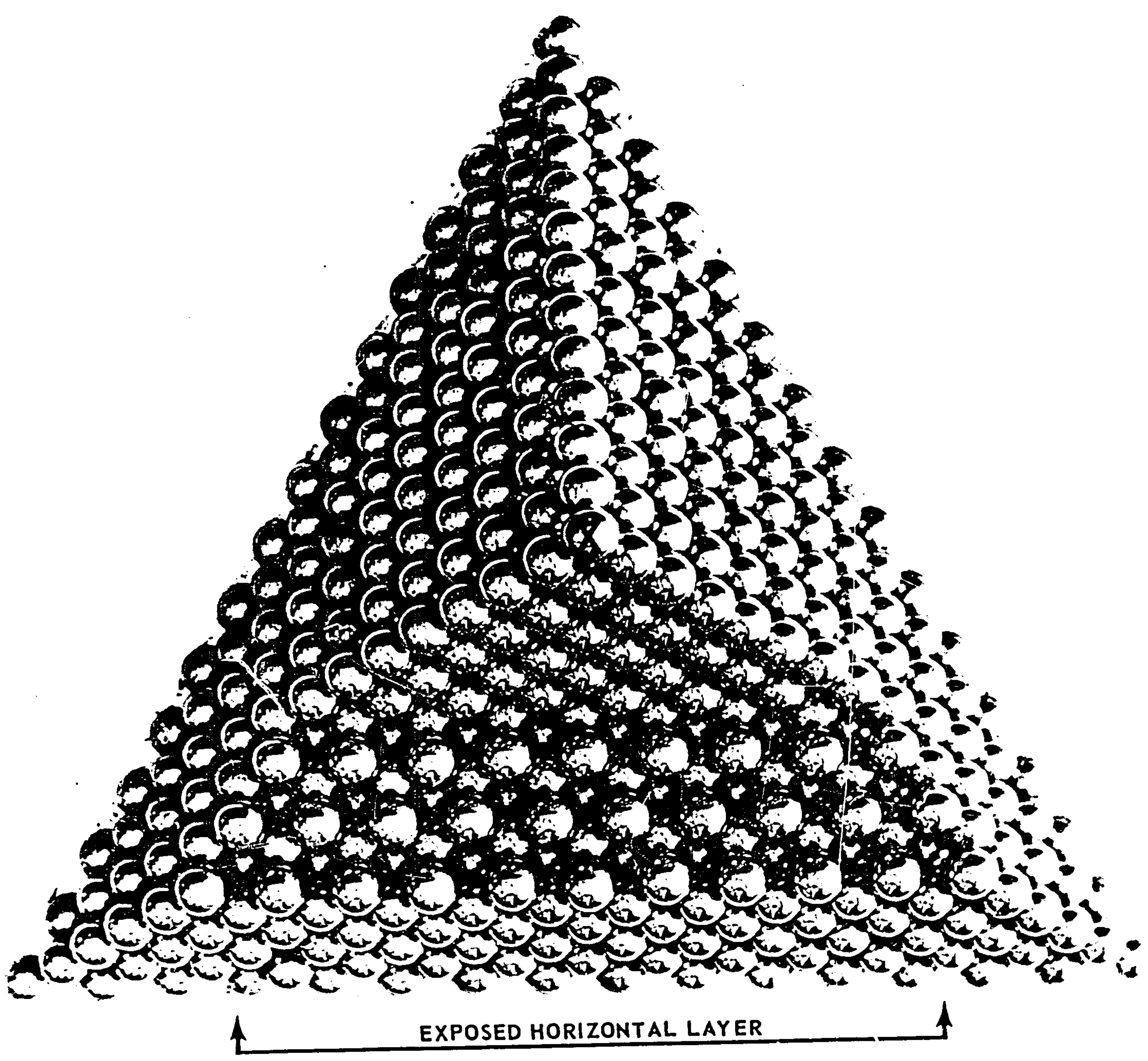
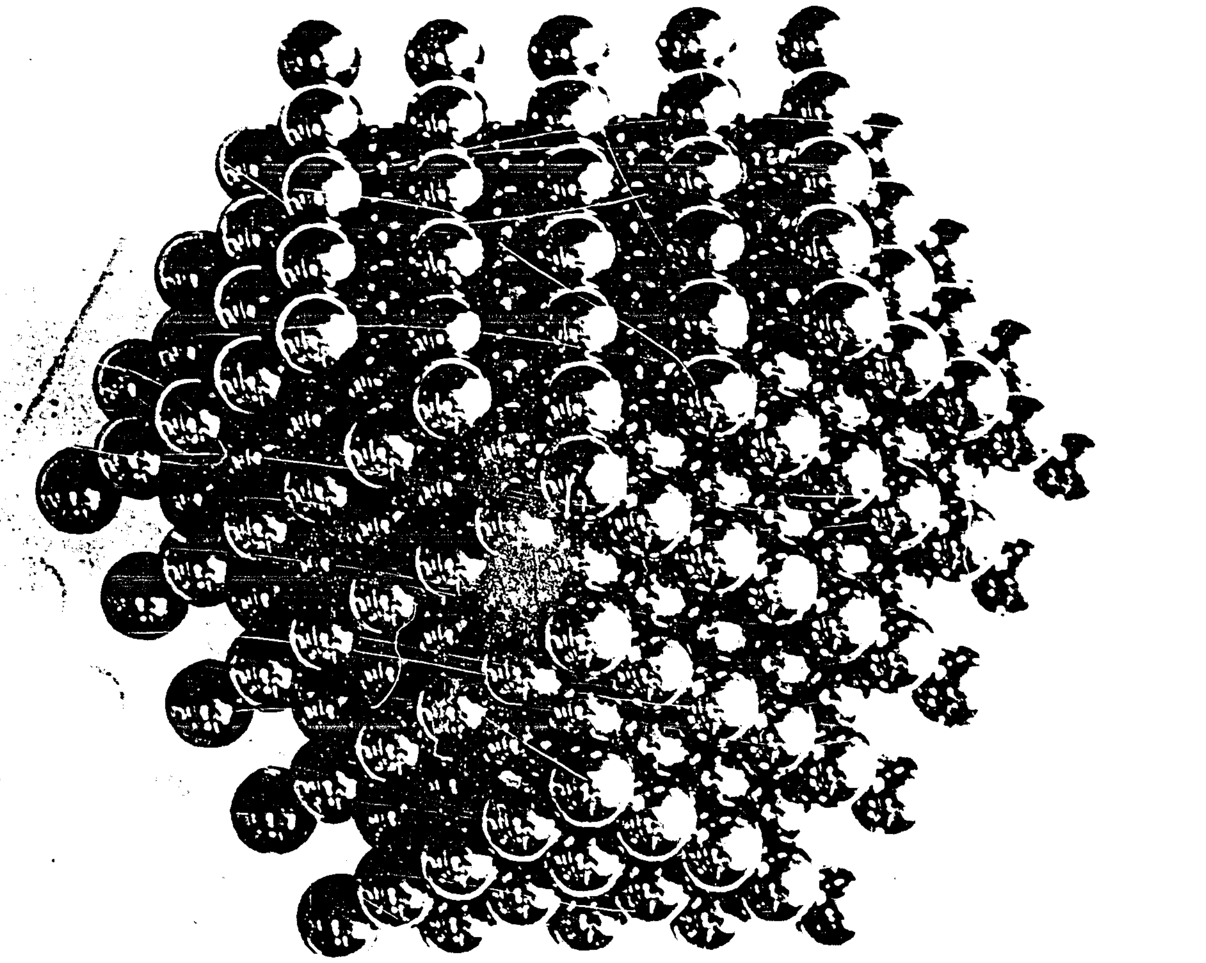
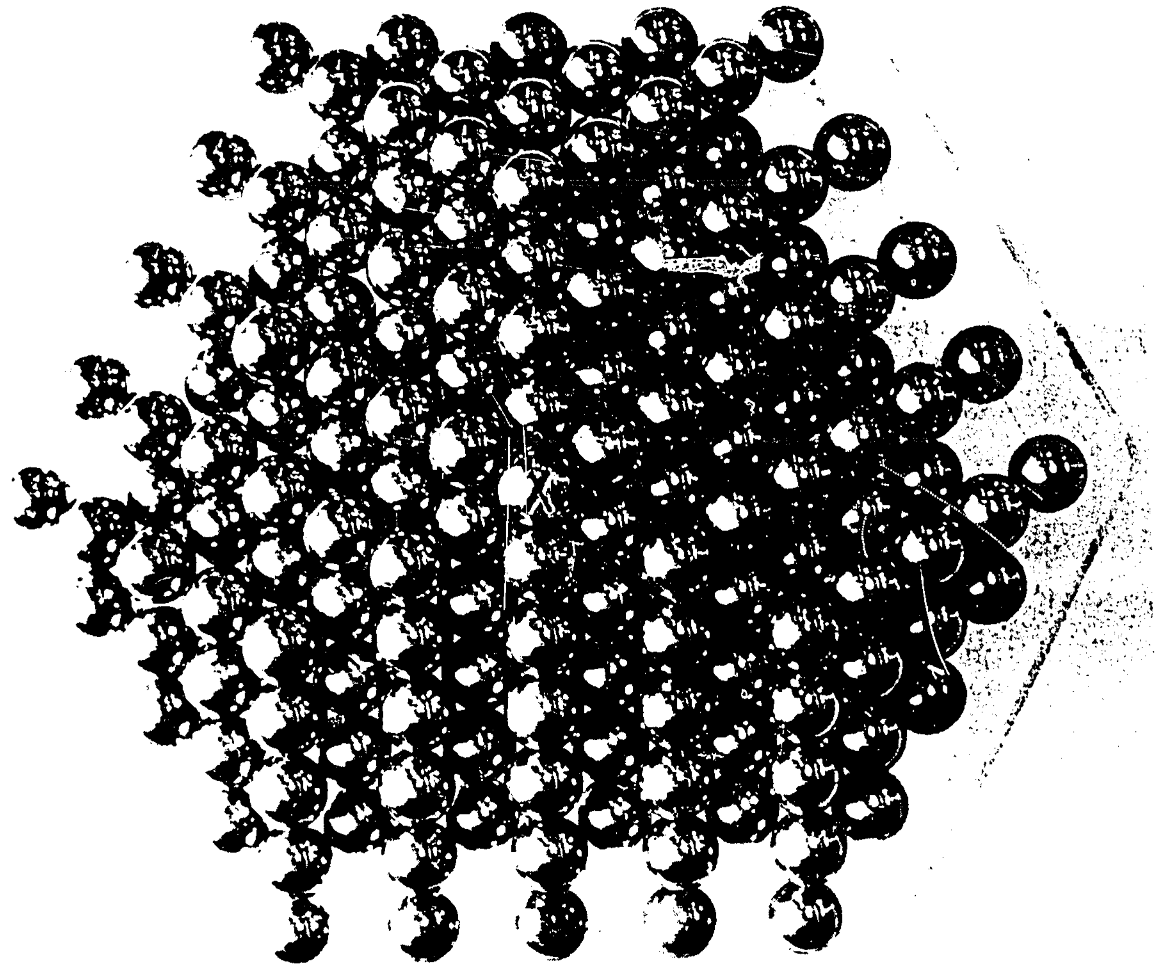


FIGURE 9. TILTED CUBIC ARRAY ON TRIANGULAR BASE (Top sphere marked X)



(a) TILTED CUBIC ARRAY ($\epsilon = 0.4764$)



(b) LIMITING CASE ($\epsilon = 0.3198$)

FIGURE 10. TWO OF THE BLOCKED PASSAGE EQUILATERAL TRIANGULAR ARRAYS ON HEXAGONAL BASES (Top spheres marked X)

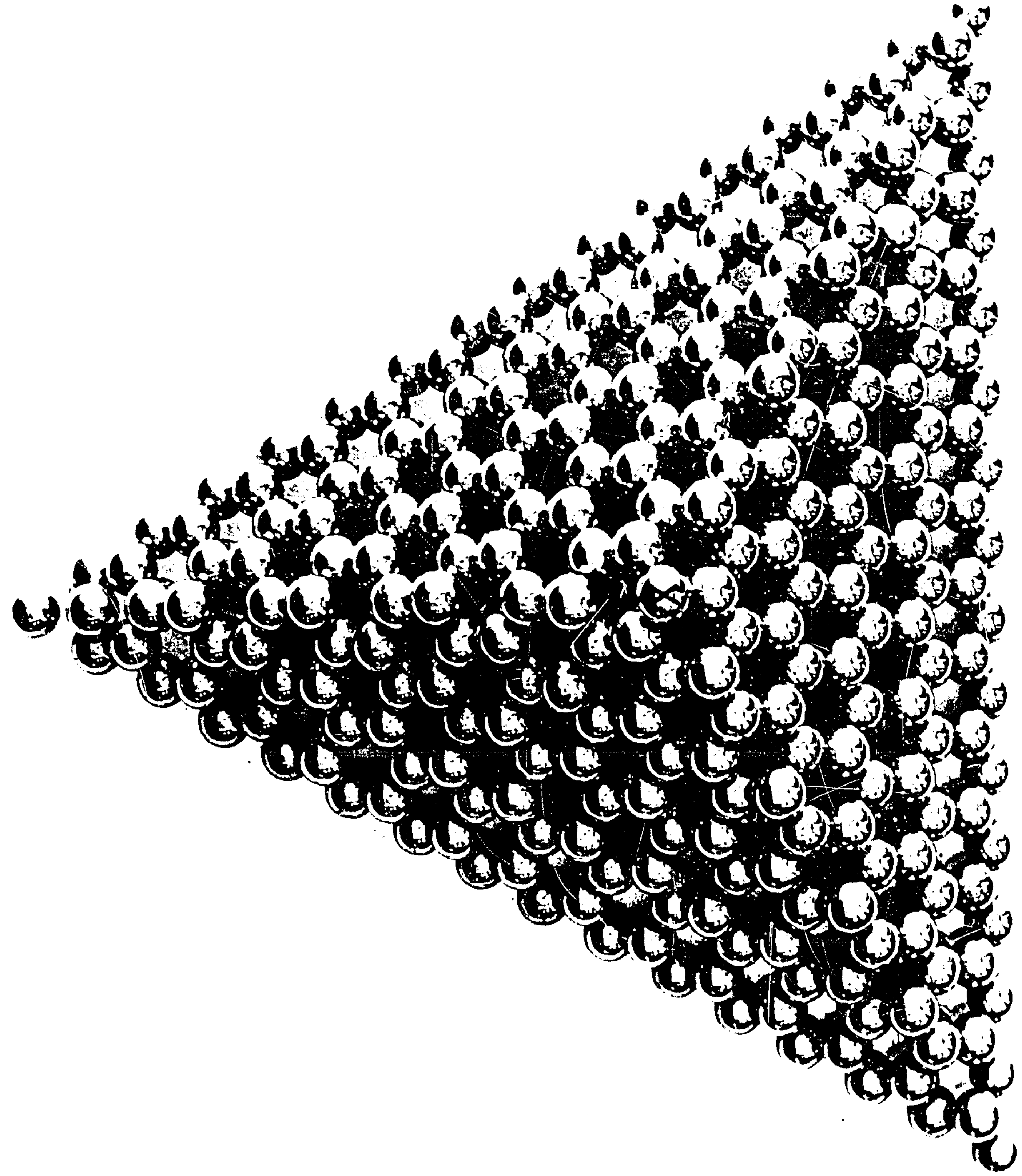
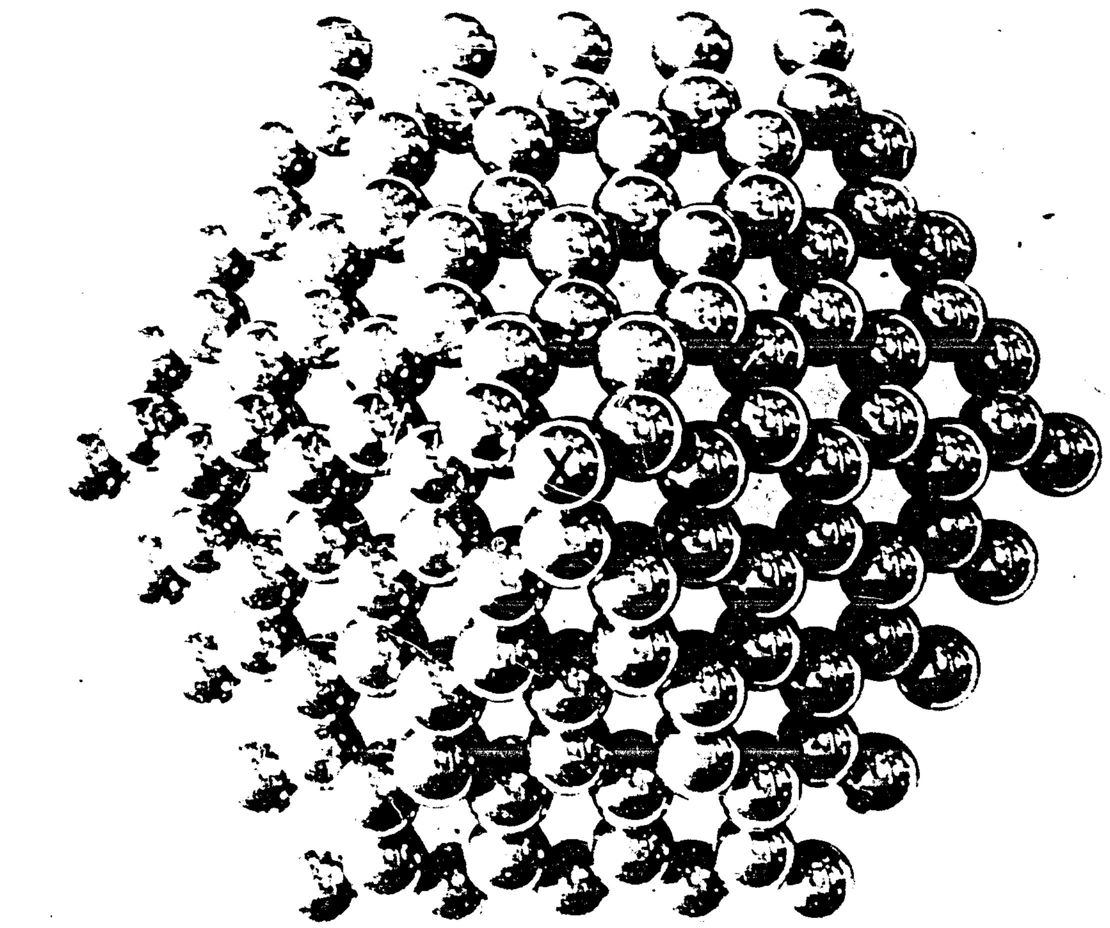
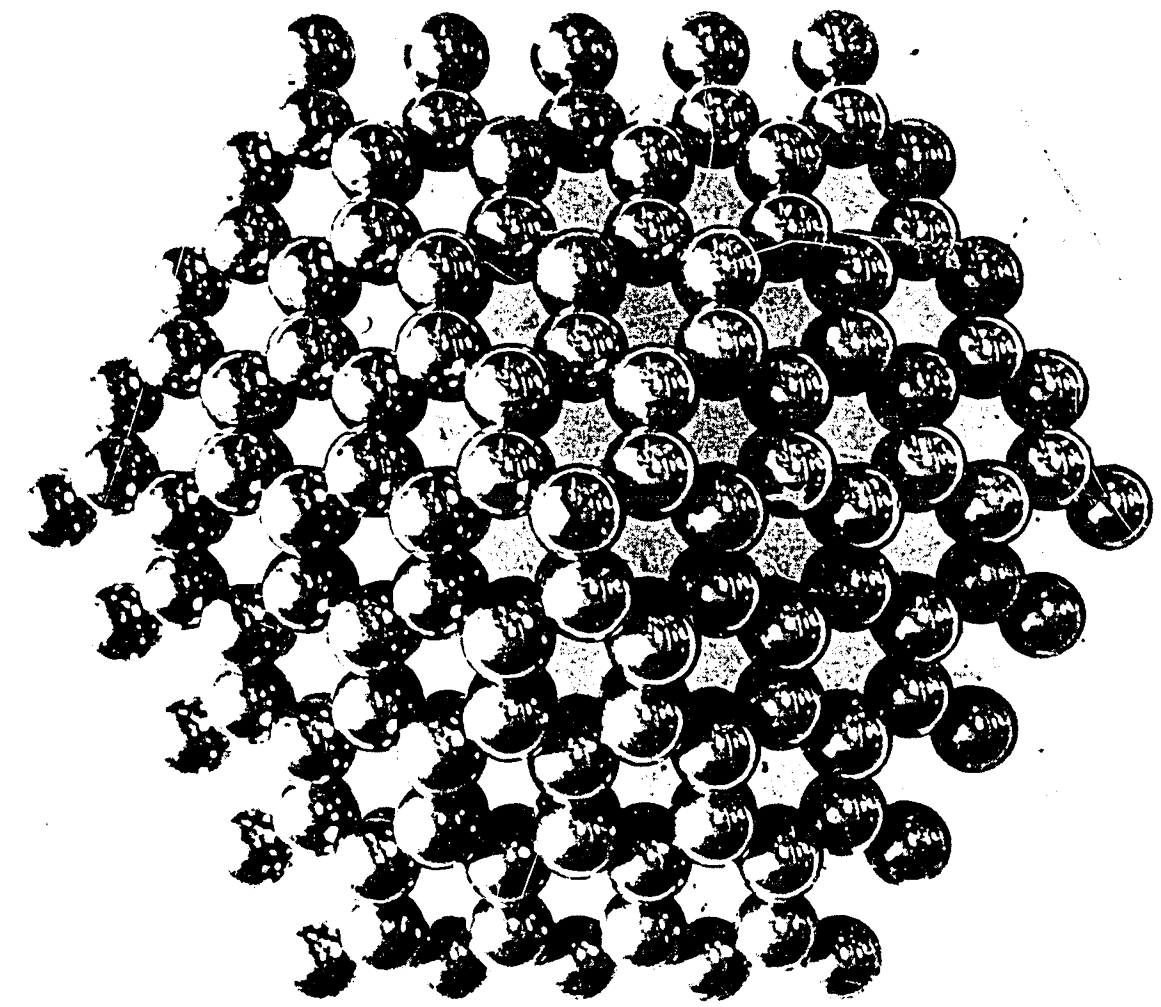


FIGURE 11. CLEAR PASSAGE EQUIVALENT OF TILTED CUBIC ARRAY ON TRIANGULAR BASE (Top sphere marked X)



(a) EQUIVALENT OF TILTED CUBIC ARRAY ($\epsilon = 0.4764$)



(b) LIMITING CASE ($\epsilon = 0.4626$)

FIGURE 12. TWO OF THE CLEAR PASSAGE EQUILATERAL TRIANGULAR ARRAYS ON HEXAGONAL BASES (Top spheres marked X)

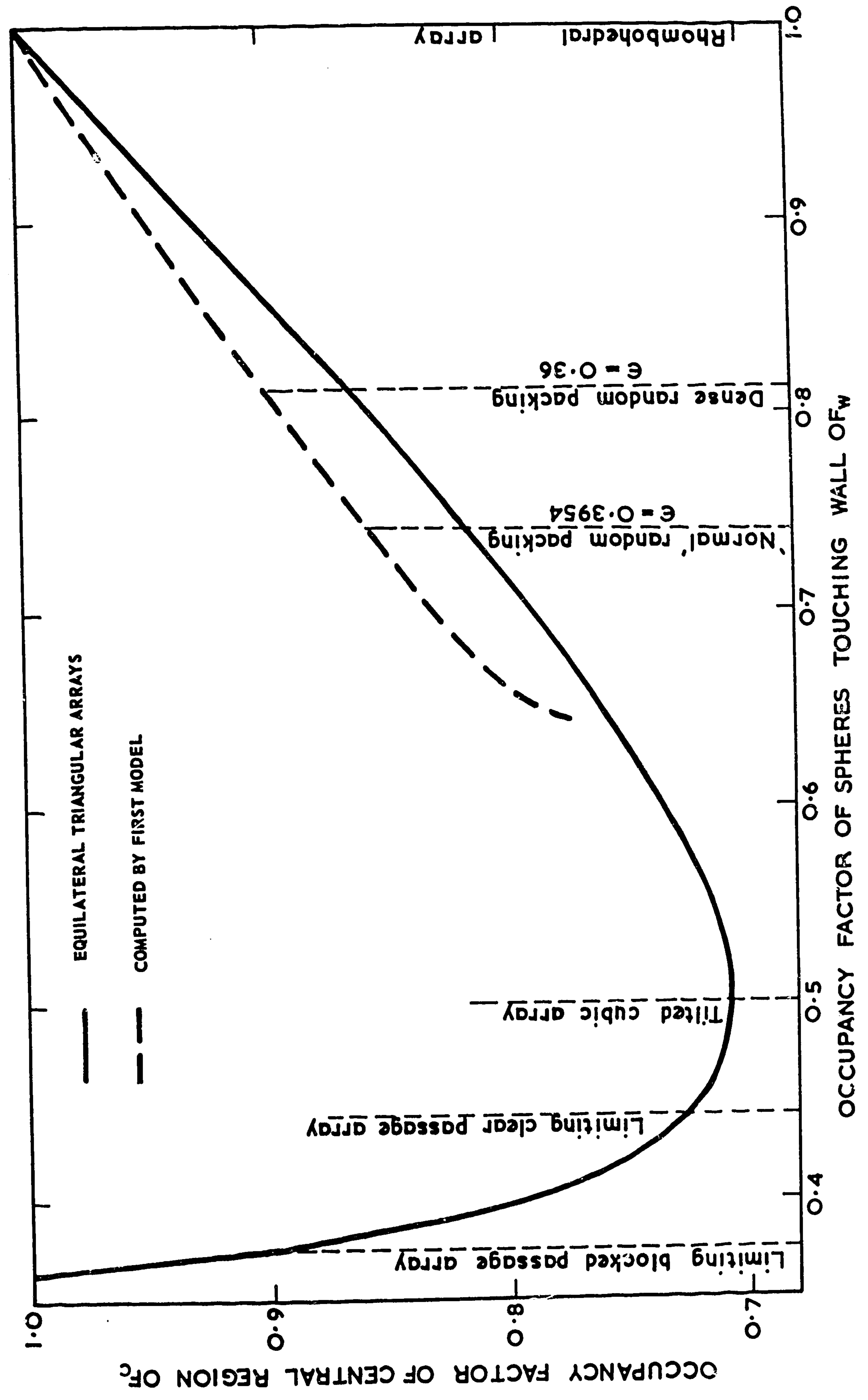


FIGURE 13. COMPARISON OF COMPUTED OCCUPANCY FACTORS OF CENTRAL REGION

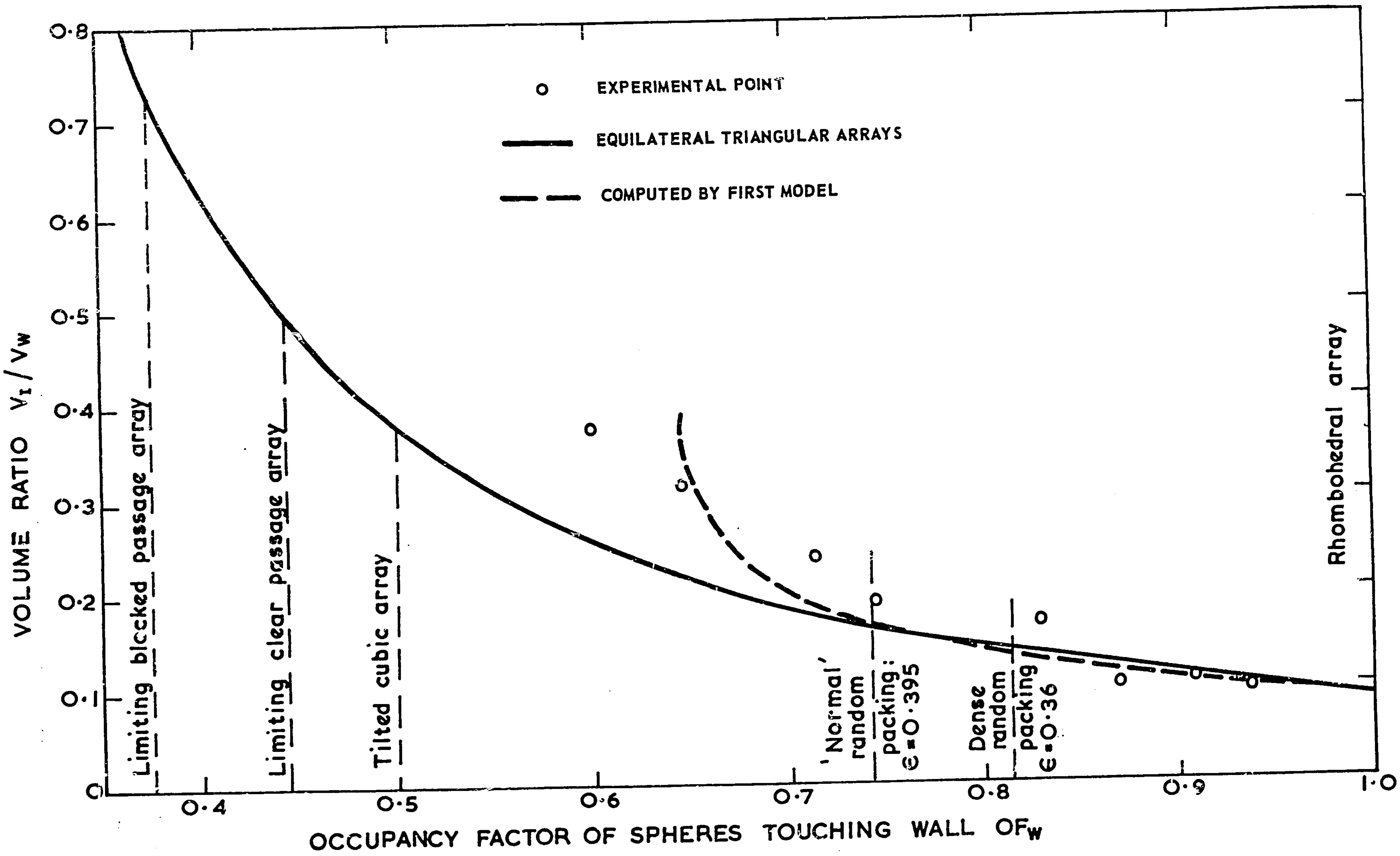


FIGURE 14. OBSERVED AND COMPUTED VOLUME RATIOS BETWEEN INTRUDING SPHERES AND SPHERES TOUCHING WALL (BEFORE ADJUSTMENT)

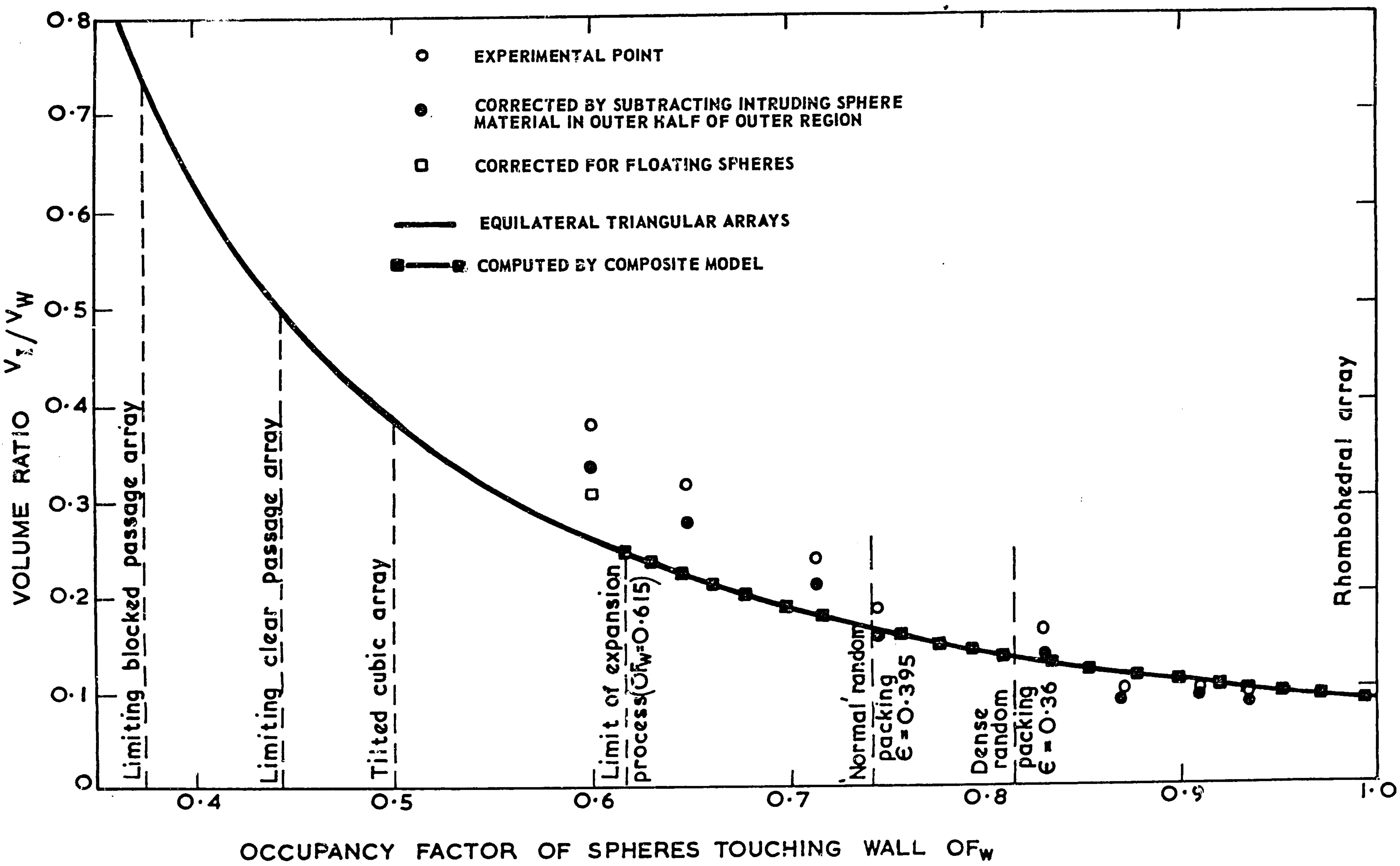


FIGURE 15. OBSERVED AND COMPUTED VOLUME RATIOS BETWEEN INTRUDING SPHERES AND SPHERES TOUCHING WALL (AFTER ADJUSTMENT)

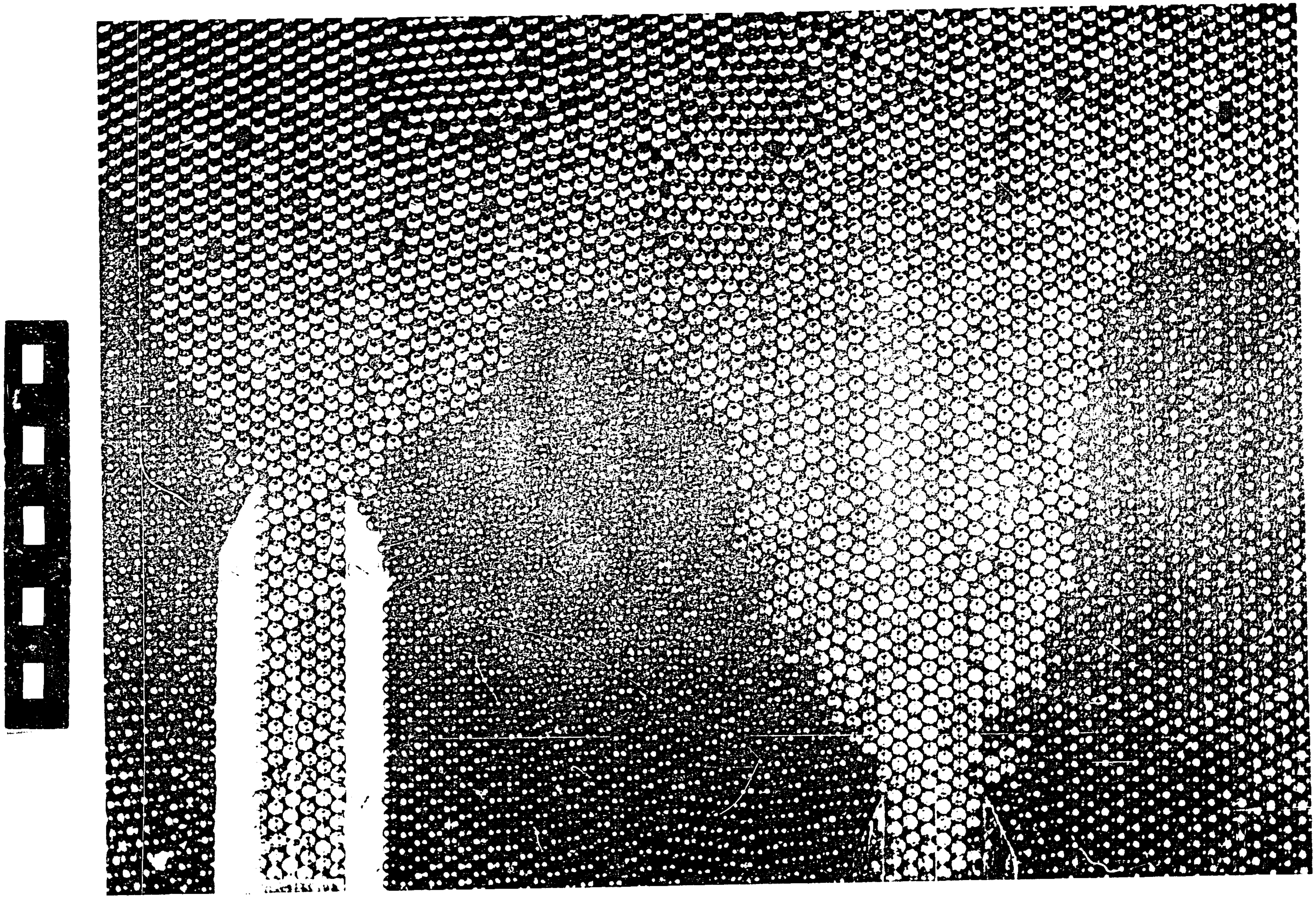


FIGURE 16. STRUCTURE RESULTING FROM FLOW IN TWO DIMENSIONAL PACKING
(After R.H. Nelmes, AAEC Unpublished)